You should attempt all the problems. Partial credit will be give for serious efforts

Part I: Algebraic Topology

(1) A $k$-fold cross cap $X_k$ ($k > 2$) is the space obtained by attaching a disk $D$ to a circle $S^1$ by the map $f: \partial D \to S^1$ given by $f(z) = z^k$ (here we view $S^1$ as the unit circle in the complex plane, i.e., $z \in \mathbb{C}$ and $|z| = 1$). For example, $X_2$ is the projective plane. Compute $H_p(X_k)$, $H_p(X_k)$, $H_p(X_k; \mathbb{Z}_k)$, $H_p(X_k; \mathbb{Z}_k)$, $H_p(X_k, S^1)$ and $H_p(X_k, S^1; \mathbb{Z}_k)$ for all $p \geq 0$. ($\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$).

(2) Let $M$ and $N$ be closed orientable surfaces. Suppose there is a degree-one map $f: M \to N$.
   (a) Prove that $f_*: \pi_1(M) \to \pi_1(N)$ is surjective.
   (b) List all possible surfaces $M$ and $N$. (Justify your answer)

(3) Let For a connected space $X$, we define the rank of $\pi_1(X)$ to be the minimal number of elements needed to generate $\pi_1(X)$. Let $M$ and $N$ be closed orientable manifolds. Determine whether or not the followings are true. Justify your answer.
   (a) If there is a degree one map $f: M \to N$, then $\text{rank}(\pi_1(M)) \geq \text{rank}(\pi_1(N))$.
   (b) If there is a covering map $f: M \to N$, then $\text{rank}(\pi_1(M)) \geq \text{rank}(\pi_1(N))$.

(4) Prove that a closed non-orientable surface can not be embedded in $\mathbb{R}^3$. 