QUALIFYING EXAM IN GEOMETRY AND TOPOLOGY, WINTER 2014

You should attempt all the problems. Partial credit will be give for serious efforts

Part I: Algebraic Topology

(1) A k-fold cross cap X_k ($k > 2$) is the space obtained by attaching a disk D to a circle S^1 by the map $f : \partial D \to S^1$ given by $f(z) = z^k$ (here we view S^1 as the unit circle in the complex plane, i.e., $z \in \mathbb{C}$ and $|z| = 1$). For example, X_2 is the projective plane. Compute $H_p(X_k)$, $H^p(X_k)$, $H_p(X_k; \mathbb{Z}_k)$, $H^p(X_k; \mathbb{Z}_k)$, $H_p(X_k, S^1)$ and $H_p(X_k, S^1; \mathbb{Z}_k)$ for all $p \geq 0$. ($\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$).

(2) Let M and N be closed orientable surfaces. Suppose there is a degree-one map $f\colon M \to N$.

  (a) Prove that $f_*\colon \pi_1(M) \to \pi_1(N)$ is surjective.

  (b) List all possible surfaces M and N. (Justify your answer)

(3) Let For a connected space X, we define the *rank* of $\pi_1(X)$ to be the minimal number of elements needed to generate $\pi_1(X)$. Let M and N be closed orientable manifolds. Determine whether or not the followings are true. Justify your answer.

  (a) If there is a degree one map $f\colon M \to N$, then $rank(\pi_1(M)) \geq rank(\pi_1(N))$.

  (b) If there is a covering map $f\colon M \to N$, then $rank(\pi_1(M)) \geq rank(\pi_1(N))$.

(4) Prove that a closed non-orientable surface can not be embedded in $\mathbb{R}^3$.