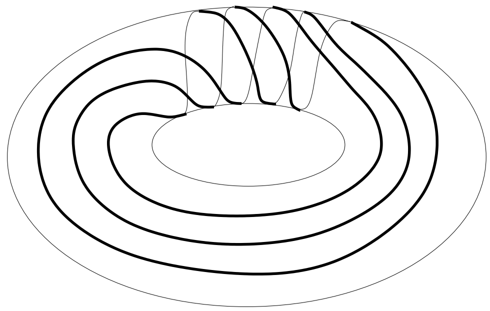
Summer 2014 Geometry / Topology Qualifying Examination

§1. Algebraic Topology

(1) Consider the knot $K \subset S^3$ depicted below. It is realized as a simple closed curve on a *standardly embedded* torus $T \subset S^3$, meaning that $S^3 \setminus T$ consists of two open solid tori.

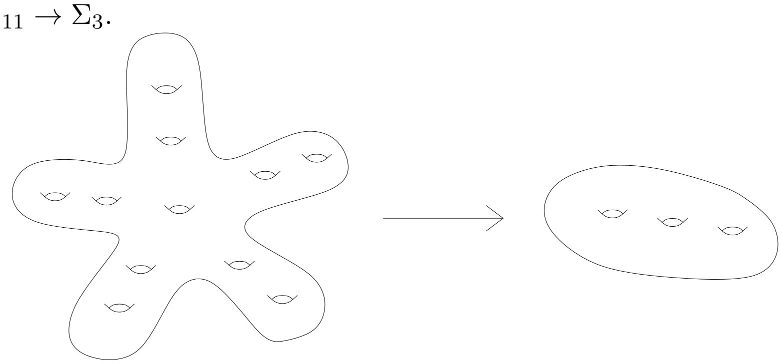


(a) Choose a basepoint $* \in S^3 \setminus K$ and determine a presentation of $\pi_1(S^3 \setminus K, *)$ involving two generators and one relator.

(b) Summarize how you would show that K is not isotopic to a trefoil knot. (Full details are appreciated but not required.)

(2) For a non-negative integer g, let Σ_g denote the closed, connected, orientable surface of genus g.

(a) Drawing inspiration from the picture below, briefly explain how to construct a 5-sheeted covering map $\Sigma_{11} \to \Sigma_3$.



(b) Generalizing part (a), for every $d, g \in \mathbb{Z}$, $d \geq 1$, $g \geq 0$, explain how to construct a d-sheeted covering map $\Sigma_h \to \Sigma_g$ for an appropriate value h. What is h as a function of d and g?

(c) Prove that if there exists a d-sheeted covering map $\Sigma_h \to \Sigma_g$, then d, g, and h are related as in the answer to part (b).

(3) Prove that if M is a compact, orientable 3-manifold, then the kernel of the inclusion map $H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q})$ is a half-dimensional subspace of the domain. (You may assume that the kernel of a linear map is isomorphic to the cokernel of its adjoint.)

(4) (a) Describe a cell decomposition of $\mathbb{R}P^n$ involving one cell of each dimension from 0 to n inclusive.

(b) Write down the associated cell chain complex of $\mathbb{R}P^5$ with $\mathbb{Z}$ coefficients. Briefly justify your calculation of the boundary maps.

(c) Calculate $H_*(\mathbb{R}P^5; \mathbb{Z})$.

(d) Suppose that X is a topological space with the property that $H_*(X; \mathbb{Z}) \approx H_*(\mathbb{R}P^5; \mathbb{Z})$ as graded abelian groups. Determine the cohomology groups of X with $\mathbb{Z}/4\mathbb{Z}$ coefficients. (Do not attempt to describe the multiplicative structure on the cohomology ring. Also note that you do not have a cell decomposition of X, just the isomorphism type of its ordinary homology groups).

§2. Differential Topology

(1) If M is a smooth manifold, show that the tangent bundle TM and the cotangent bundle T^*M are isomorphic. *(Just as with vector spaces, there is no* canonical *isomorphism. You don't have to prove this, though. Also, feel free to assume anything that you like from linear algebra.)*

(2) A *Lie homomorphism* is a smooth homomorphism between Lie groups.

(a) Show that any Lie homomorphism $\phi : G \longrightarrow H$ has constant rank: that is, there exists some $k \in \mathbb{Z}$ such that $\text{rank}(d\phi_g) = k$ for all $g \in G$.

(b) Suppose that G, H are connected n-dimensional Lie groups and $\phi : G \longrightarrow H$ is a Lie homomorphism with discrete kernel. Show that ϕ is a surjective diffeomorphism. *(In fact, ϕ is a covering map, but the proof of this is homework-level rather than exam-level.)*

(3) Write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ and let $\mathcal{G}$ be the pseudogroup generated by all diffeomorphisms ϕ between open subsets of $\mathbb{R}^n$ that take horizontal factors to horizontal factors: that is,

$$(*) \qquad \phi(x, y) = (\phi_1(x, y), \phi_2(y)),$$

for $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. Show that $\mathcal{G}$ consists of all diffeomorphisms between open subsets of $\mathbb{R}^n$ whose Jacobian matrix at every point is an $n \times n$ matrix such that the lower left $(n-k) \times k$ block is 0. *(Showing that the set of diffeomorphisms satisfying the Jacobian property is a pseudo-group is almost immediate, although you should at least say what the properties are. The real point here is to explain why it is the* minimal *pseudo-group containing all such ϕ.)*

A $\mathcal{G}$-structure on an n-manifold M is called a *codimension k foliation* of M. Since at least locally, the transition maps preserve the decomposition of $\mathbb{R}^n$ into horizontal slices, these slices piece together to give a decomposition of M into submanifolds, called the *leaves* of the foliation.

(4) Show that the antipodal map $A : S^n \longrightarrow S^n$, $A(x) = -x$ is homotopic to the identity if and only if n is odd. *(Feel free to use Lefschetz theory if you would like.)*

(5) Show that a closed 1-form ω on a manifold M is exact if and only if $\int_{S^1} f^*\omega = 0$ for every smooth map $f : S^1 \longrightarrow M$. *(Feel free to use Stokes' theorem, but you shouldn't reference deRham cohomology or anything that implicitly relies on this result.)*