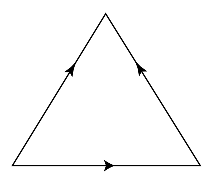
QUALIFYING EXAM IN GEOMETRY AND TOPOLOGY, SUMMER 2013

You should attempt all the problems. Partial credit will be give for serious efforts

Part I: Algebraic Topology

(1) The "dunce cap" space is the quotient of a triangle (and its interior) obtained by identifying all three edges in an inconsistent manner, as shown in the picture below.
   (a) Find the fundamental group of the "dunce cap" space.
   (b) Show that the "dunce cap" space is homotopy equivalent to a single point.



(2) Let $V \subset S^1 \times S^1$ be the union of $S^1 \times \{x\}$ and $\{x\} \times S^1$, where x is a point in S^1. Is the quotient map $q\colon S^1 \times S^1 \to S^2$ collapsing V to a point nullhomotopic? Justify your answer.

(3) (a) Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ have the same homology and cohomology groups.
   (b) Prove that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are not homotopy equivalent.

(4) Let M and N be closed orientable n-dimensional manifolds. Suppose there is a degree-one map $f\colon M \to N$. Prove that
   (a) f is surjective.
   (b) $f_*\colon \pi_1(M) \to \pi_1(N)$ is surjective.

Part II: Differential Topology

Answer all questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1
Let M be a smooth manifold and V, W smooth vector fields.
a) Prove that $\mathcal{L}_V W = [V, W]$.
b) Let V, W be the vector fields on $\mathbb{R}^2$ given by

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Find their flows.
c) Do the flows V, W commute?
d) If they do commute, find the coordinate function centered at $(1, 0)$ with V, W as the coordinate vector fields.

Question 2
Let $F : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ be given by

$$F(x) = \frac{x}{||x||^2}$$

where $||x||$ is the euclidean norm.
a) Find the differential dF_x and show that with respect to it is a composition of a reflection in the plane perpendicular to x followed by a scaling by a factor of $1/||x||^2$.
b) If ω is the euclidean volume form, find $F^*\omega$.

Question 3
a) Let $F : G \to H$ be a Lie group homomorphism and let $F_* : \mathfrak{G} \to \mathfrak{H}$ be the map between the associated Lie algebras of left-invariant vector fields defined by letting $(F_*(X))_e = dF_e(X_e)$.
Show that F_* is a Lie algebra homomorphism.
b) State the equivariant rank theorem.
c) Prove that $O(n)$ the group of orthogonal linear maps is a manifold and find its dimension.

Question 4
a) Give the definition of the integral of an n-form on an oriented n-manifold and show it is well-defined.
b) State and prove Stokes Theorem.

Question 5
a) State the Cartan Magic Formula.
b) Let M be a smooth manifold and $i_t : M \to M \times I$ be the map $i_t(x) = (x, t)$.
Show that $i_0^*, i_1^* : \Omega^*(M \times I) \to \Omega^*(M)$ are cochain homotopic, i.e., there exists a collection of linear maps $h : \Omega^p(M \times I) \to \Omega^{p-1}(M)$ such that $h \circ d + d \circ h = i_1^* - i_0^*$.
c) Prove that de Rham cohomology groups are an invariant of homotopy.