QUALIFYING EXAM IN GEOMETRY AND TOPOLOGY, SUMMER 2011

You should attempt all the problems. Partial credit will be give for serious efforts

(1) Let S be a closed non-orientable surface of genus g.

  (a) What is $H_i(S; \mathbb{Z}_2)$? (answer only)

  (b) Find out the maximal number of disjoint orientation reversing simple closed curves in S. (Justify your answer)

(2) Let X be a path-connected space and $\widetilde{X}$ a universal covering space of X. Prove that if $\widetilde{X}$ is compact, then $\pi_1(X)$ is a finite group.

(3) Let M be a compact, connected, orientable n-manifold, where n is odd.

(You may assume, if you like, that M is triangulated.)

  (a) Show that if $\partial M = \emptyset$, then $\chi(M) = 0$.

  (b) Show that if $\partial M \neq \emptyset$, then $\chi(M) = \frac{1}{2}\chi(\partial M)$.

(4) Let M be a closed 3-manifold. Suppose M is a homology sphere, i.e., M has the same $\mathbb{Z}$-coefficient homology groups as S^3, in other words, $H_n(M; \mathbb{Z}) = H_n(S^3; \mathbb{Z})$ for all n. Let k be a knot in M (i.e., k is a closed 1-dimensional submanifold of M, in other words, k is an embedded closed curve S^1 in M). Compute $H_n(M - k; \mathbb{Z})$ for all n, where $M - k$ is the complement of k.

GT Qual 2011 Part II

Show All Relevant Work!

1) The image of the map $X : \mathbf{R}^2 \to \mathbf{R}^3$ given by

$$X(\phi, \theta) = ((2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi))$$

is the torus obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the $yz-$plane about the $z-$axis. Consider the map $F : \mathbf{R}^3 \to \mathbf{R}^2$ given by $F(x, y, z) = (x, z)$ and let $f = (F$ restricted to the torus).

a) Compute the Jacobian of the map $f \circ X$. (Note that the map X descends to an embedding of $S^1 \times S^1$ into $\mathbf{R}^3$ but we don't need to obsess over the details of this.)

b) Find all regular values of f.

c) Find all level sets of f that are *not* smooth manifolds (closed embedded submanifolds).

2a) Write down the deRham homomorphism for a smooth manifold M; explain briefly why this definition is independent of the (two) choices made.

b) State the deRham Theorem for a smooth manifold M.

c) A crucial step in the proof of the deRham Theorem is: If M is covered by 2 open sets U and V, both of which and their intersection satisfy the deRham theorem, then $M = U \cup V$ satisfies the deRham theorem. Briefly explain how this crucial step is proven.

3a) If α is a differential form, then must it be true that $\alpha \wedge \alpha = 0$? If yes, then explain your reasoning. If no, then provide a counterexample.

b) If α and β are closed differential forms, prove that $\alpha \wedge \beta$ is closed.

c) If, in addition (i.e., continue to assume that α is closed), β is exact, prove that $\alpha \wedge \beta$ is exact.

4) The Chern-Simons form for a hyperbolic 3-manifold with the orthonormal framing (E_1, E_2, E_3) is the 3-form

$$Q = (\frac{1}{8\pi^2})(\omega_{12} \wedge \omega_{13} \wedge \omega_{23} - \omega_{12} \wedge \theta_1 \wedge \theta_2 - \omega_{13} \wedge \theta_1 \wedge \theta_3 - \omega_{23} \wedge \theta_2 \wedge \theta_3)$$

where $(\theta_1, \theta_2, \theta_3)$ is the dual co-frame to (E_1, E_2, E_3) (note that [Lee] uses ϵ, but here we use θ) and the ω_{ij} are the *connection* 1-forms. The connection 1-forms satisfy

$$\begin{aligned} d\theta_1 &= -\omega_{12} \wedge \theta_2 - \omega_{13} \wedge \theta_3 \\ d\theta_2 &= +\omega_{12} \wedge \theta_1 - \omega_{23} \wedge \theta_3 \\ d\theta_3 &= +\omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2 \end{aligned}$$

a) In $\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ with the Riemannian metric $g = \frac{1}{z^2}dx \otimes dx + \frac{1}{z^2}dy \otimes dy + \frac{1}{z^2}dz \otimes dz$, orthonormalize the framing $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

b) Compute the associated dual co-frame $(\theta_1, \theta_2, \theta_3)$.

c) For this orthonormal framing (and dual co-frame), in $(\mathbf{H}^3, g)$, compute the Chern-Simons form Q.