QUALIFYING EXAM IN GEOMETRY AND TOPOLOGY, SUMMER 2011

You should attempt all the problems. Partial credit will be give for serious efforts.

(1) Let $S$ be a closed non-orientable surface of genus $g$.
   (a) What is $H_i(S; \mathbb{Z}_2)$? (answer only)
   (b) Find out the maximal number of disjoint orientation reversing simple closed curves in $S$. (Justify your answer)

(2) Let $X$ be a path-connected space and $\tilde{X}$ a universal covering space of $X$. Prove that if $\tilde{X}$ is compact, then $\pi_1(X)$ is a finite group.

(3) Let $M$ be a compact, connected, orientable $n$-manifold, where $n$ is odd.
   (You may assume, if you like, that $M$ is triangulated.)
   (a) Show that if $\partial M = \emptyset$, then $\chi(M) = 0$.
   (b) Show that if $\partial M \neq \emptyset$, then $\chi(M) = \frac{1}{2} \chi(\partial M)$.

(4) Let $M$ be a closed 3-manifold. Suppose $M$ is a homology sphere, i.e., $M$ has the same $\mathbb{Z}$-coefficient homology groups as $S^3$, in other words, $H_n(M; \mathbb{Z}) = H_n(S^3; \mathbb{Z})$ for all $n$. Let $k$ be a knot in $M$ (i.e., $k$ is a closed 1-dimensional submanifold of $M$, in other words, $k$ is an embedded closed curve $S^1$ in $M$). Compute $H_n(M - k; \mathbb{Z})$ for all $n$, where $M - k$ is the complement of $k$. 
1) The image of the map $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(\phi, \theta) = ((2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi))$$

is the torus obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the $yz$–plane about the $z$–axis. Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by $F(x, y, z) = (x, z)$ and let $f = (F$ restricted to the torus).

a) Compute the Jacobian of the map $f \circ X$. (Note that the map $X$ descends to an embedding of $S^1 \times S^1$ into $\mathbb{R}^3$ but we don’t need to obsess over the details of this.)

b) Find all regular values of $f$.

c) Find all level sets of $f$ that are not smooth manifolds (closed embedded submanifolds).

2a) Write down the deRham homomorphism for a smooth manifold $M$; explain briefly why this definition is independent of the (two) choices made.

b) State the deRham Theorem for a smooth manifold $M$.

c) A crucial step in the proof of the deRham Theorem is: If $M$ is covered by 2 open sets $U$ and $V$, both of which and their intersection satisfy the deRham theorem, then $M = U \cup V$ satisfies the deRham theorem. Briefly explain how this crucial step is proven.

3a) If $\alpha$ is a differential form, then must it be true that $\alpha \wedge \alpha = 0$? If yes, then explain your reasoning. If no, then provide a counterexample.

b) If $\alpha$ and $\beta$ are closed differential forms, prove that $\alpha \wedge \beta$ is closed.

c) If, in addition (i.e., continue to assume that $\alpha$ is closed), $\beta$ is exact, prove that $\alpha \wedge \beta$ is exact.

4) The Chern-Simons form for a hyperbolic 3-manifold with the orthonormal framing $(E_1, E_2, E_3)$ is the 3-form

$$Q = \frac{1}{8\pi^2}(\omega_{12} \wedge \omega_{13} \wedge \omega_{23} - \omega_{12} \wedge \theta_1 \wedge \theta_2 - \omega_{13} \wedge \theta_1 \wedge \theta_3 - \omega_{23} \wedge \theta_2 \wedge \theta_3)$$

where $(\theta_1, \theta_2, \theta_3)$ is the dual co-frame to $(E_1, E_2, E_3)$ (note that [Lee] uses $\epsilon$, but here we use $\theta$) and the $\omega_{ij}$ are the connection 1-forms. The connection 1-forms satisfy

$$d\theta_1 = -\omega_{12} \wedge \theta_2 - \omega_{13} \wedge \theta_3$$
$$d\theta_2 = +\omega_{12} \wedge \theta_1 - \omega_{23} \wedge \theta_3$$
$$d\theta_3 = +\omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2$$

a) In $\mathbb{H}^3 = \{(x, y, z) : z > 0\}$ with the Riemannian metric $g = \frac{1}{z^2} dx \otimes dx + \frac{1}{z^2} dy \otimes dy + \frac{1}{z^2} dz \otimes dz$, orthonormalize the framing $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

b) Compute the associated dual co-frame $(\theta_1, \theta_2, \theta_3)$.

c) For this orthonormal framing (and dual co-frame), in $(\mathbb{H}^3, g)$, compute the Chern-Simons form $Q$. 