REAL ANALYSIS

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1
a) Let \( f \in L^1(m) \) and \( F(x) = \int_{-\infty}^{x} f(t) \, dt \). Then \( F \) is continuous.

b) Show for \( a > 0 \) \( \int_{-\infty}^{\infty} e^{-x^2} \cos(ax) \, dx = \sqrt{\pi} e^{-a^2/4} \).

Question 2
State and prove the Hahn-Decomposition Theorem for signed measures.

Question 3
Let \( 1 \leq p < \infty \) and \( m \) be Lebesgue measure. Let \( M \) be a closed subspace of \( L^p([0,1],m) \) such that \( M \) is contained in the space of continuous functions \( C([0,1]) \) (i.e. every element of \( M \) has a continuous function in its equivalence class, which is necessarily unique). Show that there exists a \( C_p > 0 \) such that for all \( f \in M \)
\[
||f||_u \leq C_p \cdot ||f||_p
\]
where \( ||.||_u \) is the sup norm on \( C([0,1]) \subseteq L^p([0,1],m) \).

Question 4
a) Prove the uniqueness of a (left-invariant) Haar measure on a locally compact hausdorff topological group.

b) Prove that Haar measure for a compact group or abelian group is both left and right invariant.