

# Real Analysis Qual

June 10, 2016

**Problem 1.** Let  $\sim$  be the equivalence relation on the interval  $[0, 1]$  given by  $x \sim y$  iff  $x - y \in \mathbb{Q}$ . Choose one element from each equivalence class (using the Axiom of Choice). Let  $A \subset [0, 1]$  denote the set of these chosen elements. For a given set  $B \subset \mathbb{R}$ , define  $B + x = \{y + x \mid y \in B\}$ .

- (a) Show that the sets  $A + q$ , for  $q \in \mathbb{Q} \cap [-1, 1]$ , are disjoint.
- (b) Show that  $[0, 1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1, 1]} (A + q) \subset [-1, 2]$ .
- (c) Show that  $A$  is not Lebesgue measurable (supposing it were, what would its measure be?).

**Problem 2.** Suppose  $(X, \mathcal{A}, \mu)$  is a measure space.

- (a) What does it mean for a function  $f : X \rightarrow \mathbb{R}$  to be measurable?
- (b) State the Monotone Convergence Theorem for this measure space.
- (c) Prove Fatou's Lemma, which states that

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$$

for any sequence  $(f_n)$  of non-negative, measurable functions. To get started, let  $g_n = \inf_{i \geq n} f_i$ . Observe these are increasing. What is their limit? Observe that

$$\int g_n \leq \inf_{i \geq n} \int f_i$$

(why?). Finish the proof.

**Problem 3.** Suppose  $\mu$  is a positive measure on  $(X, \mathcal{A})$ , with  $\mu(X) < \infty$ . Fix  $E_1, \dots, E_n \in \mathcal{A}$  and  $c_1, \dots, c_n \in \mathbb{R}_{\geq 0}$ . Define  $\nu : \mathcal{A} \rightarrow [0, \infty]$  by

$$\nu(A) = \sum_{i=1}^n c_i \mu(A \cap E_i).$$

- (a) Show that  $\nu$  is a measure.
- (b) Show that  $\nu$  is absolutely continuous with respect to  $\mu$ .

(c) State the Radon-Nikodym theorem with respect to  $\nu$  and  $\mu$ .

(d) Find the Radon-Nikodym derivative  $f = \frac{d\nu}{d\mu}$ .

**Problem 4.**

(a) Define the  $L^p$  norm on the measure space  $(X, \mathcal{A}, \mu)$ .

(b) What does it mean for  $H : L^p(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$  to be a bounded linear functional?

(c) Prove that a bounded linear functional is continuous with respect to the metric topology on  $L^p(X, \mathcal{A}, \mu)$  induced by the  $L^p$  norm.

## Complex Analysis Qualifying Exam – Spring 2016

Please answer the following problems. Explain your argument carefully – if you refer to a well-known theorem from class, please state the theorem precisely and explain why it applies.

Notation:  $\mathbb{D}$  = open unit disk,  $\mathbb{C}^* = \mathbb{C} - \{0\}$ ,  $\mathbb{H}$  = upper half plane.

- 1) Find a holomorphic function  $f$  on  $\mathbb{C}^*$  such that
  - $f$  is a pointwise limit of polynomial functions, but
  - $f$  is not a uniform limit of polynomial functions (that is, there is no sequence of polynomials that converges to  $f$  uniformly on compact subsets of  $\mathbb{C}^*$ ).

Prove both assertions for your choice of  $f$ .

- 2) Find a biholomorphism between  $\mathbb{H}$  and the region

$$U = \{z \in \mathbb{C} \mid |z - 1| < 1, |z - i| < 1\}.$$

It is enough to write down explicitly functions whose composition yields a biholomorphism from  $\mathbb{H}$  to  $U$  or from  $U$  to  $\mathbb{H}$ .

- 3) Let  $U \subsetneq \mathbb{C}$  be a simply connected region. For any point  $a \in U$ , the Green function of  $U$  corresponding to  $a$  is defined as  $g_a(z) = \log |\phi(z)|$ , where  $\phi : U \rightarrow \mathbb{D}$  is a biholomorphism sending  $a$  to 0.
  - a) Explain why  $g_a$  is well-defined. That is, explain briefly why such a map  $\phi$  is guaranteed to exist, and then show carefully that  $g_a$  is independent of the choice of  $\phi$ .
  - b) Prove that for any  $a, b \in U$  we have  $g_a(b) = g_b(a)$ .

- 4) Fix a lattice  $\Lambda$  and define the Weierstrass  $\sigma$ -function:

$$\sigma(z) = z \prod_{\omega \in \Lambda - \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

- a) Verify that the product converges to an odd holomorphic function with a simple zero exactly at every lattice point.
- b) Show that

$$\frac{d}{dz} \frac{\sigma'(z)}{\sigma(z)} = -\wp(z)$$

where  $\wp(z)$  is the Weierstrass  $\wp$ -function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

Justify carefully any techniques you use.