COMPLEX ANALYSIS QUALS JUNE 2015

In your proofs, you may use any theorem from the syllabus for the Complex Analysis Qualifying Exam, except of course you may not use the fact that you are trying to prove or a trivial variant of it. State clearly what theorems you use. Each problem is worth 20 points. Do all the problems. Good luck.

1.

Let H denote the upper half plane: $H = \{z \in \mathbb{C} \quad | \quad Im(z) > 0\}$.

Let $\phi : H \to \mathbb{C}$ be an analytic function with the property that $\phi(z+1) = \phi(z)$ for all $z \in H$. Prove that there exist constants $c_n \in \mathbb{C}$ such that

$$\phi(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z}.$$

2.

Find all bi-analytic functions $f : \mathbb{C} \to \mathbb{C}$. (Bi-analytic means analytic, one-to-one and onto with analytic inverse.) Prove that you have them all.

3.

Find all possible values of

$$\int_\gamma \frac{\cos(z)\ dz}{1+z^2}$$

where γ is a closed curve in $\mathbb{C} - \{i, -i\}$. Prove your answer.

4.

Let

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and let U denote the set of complex numbers z such that the real part of z is greater than 1.

(a) Prove that the series converges and defines an analytic function in U.

(b) Give a formula for the derivative $\zeta'(z)$ in U in the form of a series.

5.

Let Ω be a region in $\mathbb{C}$ and let $K \subset \Omega$ be compact. Assume that the interior K^o of K is nonempty. Let $f : \Omega \to \mathbb{C}$ be a non constant, analytic function with the property that its absolute value $|f|$ is constant on $K - K^o$. Prove: f has at least one zero in K.