

## ANALYSIS QUALIFYING EXAM

JUNE, 2014

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

**Exercise 1.** (30 points.)

Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $f : X \rightarrow \mathbb{R}$  measurable. Show that

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty pt^{p-1} \varphi(t) dt,$$

where  $\varphi(t) = \mu\{x | t < |f(x)|\}$ .

**Exercise 2.** (30 points.)

- (1) Prove that not every subset of  $[0, 1]$  is Lebesgue measurable
- (2) Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be a sequence of Lebesgue measurable functions. Prove that the set  $E = \{x | \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$  is Lebesgue measurable

**Exercise 3.** (30 points.)

Let  $X, Y$  be Banach spaces. If  $T : X \rightarrow Y$  is a linear map such that  $f \circ T \in X^*$  for all  $f \in Y^*$  then  $T$  is bounded.

**Exercise 4.** (30 points.)

Let  $(X, \mathcal{M}, \mu)$  be a finite measure space. For each of the following claims prove or give a counter example:

- (1) If a sequence  $(f_n)$  of real valued measurable functions on  $X$  converges  $\mu$  a.e., then  $(f_n)$  converges in measure.
- (2) If a sequence  $(f_n)$  of real valued measurable functions on  $X$  converges in measure., then  $(f_n)$  converges  $\mu$  a.e.
- (3) If a sequence  $(f_n)$  of real valued measurable functions on  $X$  is Cauchy in  $L^1(\mu)$ , then  $(f_n)$  converges in measure.