REAL ANALYSIS

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1 (30 points)
Let \((X, M, \mu)\) be a measure space. A measure \(\mu\) is semi-finite if for each \(E \in M\), with \(\mu(E) = \infty\), there exists an \(F \in M\) such that \(0 < \mu(F) < \infty\).
Prove that if \(\mu\) is semifinite and \(\mu(E) = \infty\), for any \(C > 0\) there exists an \(F \in M\) such that \(C < \mu(F) < \infty\).

Question 2 (20 points)
Let \((X, M, \mu)\) be a measure space and \(L^p(X) = \{f : X \to \mathbb{C} : f \text{ is measurable and } ||f||_p < \infty\}\).
Prove that \(L^p(X)\) is a Banach space for \(1 \leq p < \infty\) by proving
a) If \(f, g \in L^p(X)\) then \(||f + g||_p \leq ||f||_p + ||g||_p\)
b) \(L^p(X)\) is complete.

Question 3 (30 points)
The total variation of a complex measure \(\nu\) is the positive measure \(|\mu|\) determined by the property that if \(d\nu = fd\mu\) for some positive measure \(\mu, f \in L^1(\mu)\), then \(d|\nu| = |f|d\mu\).
Prove that this is well defined by showing the following:
a) There always exists such a measure \(\mu\).
b) The definition is independent of \(\mu\).

Question 4 (20 points)
a) Let \(||\cdot||_1, ||\cdot||_2\) be two norms on a vector space \(V\) such that \(||v||_1 \leq ||v||_2\) for all \(v \in V\). If \(V\) is complete with respect to both norms, prove that they are equivalent.
b) Let \(X, Y\) be Banach spaces and let \(T_n \in L(X, Y)\) such that \(T(x) = \lim_{n \to \infty} T_n(x)\) exists for all \(x \in X\). Prove that \(T \in L(X, Y)\).
COMPLEX ANALYSIS

You should attempt all the problems. Partial credit will be given for serious efforts.

(1) Compute the following integral:

\[ \int_{0}^{\infty} \frac{\log x}{x^2 + 1} \, dx \]

(2) Let \( A = \{a_0, a_1, \ldots, a_n\} \) be a finite set of (distinct) points in the unit disk \( D \). Define

\[ B_{\lambda}(z) = \prod_{i=0}^{n} \frac{z - a_i |a_i|}{1 - a_i z a_i}, \quad \text{for } z \in D \]

where if \( a_i = 0 \), we set \( \frac{|a_i|}{a_i} = 1 \).

(a) Prove that \( B(z) \) maps \( D \) to \( D \) and maps the unit circle to the unit circle.

(b) Let \( T : D \to D \) be a fractional linear transformation that maps the unit disk onto itself. Prove that

\[ B_{\lambda} \circ T = \lambda B_{T^{-1}(\lambda)} \]

where \( \lambda \) is a constant with \( |\lambda| = 1 \) and \( T^{-1}(\lambda) = \{T^{-1}(a_0), \ldots, T^{-1}(a_n)\} \).

(c) Let \( f : D \to D \) be an analytic function with \( f(a_i) = 0 \) for each \( a_i \in A \). Prove that

\[ |f(z)| \leq |B_{\lambda}(z)| \text{ for all } z \in D. \]

(3) The expression

\[ \{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \]

is called the Schwarzian derivative of \( f \). If \( f(z) \) has a zero or pole of order \( m \) (\( m > 1 \)) at \( z_0 \), show that \( \{f, z\} \) has a pole at \( z_0 \) of order 2 and calculate the coefficient of \( \frac{1}{(z-z_0)^2} \) in the Laurent development of \( \{f, z\} \).

(4) Let \( f \) be a bounded analytic function on the unit disk \( |z| < 1 \) and let \( \zeta \) be a point in the unit disk (i.e. \( |\zeta| < 1 \))
(a) Show that the area integral
\[
\iint_{|z|<1} \frac{f(z) \, dx \, dy}{(1 - \bar{z}\zeta)^2}, \quad z = x + yi
\]
is equal to
\[
\int_0^1 \left( \int_{|z|=1} \frac{zf(rz)}{i(z - r\zeta)^2} \, dz \right) r \, dr
\]
(Hint: use polar coordinates)

(b) Use part (a) to prove
\[
f(\zeta) = \frac{1}{\pi} \iint_{|z|<1} \frac{f(z) \, dx \, dy}{(1 - \bar{z}\zeta)^2}
\]