Algebra Qualifying Exam
June, 2012

Please answer all 10 problems and show your work. Each problem is worth 20 points. In your proofs, you may use any theorem from the syllabus for Algebra, except of course you may not use the fact you are trying to prove, or a mere variant of it. State clearly what theorems you use. Good luck.

1. Classify the groups of order 42, up to isomorphism.

2. Let $g \in \mathrm{GL}_n(F)$ be an element of finite order m, where F is an algebraically closed field in which m is nonzero. Prove that g is conjugate in $\mathrm{GL}_n(F)$ to a diagonal matrix.

3. Let G be the subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ consisting of matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$.

a) Construct an irreducible complex representation V of G such that $\dim V > 1$.

b) Show that if W is an irreducible complex representation of G which is not isomorphic to V then $\dim W = 1$.

4. Consider the 3×3 integer matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 5 \end{bmatrix}.$$

a) Determine the structure of the abelian group $\mathbb{Z}^3/A\mathbb{Z}^3$.

b) Find the rational canonical form of A.

5. Let R be a PID and M a finitely generated R-module. Prove that M is a flat R-module if and only if M is a torsion-free R-module.

6. Let $R = \mathbb{Z}/18$, and consider the R-modules $A = \mathbb{Z}/9$, $B = \mathbb{Z}/3$. Find $\mathrm{Ext}^n_R(A, B)$.

7. Let $f(x) \in \mathbb{Z}[x]$ be monic of degree $n > 0$, with roots $a_1, \ldots, a_n \in \mathbb{C}$. Let $b = \prod_{1 \le i < j \le n}(a_i^3 - a_j^3)$.

(a) Prove that $[\mathbb{Q}(b) : \mathbb{Q}] \le 2$.

(b) Prove that if $b \in \mathbb{Q}$ then $b \in \mathbb{Z}$.

8. Let K/F be any extension of fields (not necessarily algebraic). We let the automorphism group $\mathrm{Aut}(K/F)$ act on K on the right, using the notation a^s for the action of s on a. Let G be a subgroup of $\mathrm{Aut}(K/F)$ and assume that F is the field of G-invariants in K. We say that $a \in K^\times$ is a *semi-invariant* iff there exists a function $c_a : G \to F$ such that $a^s = c_a(s)a$ for all $s \in G$. Let X be the set of all semi-invariants in $K^\times$.

(a) Show that for each $a \in X$, c_a is a homomorphism from G to $F^\times$.

(b) Show that X is a group under multiplication.

(c) Suppose $a_1, \ldots, a_m \in X$ are multiplicatively independent mod $F^\times$, i.e. for all $k_i \in \mathbb{Z}$,

$$a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m} \in F^\times \implies k_i = 0 \quad \forall i.$$

Prove: $a_1, \ldots, a_m$ are algebraically independent over F.

9. Let R be a commutative ring with identity and M an R-module. For any $x \in M$ define $\mathrm{Ann}(x) = \{r \in R \mid rx = 0\}$. Let Rx denote the submodule of M generated by x. Let P be a prime ideal of R. For $r \in R$, let r_M denote the map $M \to M$ given by $r_M(x) = rx$.

(a) Show that the localization $(Rx)_P \neq 0$ if and only if $\mathrm{Ann}(x) \subset P$.

(b) Suppose $r_M \in \mathrm{End}_R(M, M)$ is nilpotent. Show that $M_P \neq 0 \implies r \in P$.

10. Let R be a commutative ring with identity, By definition, the dimension of R is the maximum index i for which $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_i$ is a chain of prime ideals in R. Find the dimension of $\mathbb{Z}[x]$ and prove your answer.