Algebra qualifying exam, Spring 2016

1. Classify the groups of order 24 having trivial center.

2. Let V be a finite dimensional vector space over a field F of characteristic p and let $T : V \to V$ be a linear transformation such that $T^p = I$, the identity transformation of V.

a) Determine the eigenvalues of T.

b) Show that there is a basis of V for which the matrix of T is upper triangular,

3. Let G be the group with presentation

$$G = \langle x, y \mid \ x^4 = 1, \ x^2 = y^3 \rangle.$$

and let $[G, G]$ be the commutator subgroup of G. Determine the structure of the group $G/[G, G]$.

4. Let $\zeta = e^{\pi i/10}$. Find all the subfields $K \subset \mathbb{Q}(\zeta)$ such that $[K : \mathbb{Q}] = 2$ and express each of them in the form $K = \mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Z}$.

5. Suppose n and m are positive integers, let $R = \mathbb{Z}[X]/(X^n)$ and let M be an R-module, let x be the image of X in R, and let (x^m) be the ideal in R generated by x^m. Compute $\mathrm{Tor}_i^R(M, (x^m))$ for all i.

6. Let k be a field. Find the minimal primes and compute the Krull dimension of $R = k[x, y, z]/(xy, xz)$.

7. Let R be an Artinian local ring. Prove that an R-module is flat if and only if it is free.

8. Suppose that R is a Noetherian ring and $\mathfrak{p} \subset R$ is a prime ideal such that $R_\mathfrak{p}$ is an integral domain. Show that there is an $f \notin \mathfrak{p}$ such that R_f is an integral domain. (Recall that $R_f = S^{-1}R$ where $S = \{1, f, f^2, f^3, \ldots\}$.)