

Algebra Qualifying Exam

Fall 2015

3 hours

1. Classify groups of order 55 up to isomorphism. Give a presentation for each of the groups in your classification.

2. Let $R = \mathbb{C}[X, Y]$ and consider the ideal $I = (X, Y)$ as an R -module.

(a) Construct an exact sequence of R -modules

$$0 \rightarrow R \rightarrow R \oplus R \rightarrow I \rightarrow 0.$$

(b) Prove that the sequence you constructed is not split.

3. Consider the ideal

$$I = (X^2 - Y, Y^2 - X) \subset \mathbb{C}[X, Y].$$

Find all maximal ideals of the quotient $\mathbb{C}[X, Y]/I$. (*Find* means give a set of generators.)

4. How many Sylow p -subgroups are there in $\mathrm{GL}_2(\mathbb{F}_p)$?

5. Suppose K is an extension of \mathbb{Q} of degree n , and let $\sigma_1, \dots, \sigma_n : K \rightarrow \mathbb{C}$ be the distinct embeddings of K into \mathbb{C} . Let $\alpha \in K$. Regarding K as a \mathbb{Q} -vector space, let $\phi : K \rightarrow K$ be the linear transformation $\phi(x) = \alpha x$. Show that the eigenvalues of ϕ are $\sigma_1(\alpha), \dots, \sigma_n(\alpha)$.

6. Let $\zeta = e^{\pi i/3} \in \mathbb{C}$.

(a) Compute the minimal polynomial of ζ over \mathbb{Q} .

(b) Find all prime ideals $\mathfrak{p} \subset \mathbb{Z}[\zeta]$ satisfying $\mathfrak{p} \cap \mathbb{Z} = 7\mathbb{Z}$, and give generators for these ideals.

7. Fix $a, b, c \in \mathbb{Q}$, and let K/\mathbb{Q} be the splitting field of

$$f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 \in \mathbb{Q}[x].$$

Show that $\mathrm{Gal}(K/\mathbb{Q}) \subset S_6$ is contained in the centralizer of a permutation with cycle type $(2, 2, 2)$.

8. Let R be a ring with identity, and let M be a left R -module. Prove that the following are equivalent:

- (i) there is a chain of submodules $M = M_0 \supset M_1 \supset \dots \supset M_n = (0)$ such that each quotient M_j/M_{j+1} is a simple R -module (*simple* means no proper nonzero submodules)
- (ii) M satisfies both the ascending chain condition and the descending chain condition for submodules.

Hint: For (i) \implies (ii), use induction on n .