Algebra Qualifying Exam
Fall 2014
You have 3 hours to answer all questions.

1. Determine the number of conjugacy classes of elements of order 4 in GL$_4(\mathbb{C})$, and give a representative of each class. Do the same for GL$_4(\mathbb{F}_7)$.

2. Find the Galois groups of $f(x) = x^5 + 7x^3 + 6x^2 + x + 5$ over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$, and $\mathbb{Q}$. You may assume without proof that $f(x) \in \mathbb{F}_3[x]$ has no irreducible quadratic factors.

3. Let $R$ be a Noetherian ring. For any ideal $J \subset R$ define
   \[ \sqrt{J} = \{ x \in R : x^k \in J \text{ for some } k \in \mathbb{Z}^+ \}. \]
   If $\sqrt{J} = J$, show that $J$ can be expressed as a finite intersection of prime ideals. Hint: among all counterexamples, a maximal one cannot be prime.

4. Classify the groups of order $2915 = 5 \cdot 11 \cdot 53$.

5. Suppose $R$ is a PID and $A$ and $B$ are $R$-modules. Let $B_{\text{tors}} \subset B$ be the submodule of $R$-torsion elements. Prove that
   \[ \text{Tor}^R_1(A, B) \cong \text{Tor}^R_1(A, B_{\text{tors}}). \]

6. Suppose $R$ is a Noetherian ring and $p \subset R$ is a prime ideal. Show that there is an $r \not\in p$ such that $S^{-1}R \to R_p$ is injective, where $S = \{1, r, r^2, r^3, \ldots \}$.

7. Suppose $R$ is a commutative local ring, and $M$ and $N$ are $R$-modules satisfying
   \[ M \otimes_R N = 0. \]
   (a) If $M$ and $N$ are finitely generated, show that either $M = 0$ or $N = 0$.
   (b) Show by example that (a) is false if we drop the hypothesis that $M$ and $N$ are finitely generated.

8. Let $\zeta_8 \in \mathbb{C}$ be a primitive eighth root of unity. The ring of integers in $\mathbb{Q}[\zeta_8]$ is $\mathbb{Z}[\zeta_8]$ (you may assume this without proof). If $p$ is a prime, determine the number of primes of $\mathbb{Z}[\zeta_8]$ above $p$ when
   (a) $p = 2$,
   (b) $p \equiv 1 \pmod{8}$,
   (c) $p \equiv 3, 5, 7 \pmod{8}$.