Algebra Qualifying Exam
Fall 2014
You have 3 hours to answer all questions.

1. Determine the number of conjugacy classes of elements of order 4 in $\mathrm{GL}_4(\mathbb{C})$, and give a representative of each class. Do the same for $\mathrm{GL}_4(\mathbb{F}_7)$.

2. Find the Galois groups of $f(x) = x^5 + 7x^3 + 6x^2 + x + 5$ over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, and $\mathbb{Q}$. You may assume without proof that $f(x) \in \mathbb{F}_3[x]$ has no irreducible quadratic factors.

3. Let R be a Noetherian ring. For any ideal $J \subset R$ define

$$\sqrt{J} = \{x \in R : x^k \in J \text{ for some } k \in \mathbb{Z}^+\}.$$

If $\sqrt{J} = J$, show that J can be expressed as a finite intersection of prime ideals. Hint: among all counterexamples, a maximal one cannot be prime.

4. Classify the groups of order $2915 = 5 \cdot 11 \cdot 53$.

5. Suppose R is a PID and A and B are R-modules. Let $B_{\text{tors}} \subset B$ be the submodule of R-torsion elements. Prove that

$$\mathrm{Tor}_1^R(A, B) \cong \mathrm{Tor}_1^R(A, B_{\text{tors}}).$$

6. Suppose R is a Noetherian ring and $\mathfrak{p} \subset R$ is a prime ideal. Show that there is an $r \notin \mathfrak{p}$ such that $S^{-1}R \to R_{\mathfrak{p}}$ is injective, where $S = \{1, r, r^2, r^3, \ldots\}$.

7. Suppose R is a commutative local ring, and M and N are R-modules satisfying

$$M \otimes_R N = 0.$$

(a) If M and N are finitely generated, show that either $M = 0$ or $N = 0$.

(b) Show by example that (a) is false if we drop the hypothesis that M and N are finitely generated.

8. Let $\zeta_8 \in \mathbb{C}$ be a primitive eighth root of unity. The ring of integers in $\mathbb{Q}[\zeta_8]$ is $\mathbb{Z}[\zeta_8]$ (you may assume this without proof). If p is a prime, determine the number of primes of $\mathbb{Z}[\zeta_8]$ above p when

(a) $p = 2$,

(b) $p \equiv 1 \pmod 8$,

(c) $p \equiv 3, 5, 7 \pmod 8$.