Algebra Qualifying Exam, Fall 2013
You have 3 hours to answer all problems.

1. Classify, up to isomorphism, all groups of order $385 = 5 \cdot 7 \cdot 11$.

2. Determine the Galois group of the polynomial $x^5 - 2 \in \mathbb{Q}[X]$.

3. Let $R$ be a local ring with maximal ideal $\mathfrak{m}$. Suppose that $f : A \to B$ is a homomorphism of finitely generated free $R$-modules with the property that the induced map $A/\mathfrak{m}A \to B/\mathfrak{m}B$ is an isomorphism. Show that $f$ is itself an isomorphism.

4. The ring of integers of $\mathbb{Q}[\sqrt{7}]$ is $\mathbb{Z}[\sqrt{7}]$. For each of the following primes $p \in \mathbb{Z}$, describe how the ideal $p\mathbb{Z}[\sqrt{7}]$ factors as a product of prime ideals ("describe" means give the number of prime factors, their multiplicities in the factorization, and the cardinalities of the residue fields):
   
   (a) $p = 2$
   
   (b) $p = 7$
   
   (c) $p = 17$.

5. Let $A$ be an $n \times n$ matrix with entries in an algebraically closed field. Show that $A$ is similar to a diagonal matrix if and only if the minimal polynomial of $A$ has no repeated roots.

6. Let $R$ be a commutative ring with $1$, $N$ an $R$-module, and for every maximal ideal $m \subset R$ let $N_m$ be the localization of $N$ at $m$. Prove that the natural map $N \to \prod_m N_m$ is injective.

7. Let $k$ be a field, $R = k[x, y]$ and $I = (x, y)$.
   
   (a) Prove that $I$ is neither flat nor projective as an $R$-module.
   
   (b) Compute $\text{Ext}^1_R(R/I, I)$.

8. Let $k$ be an algebraically closed field. Consider the affine variety $V = k^2$ with coordinates $x, y$, and the affine variety $W = k^2$ with coordinates $s, t$. Suppose $f : V \to W$ a morphism, and denote by $R$ the image of the induced pull-back map $f^* : k[s, t] \to k[x, y]$. For each of the following statements, give a proof or a counterexample.
   
   (a) If $f$ has Zariski dense image, then $f$ is surjective.
   
   (b) If $k[x, y]/R$ is an integral extension of rings, then $f$ is surjective.