Algebra qualifying exam
September 6, 2011

There are eight problems. All problems have equal weight. Show all of your work.

1. For which primes \( p \) does there exist a nonabelian group of order \( 4p \)? For each such prime give an example of such a group.

2. Let \( G = \text{GL}_2(\mathbb{F}_{11}) \) be the group of \( 2 \times 2 \) invertible matrices over the field of 11 elements.
   a) Show that the elements of order three in \( G \) form a single conjugacy class in \( G \).
   b) Find the number of Sylow 3-subgroups of \( G \).

3. Let \( G \) be a cyclic group of order \( m \) and let \( p \) be a prime not dividing \( m \).
   1. Construct all of the simple modules over the group ring \( \mathbb{F}_p[G] \).
   2. Give the number of simple \( \mathbb{F}_p[G] \)-modules and their dimensions as \( \mathbb{F}_p \)-vector spaces, in terms of \( p \) and \( m \).

4. Suppose \( R \) is a commutative ring, and that

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]

is an exact sequence of \( R \)-modules. Prove that \( B \) is Noetherian if and only if both \( A \) and \( C \) are Noetherian.
5. Let $K \subset \mathbb{C}$ be the splitting field over $\mathbb{Q}$ of the cyclotomic polynomial

$$f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \in \mathbb{Z}[x].$$

Find the lattice of subfields of $K$ and for each subfield $F \subset K$ find polynomial $g(x) \in \mathbb{Z}[x]$ such that $F$ is the splitting field of $g(x)$ over $\mathbb{Q}$.

6. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree five with exactly three real roots, and let $K$ be the splitting field of $f$. Prove that $\text{Gal}(K/\mathbb{Q}) \simeq S_5$.

7. Let $k$ be a field, and let $R = k[x,y]/(y^2 - x^3 - x^2)$.

   a) Prove that $R$ is an integral domain.

   b) Compute the integral closure of $R$ in its quotient field.

      [Hint: Let $t = \overline{y}/\overline{x}$, where $\overline{x}$ and $\overline{y}$ are the images of $x$ and $y$ in $R$.]

8. Let $p$ be a prime and let $G$ be the group of upper triangular matrices over the field $\mathbb{F}_p$ of $p$ elements:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{F}_p \right\}.$$

Let $Z$ be the center of $G$ and let $\rho : G \to \text{GL}(V)$ be an irreducible complex representation of $G$. Prove the following.

   a) If $\rho$ is trivial on $Z$ then $\dim V = 1$.

   b) If $\rho$ is nontrivial on $Z$ then $\dim V = p$.

      [Hint: Consider the subgroup of matrices in $G$ having $y = 0$.]