

**Algebra qualifying exam**  
**September 6, 2011**

There are eight problems. All problems have equal weight. Show all of your work.

1. For which primes  $p$  does there exist a nonabelian group of order  $4p$ ? For each such prime give an example of such a group.
  
2. Let  $G = \text{GL}_2(\mathbb{F}_{11})$  be the group of  $2 \times 2$  invertible matrices over the field of 11 elements.
  - a) Show that the elements of order three in  $G$  form a single conjugacy class in  $G$ .
  - b) Find the number of Sylow 3-subgroups of  $G$ .
  
3. Let  $G$  be a cyclic group of order  $m$  and let  $p$  be a prime not dividing  $m$ .
  1. Construct all of the simple modules over the group ring  $\mathbb{F}_p[G]$ .
  2. Give the number of simple  $\mathbb{F}_p[G]$ -modules and their dimensions as  $\mathbb{F}_p$ -vector spaces, in terms of  $p$  and  $m$ .

4. Suppose  $R$  is a commutative ring, and that

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of  $R$ -modules. Prove that  $B$  is Noetherian if and only if both  $A$  and  $C$  are Noetherian.

5. Let  $K \subset \mathbb{C}$  be the splitting field over  $\mathbb{Q}$  of the cyclotomic polynomial

$$f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \in \mathbb{Z}[x].$$

Find the lattice of subfields of  $K$  and for each subfield  $F \subset K$  find polynomial  $g(x) \in \mathbb{Z}[x]$  such that  $F$  is the splitting field of  $g(x)$  over  $\mathbb{Q}$ .

6. Let  $f(x) \in \mathbb{Q}[x]$  be an irreducible polynomial of degree five with exactly three real roots, and let  $K$  be the splitting field of  $f$ . Prove that  $\text{Gal}(K/\mathbb{Q}) \simeq S_5$ .

7. Let  $k$  be a field, and let  $R = k[x, y]/(y^2 - x^3 - x^2)$ .

a) Prove that  $R$  is an integral domain.

b) Compute the integral closure of  $R$  in its quotient field.

[Hint: Let  $t = \bar{y}/\bar{x}$ , where  $\bar{x}$  and  $\bar{y}$  are the images of  $x$  and  $y$  in  $R$ .]

8. Let  $p$  be a prime and let  $G$  be the group of upper triangular matrices over the field  $\mathbb{F}_p$  of  $p$  elements:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{F}_p \right\}.$$

Let  $Z$  be the center of  $G$  and let  $\rho : G \rightarrow \text{GL}(V)$  be an irreducible complex representation of  $G$ . Prove the following.

a) If  $\rho$  is trivial on  $Z$  then  $\dim V = 1$ .

b) If  $\rho$  is nontrivial on  $Z$  then  $\dim V = p$ .

[Hint: Consider the subgroup of matrices in  $G$  having  $y = 0$ .]