Algebra qualifying exam
June 1, 2011

1. Classify the groups of order 105, up to isomorphism, and give a presentation of each group.

2.

a) Compute $\operatorname{Ext}^i_{\mathbb{Z}}\big(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}\big)$ for all i.

b) Compute $\operatorname{Tor}_i^{\mathbb{Z}}\big(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}\big)$ for all i.

3. Let W denote the unique irreducible two dimensional complex representation of the symmetric group S_3. Determine the dimensions and multiplicities of the irreducible constituents of $\operatorname{Ind}_{S_3}^{S_4} W$.

4. Suppose R is a commutative local ring, and M is a finitely generated R-module. Prove that M is projective if and only if M is free. Hint: you may use any form of Nakayama's Lemma you like, provided you first state it correctly.

5. Let $f \in \mathbb{Z}[x]$ be an irreducible monic polynomial, let K be a splitting field of f and let $\alpha \in K$ be a root of f. Assume the Galois group $\operatorname{Gal}(K/\mathbb{Q})$ is abelian.

a) Prove that $K = \mathbb{Q}(\alpha)$.

b) Assume there is a prime p such that the image of f in $\mathbb{F}_p[x]$ is irreducible. Determine the structure of $\operatorname{Gal}(K/\mathbb{Q})$.

6. Let $K = \mathbb{C}(x)$ be the field of rational functions in one variable x. Fix an integer $n \geq 2$ and let $F \subset K$ be the field of rational functions fixed by the two automorphisms

$$\sigma : x \mapsto e^{2\pi i/n} x, \qquad \tau : x \mapsto x^{-1}.$$

a) Determine the structure of the Galois group $\operatorname{Gal}(K/F)$.

b) Show that $F = \mathbb{C}(t)$, where $t = x^n + x^{-n}$, and determine the minimal polynomial of x over F.

7. Show that every finite subgroup of $\mathrm{GL}_2(\mathbb{Q})$ has order $2^a 3^b$ for some a and b.

8. Let F be a subfield of $\mathbb{C}$ that is finite and Galois over $\mathbb{Q}$. Suppose $\alpha \in F$ is an algebraic integer with the property that every Galois conjugate has complex absolute value 1. Prove that α is a root of unity. [Hint: show that the set of all such α in F is finite.]