1 Let $\Sigma_g$ denote the closed, orientable, surface of genus $g$.

Prove that if $\Sigma_g$ is a covering space of $\Sigma_h$, then there is a $d \in \mathbb{Z}^+$ satisfying

$$g = d(h - 1) + 1.$$  

(2) Let $X$ be a closed (i.e., compact & boundaryless), orientable $2k$–dimensional manifold. Prove that if $H_{k-1}(X; \mathbb{Z})$ is torsion-free, then so is $H_k(X; \mathbb{Z})$.

(3) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2–torus, concretely identified as the quotient space of the Euclidean plane by the standard integer lattice. Then any $2 \times 2$ integer matrix $A$ induces a map

$$\phi : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$$

by left (matrix) multiplication.

(a) Show that (with respect to a suitable basis) the induced contravariant map

$$\phi^* : H^1(T^2; \mathbb{Z}) \to H^1(T^2; \mathbb{Z})$$

on the cellular cohomology is left multiplication by the transpose of $A$.

(b) Since $T^2$ is a closed, $\mathbb{Z}$–oriented manifold, it has a fundamental class, $[T^2] \in H_2(T^2; \mathbb{Z})$. Prove that

$$\phi_* [T^2] = \text{det}(A) \cdot [T^2].$$

(Hint: Use part (a) and the naturality of the cup product under induced maps on homology/cohomology.)

(4) The closed, orientable surface $\Sigma_g$ of genus $g$, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region $R$ (often called a genus $g$ solid handlebody).

Two copies of $R$, glued together by the identity map between their boundary surfaces, form a closed 3–manifold $X$. Compute $H_*(X; \mathbb{Z})$. 

1) Consider stereographic projection of the unit circle $S^1$ in $\mathbb{R}^2$ to $\mathbb{R}$ from the North Pole ($\sigma$) and from the South Pole ($\tilde{\sigma}$).

   a) Show that $\tilde{\sigma} \circ \sigma^{-1}(x) = 1/x$

   b) Consider the smooth vector field $\frac{d}{dx}$ on $\mathbb{R}$. Using $\sigma$, this induces a smooth vector field on the circle minus the North Pole. Can it be extended to a smooth vector field on all of $S^1$?

2a) A smooth map $F : M \to N$ is a submersion if...

   b) Let $M$ be a compact, smooth 3-manifold. Prove that there is no submersion $F : M \to \mathbb{R}^3$.

3) Consider $D$ the open unit disk in $\mathbb{R}^2$ with Riemannian metric

   $g = \left(\frac{2}{1 + x^2 + y^2}\right)^2 dx \otimes dx + \left(\frac{2}{1 + x^2 + y^2}\right)^2 dy \otimes dy$

   a) Write down an (oriented) orthonormal frame $(E_1, E_2)$ for $D$ with respect to this metric.

   b) Write down the associated dual coframe $(\epsilon^1, \epsilon^2)$.

   c) Compute $\epsilon^1 \wedge \epsilon^2$. Is this the Riemannian volume form (that is, does it agree with the volume formula $\sqrt{\det(g_{ij})} dx \wedge dy$)?

   d) Compute the volume (area?) of $D$ with respect to this metric.

   e) What have you computed?

4) Suppose that $f_0$ and $f_1$ are smoothly homotopic maps from $X$ to $Y$ and that $X$ is a compact k-dimensional manifold without boundary.

   a) Complete the sentence “$f_0$ and $f_1$ are smoothly homotopic maps from $X$ to $Y$ means that there exists a function $F$ from ...”

   b) Prove that if $\omega$ is a closed $k$–form on $Y$ then $\int_X f_0^*(\omega) = \int_X f_1^*(\omega)$. 