Classification of Numerical Semigroups

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Numerical Semigroup

A *numerical semigroup* is a subset $\Gamma \subset \mathbb{Z}^{\geq 0}$ such that

- Γ contains 0,
- Γ is closed under addition,
- the complement of Γ in $\mathbb{Z}^{\geq 0}$ is finite, denoted as *gap sequence*, and its elements are called *gaps*.

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Example 1: $\Gamma = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \dots\}$.
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Example 2: $\Gamma = \{0, 4, 5, 7, 8, 9, \dots\}$.
$\mathbb{Z}^{\geq 0} \setminus \Gamma = \{1, 2, 3, 6\}$.

Generators, Genus, and Conductor

Every numerical semigroup can be represented uniquely by a set of minimal *generators*.

Ex.1: $\langle 4,5 \rangle = \{0,4,5,8,9,10,12,\dots\} = \mathbb{Z} \setminus \{1,2,3,6,7,11\}$.

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We define

- The *genus* $g(\Gamma)$ to be the number of elements in the gap sequence.
- The *Frobenius number* $F(\Gamma)$ is the largest element in the gap sequence.
- The *conductor* $\mathrm{Cond}(\Gamma) = F(\Gamma) + 1$.

(Sylvester, 1884) If $\Gamma = \langle a, b\rangle$, where a and b are coprime, then

$$g(\Gamma) = (a-1)(b-1)/2$$
$$F(\Gamma) = (a-1)(b-1) - 1$$

Symmetric Numerical Semigroup

Numerical semigroups arise in algebraic geometry in the context of curve singularities. Namely, to a semigroup Γ, one associates a *unibranch* curve singularity whose algebra of functions is

$$\mathbb{C}[\Gamma] := \mathbb{C}[t^a \mid a \in \Gamma].$$

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$$\mathbb{C}[\Gamma] := \mathbb{C}[t^a \mid a \in \Gamma].$$

The Γ-singularity, Spec $\mathbb{C}[t]$, is called *Gorenstein* if Γ is a *symmetric* semigroup, i.e.,

$$n \in \{gaps\} \Leftrightarrow F(\Gamma) - n \notin \{gaps\}.$$

$\langle 4, 5 \rangle = \mathbb{Z} \setminus \{1, 2, 3, 6, 7, 11\}$ is symmetric.
$\langle 4, 5, 7 \rangle = \mathbb{Z} \setminus \{1, 2, 3, 6\}$ is non-symmetric.

Open Problem

Classify numerical semigroups that satisfy the inequality

$$\frac{(2g-1)^2}{\sum_{i=1}^{g} b_i} \leq \frac{11}{2}.$$

In our work, we classified those satisfying a stronger condition

$$\frac{(2g-1)^2}{\sum_{i=1}^{g} b_i} \leq 4,$$

which is equivalent to

$$\sum_{i=1}^{g} b_i > g(g-1).$$

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$$\begin{aligned} \implies \sum_{i=1}^{g} b_i &= k^2 \\ &> k(k-1) \\ &= g(g-1). \end{aligned}$$

Key Observation

Lemma

Suppose Γ *is a semigroup of genus* g *and* $\{b_1, \ldots, b_g\}$ *are gaps of* Γ*. Then* $b_i \leq 2i - 1$*.*

Proof.

If b_i is a gap then every pair $(k, b_i - k)$, for $1 \leq k \leq \lfloor b_i/2 \rfloor$ must have at least one gap. Otherwise b_i would not be a gap.

Suppose $b_i \geq 2i$. Then there exist at least i such pairs. Since each such pair must contain at least one gap there must be at least i gaps before b_i. A contradiction! □

When the i^{th} gap satisfies $b_i = 2i - 1$ we will refer to it as a *maximal* gap.

Corollaries and Results for Unibranch Case

Let n be the minimal generator of Γ.

Corollary

If $b_i = 2i - 1$ is maximal or $b_i = 2i - 2$, then $b_{i-1} = b_i - n$.

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Suppose Γ satisfies the inequality $(2g-1)^2 \leq 4\sum_{i=1}^{g} b_i$, then we must have $n \leq 4$.

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Theorem

The non-hyperelliptic numerical semigroups that satisfy $(2g-1)^2 \leq 4\sum_{i=1}^{g} b_i$ are

- *Symmetric:* $\langle 3,4\rangle, \langle 3,5\rangle, \langle 3,7\rangle, \langle 4,5,6\rangle$.
- *Non-symmetric:* $\langle 3,4,5\rangle$, $\langle 3,5,7\rangle$.

Setting up the Multibranch Case

- Let $S = \mathbb{C}[t_1] \times \mathbb{C}[t_2] \times \cdots \times \mathbb{C}[t_b]$. Consider a $\mathbb{C}^*$-action on S given by
$$\lambda \cdot t_i = \lambda^{\alpha_i} t_i.$$
- We define S_d as the degree d weighted piece of S, that is:

$$S_d = \left\{ \left(c_1 \cdot t_1^{d_1}, ..., c_b \cdot t_b^{d_b} \right) \mid c_i \in \mathbb{C}, c_i \neq 0 \implies \alpha_i \cdot d_i = d \right\}.$$

Multibranch Case

Consider a $\mathbb{C}$-subalgebra $R \subset S$ such that

- R is $\mathbb{C}^*$-invariant.
- The only degree 0 element of R is $(1, 1, \ldots, 1)$.
- $\dim_{\mathbb{C}}(S/R) < \infty$.

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If $x = (c_1 t_1^{d_1}, \dots, c_b t_b^{d_b}) \in R$ is scaled by $\mathbb{C}^*$-action, we call the number

$$w(x) := \alpha_1 d_1 = \cdots = \alpha_b d_b$$

the $\mathbb{C}^*$*-weight* (or *degree*) of x. We set

$$R_d := R \cap S_d$$

to be the subspace of R consisting of the elements of $\mathbb{C}^*$-weight d.

Multibranch Case

We define for R

- A *generalized semigroup*

$$\tilde{\Gamma}_R := \{(d, v_d) \mid d \in \mathbb{Z}, v_d = \dim_{\mathbb{C}} R_d\}.$$

- A *numerical conductor*

$$\mathrm{cond}(R) := \min\{d \mid v_n = \dim_{\mathbb{C}} S_n \ \forall\ n \geq d\}.$$

- The module of *Rosenlicht differentials*

$$\omega_R := \left\{ w = \left(u_1 \frac{dt_1}{t_1^{m_1}}, \ldots, u_b \frac{dt_b}{t_b^{m_b}} \right) \ \middle| \ \sum_{i=1}^{b} \mathrm{Res}_{t_i}\ f \cdot w = 0 \ \forall\ f \in R \right\}$$

We say that R is *Gorenstein* if ω_R is a principal R-module.

Linking R and ω_R

Let $(\omega_R)_d$ be the dth graded piece of the module ω_R, and $-B$ be the degree of the generator of ω_R.

We show the following:

- $\dim(\omega_R)_d = \dim S_d - \dim R_d$
- $\dim(\omega_R)_{-B+d} = \dim R_d$
- Putting the above two statements together this shows: $\dim R_d + \dim R_{B-d} = \dim S_d = \dim S_{B-d}$.

The above formula is an analog of "the symmetry condition" in the unibranch case.

Looking forward

We define the following:

- The analog of "the sum of gaps":

$$\chi_1 = \sum_{d=0}^{B} \dim(\omega_R)_{-B+d} \cdot (d - B) = \sum_{d=0}^{B} \dim R_d \cdot (d - B)$$

- The analog of "the sum of gaps plus $(2g-1)^2$":

$$\chi_2 = \sum_{d=0}^{2B} \dim R_d \cdot (d - 2B).$$

Open Problem

Classify multibranch Gorenstein algebras R such that

$$\frac{\chi_2}{\chi_1} \leq 5.$$