Classification of Numerical Semigroups

Andrew Ferdowsian, Jian Zhou
Advised by: Maksym Fedorchuk
Supported by: BC URF program

Boston College

September 16, 2015
A *numerical semigroup* is a subset $\Gamma \subset \mathbb{Z}^{\geq 0}$ such that

- $\Gamma$ contains 0,
- $\Gamma$ is closed under addition,
- the complement of $\Gamma$ in $\mathbb{Z}^{\geq 0}$ is finite, denoted as *gap sequence*, and its elements are called *gaps*.

We can regard $\Gamma \subset \mathbb{Z}^{\geq 0}$ as a group under addition, but without the property of inverses.
A numerical semigroup is a subset $\Gamma \subset \mathbb{Z}^{\geq 0}$ such that

- $\Gamma$ contains 0,
- $\Gamma$ is closed under addition,
- the complement of $\Gamma$ in $\mathbb{Z}^{\geq 0}$ is finite, denoted as gap sequence, and its elements are called gaps.

We can regard $\Gamma \subset \mathbb{Z}^{\geq 0}$ as a group under addition, but without the property of inverses.

Example 1: $\Gamma = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \ldots \}$.

$\mathbb{Z}^{\geq 0} \setminus \Gamma = \{1, 2, 3, 6, 7, 11\}$. 
A *numerical semigroup* is a subset $\Gamma \subset \mathbb{Z}_{\geq 0}$ such that

- $\Gamma$ contains 0,
- $\Gamma$ is closed under addition,
- the complement of $\Gamma$ in $\mathbb{Z}_{\geq 0}$ is finite, denoted as *gap sequence*, and its elements are called *gaps*.

We can regard $\Gamma \subset \mathbb{Z}_{\geq 0}$ as a group under addition, but without the property of inverses.

**Example 1:** $\Gamma = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \ldots \}$.  
$\mathbb{Z}_{\geq 0} \setminus \Gamma = \{1, 2, 3, 6, 7, 11\}$.

**Example 2:** $\Gamma = \{0, 4, 5, 7, 8, 9, \ldots \}$.  
$\mathbb{Z}_{\geq 0} \setminus \Gamma = \{1, 2, 3, 6\}$. 
Generators, Genus, and Conductor

Every numerical semigroup can be represented uniquely by a set of minimal *generators*.

Ex.1: \( \langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12, \ldots \} = \mathbb{Z} \setminus \{1, 2, 3, 6, 7, 11\} \).

Ex.2: \( \langle 4, 5, 7 \rangle = \{0, 4, 5, 7, \ldots \} = \mathbb{Z} \setminus \{1, 2, 3, 6\} \).
Every numerical semigroup can be represented uniquely by a set of minimal *generators*.

Ex.1: $\langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12, \ldots \} = \mathbb{Z} \setminus \{1, 2, 3, 6, 7, 11\}$.
Ex.2: $\langle 4, 5, 7 \rangle = \{0, 4, 5, 7, \ldots \} = \mathbb{Z} \setminus \{1, 2, 3, 6\}$.

We define

- The *genus* $g(\Gamma)$ to be the number of elements in the gap sequence.
- The *Frobenius number* $F(\Gamma)$ is the largest element in the gap sequence.
- The *conductor* $\text{Cond}(\Gamma) = F(\Gamma) + 1$.

(Sylvester, 1884) If $\Gamma = \langle a, b \rangle$, where $a$ and $b$ are coprime, then

\[
g(\Gamma) = \frac{(a - 1)(b - 1)}{2} \\
F(\Gamma) = (a - 1)(b - 1) - 1
\]
Numerical semigroups arise in algebraic geometry in the context of curve singularities. Namely, to a semigroup \( \Gamma \), one associates a unibranch curve singularity whose algebra of functions is

\[
\mathbb{C}[\Gamma] := \mathbb{C}[t^a \mid a \in \Gamma].
\]

\(<4,5>\) = \(\mathbb{Z}\)\{1, 2, 3, 6, 7, 11\} is symmetric.

\(<4,5,7>\) = \(\mathbb{Z}\)\{1, 2, 3, 6\} is non-symmetric.
Numerical semigroups arise in algebraic geometry in the context of curve singularities. Namely, to a semigroup $\Gamma$, one associates a unibranch curve singularity whose algebra of functions is

$$\mathbb{C}[\Gamma] := \mathbb{C}[t^a \mid a \in \Gamma].$$

The $\Gamma$-singularity, $\text{Spec } \mathbb{C}[t]$, is called Gorenstein if $\Gamma$ is a symmetric semigroup, i.e.,

$$n \in \{\text{gaps}\} \iff F(\Gamma) - n \not\in \{\text{gaps}\}.$$ 

$\langle 4, 5 \rangle = \mathbb{Z} \setminus \{1, 2, 3, 6, 7, 11\}$ is symmetric.

$\langle 4, 5, 7 \rangle = \mathbb{Z} \setminus \{1, 2, 3, 6\}$ is non-symmetric.
Open Problem

Classify numerical semigroups that satisfy the inequality

\[
\frac{(2g - 1)^2}{\sum_{i=1}^{g} b_i} \leq \frac{11}{2}.
\]

In our work, we classified those satisfying a stronger condition

\[
\frac{(2g - 1)^2}{\sum_{i=1}^{g} b_i} \leq 4,
\]

which is equivalent to

\[
\sum_{i=1}^{g} b_i > g(g - 1).
\]
A numerical semigroup $\Gamma$ is called hyperelliptic if $\Gamma = \langle 2, m \rangle$, where $m = 2k + 1$ is odd.
A numerical semigroup $\Gamma$ is called \textit{hyperelliptic} if $\Gamma = \langle 2, m \rangle$, where $m = 2k + 1$ is odd. In this case, we have \{1, 3, \ldots, 2k − 1\} as gaps, and so

\[
g = k = \frac{m - 1}{2},
\]
\[
g \sum_{i=1}^{g} b_i = 1 + 3 + \cdots + (2k - 1).
\]
Hyperelliptic Case

A numerical semigroup $\Gamma$ is called *hyperelliptic* if $\Gamma = \langle 2, m \rangle$, where $m = 2k + 1$ is odd. In this case, we have $\{1, 3, \ldots, 2k - 1\}$ as gaps, and so

$$g = k = \frac{m - 1}{2},$$

$$\sum_{i=1}^{g} b_i = 1 + 3 + \cdots + (2k - 1).$$

$$\implies \sum_{i=1}^{g} b_i = k^2$$

$$> k(k - 1)$$

$$= g(g - 1).$$
Lemma

Suppose $\Gamma$ is a semigroup of genus $g$ and $\{b_1, \ldots, b_g\}$ are gaps of $\Gamma$. Then $b_i \leq 2i - 1$.

Proof.

If $b_i$ is a gap then every pair $(k, b_i - k)$, for $1 \leq k \leq \lfloor b_i/2 \rfloor$ must have at least one gap. Otherwise $b_i$ would not be a gap.

Suppose $b_i \geq 2i$. Then there exist at least $i$ such pairs. Since each such pair must contain at least one gap there must be at least $i$ gaps before $b_i$. A contradiction!

When the $i^{th}$ gap satisfies $b_i = 2i - 1$ we will refer to it as a maximal gap.
Corollaries and Results for Unibranch Case

Let $n$ be the minimal generator of $\Gamma$.

**Corollary**

If $b_i = 2i - 1$ is maximal or $b_i = 2i - 2$, then $b_{i-1} = b_i - n$.

**Corollary**

Suppose $\Gamma$ satisfies the inequality $(2g - 1)^2 \leq 4 \sum_{i=1}^{g} b_i$, then we must have $n \leq 4$. 

Let $n$ be the minimal generator of $\Gamma$.

**Corollary**

If $b_i = 2i - 1$ is maximal or $b_i = 2i - 2$, then $b_{i-1} = b_i - n$.

**Corollary**

Suppose $\Gamma$ satisfies the inequality $(2g - 1)^2 \leq 4 \sum_{i=1}^{g} b_i$, then we must have $n \leq 4$.

**Theorem**

The non-hyperelliptic numerical semigroups that satisfy $(2g - 1)^2 \leq 4 \sum_{i=1}^{g} b_i$ are

- **Symmetric**: $\langle 3, 4 \rangle$, $\langle 3, 5 \rangle$, $\langle 3, 7 \rangle$, $\langle 4, 5, 6 \rangle$.
- **Non-symmetric**: $\langle 3, 4, 5 \rangle$, $\langle 3, 5, 7 \rangle$. 
Setting up the Multibranch Case

Let $S = \mathbb{C}[t_1] \times \mathbb{C}[t_2] \times \cdots \times \mathbb{C}[t_b]$. Consider a $\mathbb{C}^*$-action on $S$ given by

$$\lambda \cdot t_i = \lambda^{\alpha_i} t_i.$$ 

We define $S_d$ as the degree $d$ weighted piece of $S$, that is:

$$S_d = \left\{ (c_1 \cdot t_1^{d_1}, \ldots, c_b \cdot t_b^{d_b}) \mid c_i \in \mathbb{C}, c_i \neq 0 \implies \alpha_i \cdot d_i = d \right\}.$$
Multibranch Case

Consider a $\mathbb{C}$-subalgebra $R \subset S$ such that

- $R$ is $\mathbb{C}^*$-invariant.
- The only degree 0 element of $R$ is $(1, 1, \ldots, 1)$.
- $\dim_{\mathbb{C}}(S/R) < \infty$.

Geometrically, $R$ is the algebra of functions on a connected affine curve with $b$ branches and a good $\mathbb{C}^*$-action.
Multibranch Case

Consider a $\mathbb{C}$-subalgebra $R \subset S$ such that
- $R$ is $\mathbb{C}^*$-invariant.
- The only degree 0 element of $R$ is $(1, 1, \ldots, 1)$.
- $\dim_{\mathbb{C}}(S/R) < \infty$.

Geometrically, $R$ is the algebra of functions on a connected affine curve with $b$ branches and a good $\mathbb{C}^*$-action.

If $x = (c_1 t_1^{d_1}, \ldots, c_b t_b^{d_b}) \in R$ is scaled by $\mathbb{C}^*$-action, we call the number

$$w(x) := \alpha_1 d_1 = \cdots = \alpha_b d_b$$

the $\mathbb{C}^*$-weight (or degree) of $x$. We set

$$R_d := R \cap S_d$$

to be the subspace of $R$ consisting of the elements of $\mathbb{C}^*$-weight $d$. 
Multibranch Case

We define for $R$

- A generalized semigroup
  $$\tilde{\Gamma}_R := \{(d, v_d) \mid d \in \mathbb{Z}, v_d = \dim_{\mathbb{C}} R_d\}.$$

- A numerical conductor
  $$\text{cond}(R) := \min\{d \mid v_n = \dim_{\mathbb{C}} S_n \quad \forall \quad n \geq d\}.$$

- The module of Rosenlicht differentials
  $$\omega_R := \left\{ w = \left(u_1 \frac{dt_1}{t_1^{m_1}}, \ldots, u_b \frac{dt_b}{t_b^{m_b}}\right) \mid \sum_{i=1}^{b} \text{Res}_{f_i} f \cdot w = 0 \quad \forall \quad f \in R \right\}$$

We say that $R$ is Gorenstein if $\omega_R$ is a principal $R$-module.
Let \((\omega_R)_d\) be the \(d\)th graded piece of the module \(\omega_R\), and \(-B\) be the degree of the generator of \(\omega_R\).

We show the following:

- \(\dim(\omega_R)_d = \dim S_d - \dim R_d\)
- \(\dim(\omega_R)_{-B+d} = \dim R_d\)

Putting the above two statements together this shows:

\[
\dim R_d + \dim R_{B-d} = \dim S_d = \dim S_{B-d}.
\]

The above formula is an analog of “the symmetry condition” in the unibranch case.
Looking forward

We define the following:

- The analog of “the sum of gaps”:
  \[ \chi_1 = \sum_{d=0}^{B} \dim(\omega_R)_{B+d} \cdot (d - B) = \sum_{d=0}^{B} \dim R_d \cdot (d - B) \]

- The analog of “the sum of gaps plus \((2g - 1)^2\)”:
  \[ \chi_2 = \sum_{d=0}^{2B} \dim R_d \cdot (d - 2B) \]

Open Problem

Classify multibranch Gorenstein algebras \( R \) such that

\[ \frac{\chi_2}{\chi_1} \leq 5. \]