Extremality of some F-nef Functions

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F-nef Functions

Given an abelian group G, a function $f : G \to \mathbb{Z}$ is said to be F-nef if f satisfies the inequality:

$$f(a) + f(b) + f(c) + f(a+b+c) \geq f(a+b) + f(a+c) + f(b+c)$$

for all $a, b, c \in G$.

F-nef Cone

Functions: Assign $f : \mathbb{Z}_m \to \mathbb{Z}$ to $\vec{v} \in \mathbb{R}^m$

E.g. $v = (1, 0, 0, 0, 0, 0, ...)^T$

Triples: Assign (a, b, c) to it's normal vector.

E.g. $(0, 0, 0) \to v = (1, 0, 0, 0, 0, 0, ...)^T$ corresponds to the inequality $f(0) \geq 0$

The F-nef Cone (continued

By taking all vectors satisfying the inequality over all linearly independent triples, one can create the cone of all F-nef functions for a fixed n.

A function (reprsented by it's vector) is an extremal ray of the F-nef cone if for $m - 1$ linearly independent inequalities, the inequalities are actually equalities.

First Extremal Ray

$A_m : \mathbb{Z}_m \to \mathbb{Z}$ defined by $A_m(i) = i(m - i)$ is F-nef and an extremal ray. At first glance this is hard to check.

```
[1, 0, 0, 0, 0, 0, 0]
[0, 3, -3, 1, 0, 0, 0]
[0, 2, 0, -2, 1, 0, 0]
[0, 2, -1, 1, -2, 1, 0]
[0, 2, -1, 0, 1, -2, 1]
[1, 2, -1, 0, 0, 1, -2]
[0, 1, 2, -2, -1, 1, 0]
[0, 1, 1, 0, -1, -1, 1]
[1, 1, 1, -1, 1, -1, -1]
[1, 1, 0, 2, -2, 0, -1]
[0, 0, 3, 0, -3, 0, 1]
[1, 0, 2, 1, -1, -2, 0]
[1, -2, 1, 0, 0, -1, 2]
[1, -1, -1, 1, -1, 1, 1]
[0, 1, -2, 1, 0, -1, 2]
[1, -1, 0, -2, 2, 0, 1]
[1, 0, -2, -1, 1, 2, 0]
[0, 1, -1, -1, 0, 1, 1]
[0, 0, 1, -2, 1, -1, 2]
[0, 1, 0, -3, 0, 3, 0]
[0, 0, 1, -1, -2, 2, 1]
[0, 0, 0, 1, -2, 0, 2]
[0, 0, 0, 0, 1, -3, 3]
```

It can be easily checked that A_m satisfies the triples $(1, 1, x)$ where $0 \leq x \leq m - 2$.
So all we need to do is row reduce the associated matrix:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 3 & -3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & \ddots & & & \vdots & \\
0 & 2 & -1 & 0 & 0 & 0 & & 1 & -2 & 1 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\
1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2
\end{bmatrix}$$

Row Reduction

This problem looks hard, so instead, row reduce:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 3 & -3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & \ddots & & & \vdots & \\
0 & 2 & -1 & 0 & 0 & 0 & & 1 & -2 & 1 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\
1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2
\end{bmatrix}$$

Second extremal ray

$B_m : \mathbb{Z}_m \to \mathbb{Z}$ defined by $B_m(i) = A_m(i)$ for $i \neq 0 \in \mathbb{Z}_m$ and $B_m(0) = m$ is extremal for all $m > 2$
It satisfies the triples $(1, 1, m-1)$ and $(m-1, m-1, 1)$

$$\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 3 \\
0 & 3 & -3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 1 & -2 & 1 & & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 1 & -2 & & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & \ddots & & & \vdots & \\
0 & 2 & -1 & 0 & 0 & 0 & & 1 & -2 & 1 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\
-2 & 3 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix}$$

Second Extremal ray (continued)

Subtracting the first row from the last row yields:

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 3 \\ 0 & 3 & -3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 & -2 & 1 & & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & -2 & & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\ & & \vdots & & & & \ddots & & & \vdots & \\ 0 & 2 & -1 & 0 & 0 & 0 & & 1 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 \end{bmatrix}$$

Which shows these vectors are linearly independent.

Extra Results

$[0, 1, 0, 1, 0, 1..., 0, 1]^T$ and $[3, 4, 3, 4, 3, 4, ...3, 4]^T$ are extremal.

In fact if $\phi : \mathbb{Z}_m \to \mathbb{Z}_m$ is an isomorphism and f is extremal, then $f \circ \phi$ is also extremal.

This isn't always true when ϕ is a homomorphism for example, if $\phi : \mathbb{Z}_{2k} \to \mathbb{Z}$, defined by $\phi(x) = kx$, then $B_m \circ \phi = (2k, k^2, 2k, k^2, ..., 2k, k^2)^T$ is never an extremal ray check by parity).
Homomorphisms can help you find extremal rays.

End of Presentation.

Any Questions?