STRING CONES AND TORIC VARIETIES

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ABSTRACT. Alexeev and Brion provide, given a reduced expression, w_0, and a dominant weight λ, a way to calculate a toric degeneration of $G/P(\lambda)$ with toric limit isomorphic to the toric variety obtained from the polytope $Q_{w_0}(\lambda)$. Here, we will investigate what these polytopes look like as $Q_{w_0}(\lambda)$ varies over w_0 and λ.

CONTENTS

1. CONVEX POLYTOPES

We begin with some brief notes on polytopes. What we will soon discover is that convex polytopes play a crucial role in algebraic geometry. In particular, we will observe the close relationship between toric varieties and a certain type of cone.

Definition 1.1. A set $K \subset R^d$ is **convex** if and only if for each pair of distinct points $a, b \in K$ the closed segment with endpoints a and b is contained in K. This is equivalent to saying that K is convex if and only if $a, b \in K$ and $0 \leq \lambda \leq 1$ imply $\lambda a + (1-\lambda)b \in K$.

We provide a useful way to turn an arbitrary set into a convex set.

Definition 1.2. The **convex hull**, conv A, of a subset A of $\mathbb{R}^d$ is the intersection of all the convex sets in $\mathbb{R}^d$ that contain A, and it is clear that conv A is the smallest such convex set in $\mathbb{R}^d$.

Definition 1.3. A **convex polyhedral cone** in $\mathbb{R}^n$ is a subset of the form

$$\sigma = \text{Cone}\,(S) = \left\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \geq 0 \right\} \subset \mathbb{R}^n$$

where $S \in \mathbb{R}^n$ is finite. We say that σ is generated by S, and we set $\text{Cone}(\varnothing) = \{0\}$.

Definition 1.4. A **polytope** in $\mathbb{R}^n$ is a subset of the form

$$P = \text{conv}(S) = \left\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \geq 0 \right\} \subset \mathbb{R}^n$$

where $S \in \mathbb{R}^n$ is finite. We say that P is the convex hull of S.

Since these are subsets of Euclidean space, we can talk about dimension. The dimension of a convex polyhedral cone σ is the dimension of the smallest subspace R_σ containing σ. Further, the dimension of a polytope $P \subset \mathbb{R}^n$ is the dimension of the smallest affine space containing P.

Examples 1.5. Polytopes include regular n-gons in $\mathbb{R}^2$ and cubes, tetrahedra, octahedra, etc. in $\mathbb{R}^3$.

2. Algebraic Geometry

We will now cover some algebraic preliminaries that will be used throughout the paper. We start with a discussion of affine varieties, focusing first on the affine variety $(C^*)^n$, which is a group under component-wise multiplication. The idea of a variety that also has a group structure is a key idea, and we make the following important definition:

Definition 2.1. A **torus** is an affine variety that is isomorphic to $(C^*)^n$, where T inherits a group structure from the isomorphism.

We now present two basic facts about tori

Proposition 2.2.

(1) *If T_1 and T_2 are tori, and $\phi : T_1 \to T_2$ is a morphism that is also a group homomorphism, then the image of ϕ is a torus and is closed in T_2*
(2) *If T is a torus and $H \subseteq T$ is an irreducible subvariety of T that is a subgroup, then H is also a torus.*

To further the discussion of tori, we will need to define a certain type of algebraic structure.

Definition 2.3. A **lattice** is a free abelian group of finite rank. A lattice of rank n is isomorphic to $\mathbb{Z}^n$.

Definition 2.4. A **character** of a torus T is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. For example, $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$ gives a character $\chi_m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi_m(t, \dots, t) = t_1^{a1} \cdots t_n^{a_n}$$

It is a fact that all characters of $(C^*)^n$ arise in this way, as this is the only way for a character to satisfy the conditions of being both a morphism of varieties and also a group homomorphism. Then we have that the characters of $(C^*)^n$ form a group isomorphic to $\mathbb{Z}^n$. In particular, for an arbitrary torus, its characters form a lattice M of rank equal to the dimension of T.

A similar notion that is in a sense dual to the character is the one-parameter subgroup.

Definition 2.5. A **one-parameter subgroup** of a torus T is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. For example, $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$ gives a one parameter subgroup $\lambda^u : \mathbb{C}^*(\mathbb{C}^*)^n$ defined by

$$\lambda^u(t) = (t^{b_1}, \dots, t^{b_n})$$

As with characters, all one-parameter subgroups of $(\mathbb{C}^*)^n$ arise in this way, forming a group isomorphic to $\mathbb{Z}^n$. Similarly, this turns the one-parameter subgroups of $(\mathbb{C}^*)^n$ into a lattice N of rank equal to the dimension of T. An element $u \in N$ gives the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T$.

There is a natural bilinear pairing between characters and one-parameter subgroups $\langle , \rangle : M \times N \to \mathbb{Z}$ given as follows. If we have a character χ_m and a one-parameter subgroup λ^u, then the composition $\chi_m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ is in fact a character of $\mathbb{C}^*$, which is given by $t \mapsto t^l$ for some $l \in \mathbb{Z}$. We say that $\langle m, u \rangle = l$.

Explicitly, if $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$, $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$, then $\langle m, u \rangle$ is the usual dot product

$$\langle m, u \rangle = \sum_{i=1}^{n} a_i b_i$$

From this, we identify N with $\hom_{\mathbb{Z}}(M, \mathbb{Z})$ and M with $\hom_{\mathbb{Z}}(N, \mathbb{Z})$. Therefore, we obtain $N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T$ via $u \otimes t \mapsto \lambda^u(t)$. Hence we usually write a torus as T_N .

Remark 2.6. Picking an isomorphism $T_N \simeq (\mathbb{C}^*)^n$ induces dual bases of M and N that turn characters into Laurent monomials, one-parameter subgroups into monomial curves, and the pairing into the dot product.

With a brief exposition of tori out of the way, we can also talk about varieties that contain tori. That is,

Definition 2.7. An **affine toric variety** is an irreducible affine variety V containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action (i.e. a morphism) of T_N on V.

We now give typical examples of affine toric varieties

Definition 2.8. The plane curve $C = V(x^3 - y^2) \subseteq \mathbb{C}^2$ is an affine toric variety with torus

$$C \setminus \{0\} = C \cap (C^*)^2 = \{(t^2, t^3) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^*$$

where the isomorphism is $t \mapsto (t^2, t^3)$.

Example 2.9. The variety $V = V(xyzq) \subseteq \mathbb{C}^4$ is an affine toric variety with torus

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1} \mid t \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^3$$

where the isomorphism is $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$.

Definition 2.10. Given a torus T_N with character lattice M,a set $A = \{m_1, \dots, m_s\} \subseteq M$ gives characters $\chi^{m_i} : T_N \to \mathbb{C}^*$. Then consider the map

$$\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$$

defined by

$$\Phi_{\mathscr{A}}(t) = (\chi^{m_1}(t), \dots, \chi^{m_s}(t)) \in \mathbb{C}^s$$

Then given such a set $A \subseteq M$, we can define the affine toric variety $Y_{\mathscr{A}}$ to be the Zariski closure of the image of the map $\Phi_{\mathscr{A}}$. It is a fact that every affine toric variety is isomorphic to $Y_{\mathscr{A}}$ for some finite subset $\mathscr{A}$ of a lattice. This fact relies on the study of something called toric ideals. It can be shown that we can construct a toric ideal from $Y_{\mathscr{A}}$. This construction is useful enough to where we form a generalization called a lattice ideal, where prime lattice ideals are precisely the toric ideals.

Examples 2.11. The ideals $\langle x^3 - y^2 \rangle \subseteq \mathbb{C}[x, y]$ and $\langle xz - yw \rangle \subseteq \mathbb{C}[x, y, z, w]$ are examples of toric ideals.

It is also true that an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_s]$ is toric if and only if it is prime and generated by binomials. This (non-rigorous) discussion of toric ideals just serves to highlight the fundamental fact that all affine toric varieties arise from toric ideals.

Toric varities also arise from other places, and we will shift focus now in order to observe this.

Definition 2.12. A **semigroup** is a set S with an associative binary operation and an identity element. Further, an **affine semigroup** is a semigroup such that

- The binary operation on S is commutative. To standardize the notation, we will denote the operation as $+$ and the identity element as 0. Therefore, a finite set $\mathscr{A} \subseteq S$ gives
$$\mathbb{N}\mathscr{A} = \left\{ \sum_{m \in \mathscr{A}} a_m m \mid a_m \in \mathbb{N} \right\} \subseteq S$$
- The semigroup is finitely generated. That is to say that there is a finite set $\mathscr{A} \subseteq S$ such that $\mathbb{N}\mathscr{A} = S$.
- The semigroup can be embedded in a lattice M.

Example 2.13. A basic example of an affine semigroup is $\mathbb{N}^n \subseteq \mathbb{Z}^n$. In general, if we have a lattice M and a finite set $\mathscr{A} \subseteq M$, $\mathbb{N}\mathscr{A} \subseteq M$ is an affine semigroup. In fact, all affine semigroups are of this form, up to isomorphism.

We make another useful algebraic definition.

Definition 2.14. An **algebra** is a vector space with basis S equipped with bilinear map,
$$A \times A \to A, \qquad (x, y) \to xy$$
where xy is called the product of x and y which is inherited from the structure of S.

For example, given an affine semigroup S, we can define the **semigroup algebra**, $\mathbb{C}[S]$. This is a $\mathbb{C}$-vector space which has elements of S as a basis, where to the element $m \in S$, we associate the basis vector χ^m. In particular, elements of $\mathbb{C}[S]$ consist of formal sums
$$\sum_{m \in S} a_m \chi^m$$
where only finitely many $a_m \in \mathbb{C}$ are nonzero. We see that the multiplication of two formals sums is inherited from the semigroup structure, as the rules for multiplication are given by
$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}$$
using the distributive law, and where χ^0 is the multiplicative identity element.

Example 2.15. If $S = \mathbb{N}\mathscr{A}$ for $\mathscr{A} = \{m_1, ..., m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$.

Example 2.16. $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives that $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n]$ where $x_i = \chi^{e_i}$ where e_i is the standard basis for $\mathbb{Z}^n$.

Proposition 2.17. *Let $S \subset M$ be an affine semigroup. Then*

(1) $\mathbb{C}[S]$ is an integral domain and a finitely generated $\mathbb{C}$-algebra

(2) $\mathrm{Spec}(\mathbb{C}[S])$ is an affine toric variety whose torus has character lattice $\mathbb{Z}S$, and if $S = \mathbb{N}\mathscr{A}$ for a finite set $A \subseteq M$, then $\mathrm{Spec}(\mathbb{C}[S]) = Y_{\mathscr{A}}$.

We give an example that ties all of this together.

Example 2.18. Consider the affine semigroup $S \in \mathbb{Z}$ generated by 2 and 3, so that $S = \{0, 2, 3, \ldots\}$. Setting $A = \{2, 3\}$, then $\phi_{\mathscr{A}}(t) = (t^2, t^3)$ and the toric ideal is $I(Y_{\mathscr{A}}) = \langle x^3 - y^2 \rangle$. Hence,

$$\mathbb{C}[S] = \mathbb{C}[t^2, t^3] \simeq \mathbb{C}[x, y]/\langle x^3 - y^2 \rangle$$

and the affine toric variety $Y_{\mathscr{A}}$ is the curve $x^3 - y^2$, as in an example above.

Furthermore, the following proposition highlights the relationship between the ideas weve been discussing.

Proposition 2.19. *Let V be an affine variety. The following are equivalent:*

(1) V is an affine toric variety

(2) $V = Y_{\mathscr{A}}$ for a finite set A in a lattice

(3) V is an affine variety defined by a toric ideal

(4) $V = \mathrm{Spec}(\mathbb{C}[S])$ for an affine semigroup S.

With the developments of this section in mind, we now return to the discussion of cones from the previous section, focusing now on the algebraic geometry of tori.

Example 2.20. We saw in an example above that $V = V(xy - zw) \subseteq \mathbb{C}^4$ is a toric variety with toric ideal $\langle xy - zw \rangle \subseteq \mathbb{C}[x, y, z, w]$. The torus is $(\mathbb{C}^*)^3$ via the map $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$. In particular, we have that

$$\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$$

defined by

$$\Phi_{\mathscr{A}}(t) = (\chi^{m_1}(t), \ldots, \chi^{m_4}(t)) \subseteq \mathbb{C}^4$$

with $m_1 = (1, 0, 0)$, $m_2 = (0, 1, 0)$, $m_3 = (0, 0, 1)$, and $m_4 = (1, 1, 1)$. In particular, the lattice points used in the map can be represented as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

From this, we have a semigroup $S \subseteq \mathbb{Z}^3$ consisting of the $\mathbb{N}$-linear combinations of the column vectors. Therefore, the elements of S are the lattice points lying in the polyhedral region in $\mathbb{R}^3$ generated by the column vectors

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}.$$

This type of polyhedral region is called a **rational polyhedral cone**. It is a fact that S consists of all lattice points lying in this cone.

Furthering the discussion of cones, if we have a cone, we can talk about its dual. This will turn out to be useful.

Definition 2.21. Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its **dual cone** is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}$$

If $\sigma \subseteq N_{\mathbb{R}}$ is a polyhedral cone, then two useful properties are that $\sigma^{\vee}$ is a polyhedral cone in $M_{\mathbb{R}}$ and also that $(\sigma^{\vee})^{\vee} = \sigma$.

Now, thinking about cones in terms of inequalities is sometimes confusing, and perhaps a more intuitive way to think about them is in terms of hyperplanes and half-spaces.

Given $m \neq 0$ in $M_{\mathbb{R}}$,

Definition 2.22. We define the **hyperplane**

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle = 0\} \subset N_{\mathbb{R}}$$

Definition 2.23. And we define the **closed half-space**

$$H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle \geq 0\} \subset N_{\mathbb{R}}$$

In particular, if $m_1, \ldots, m_s$ generate $\sigma^{\vee}$, then we have that

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$$

so that *every polyhedral cone is an intersection of finitely many closed half-spaces.*

Finally, again let N and M be dual lattices with associated vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Then a polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is **rational** if $\sigma = \text{Cone}(S)$ for some finite set $S \subseteq N$.

A main result is that, given a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, the lattice points

$$S\sigma = \sigma^{\vee} \cap M \subseteq M$$

form a semigroup. An important fact is

Proposition 2.24 (Gordon's Lemma)**.** *$S\sigma = \sigma^{\vee} \cap M$ is finitely generated and hence is an affine semigroup*

Finally, to bring this all together we have that if $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ is a rational polyhedral cone with semigroup $S\sigma = \sigma^{\vee} \cap M$, then

$$U_{\sigma} = \text{Spec}(\mathbb{C}[S\sigma]) = \text{Spec}(\mathbb{C}[\sigma^{\vee} \cap M])$$

is an affine toric variety.

Example 2.25. Let $\sigma = \text{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq N_{\mathbb{R}} = \mathbb{R}^3$ with $N = \mathbb{Z}^3$. We calculate that $\sigma^{\vee} = \text{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3)$. This is precisely the cone whose lattice points are generated by the columns of the matrix given in example 2.20. It follows that $U\sigma$ is the affine toric variety $V(xy - zw)$.

In these past two sections, we have only begun to scratch the surface of the deep relationship between cones, semigroups, and toric varieties. There is still much to be said on the subject, but in the next couple sections we will be switching gears.

3. Representation Theory

As we switch our focus from cones and algebraic geometry to the development of the mysterious objects called string cones, were in fact going to need a few basic facts about representation theory.

Definition 3.1. Suppose G is a finite group. A **linear representation** of G in a vector space V is a homomorphism $\rho : G \to GL(V)$. In particular, we associate an element $\rho(s) \in GL(V)$ to each element $s \in G$ so that we have

$$\rho(st) = \rho(s)\rho(t) \qquad \text{for } s, t \in G$$

Representations first arose as group actions where a group would permute the elements of a set. In light of this, for an element $s \in G$ and $v \in GL(V)$, we often write $\rho_s v$ or even sv instead of $\rho(s)v$, as an abuse of notation.

Definitions 3.2. A **subrepresentation** of a representation V is a vector subspace W of V which is invariant under G. A representation V is called irreducible if there is no proper nonzero invariant subspace W of V.

Theorem 3.3. *Every representation is a direct sum of irreducible representations*

This fact is called **complete reducibility**. In particular, if W is a finite dimensional representation of GL_n, then there exist irreducible subrepresentations $W_1, \dots, W_r$ (not necessarily distinct, nor unique) such that $W = W_1 \oplus \cdots \oplus W_r$. Therefore, in order to understand the (finite dimensional) representations of GL_n, we need only determine all of the irreducible representations.

Lemma 3.4. *The tensor product of two representations is a representation.*

We note that the tensor product of two irreducible representations is not in general irreducible.

As we saw, a useful tool in understanding representations is to break them down in terms of the representations that lie inside of them. However, were we to simply decompose a representation into irreducibles via Theorem 3.3, we would be faced with the fact that this direct sum is not unique (take all of the $_s$ to be 1, for example, where in the decomposition $V = W_1 \oplus \cdots \oplus W_n$, the W_i are lines, and there are many ways to decompose a vector space as a direct product of lines) it is natural, then, to establish a canonical decomposition of representations. In particular, suppose we have an arbitrary representation $\rho : G \to GL(V)$ and irreducible representations $W_1, \dots, W_h$ of G with degrees $n_1, \dots, n_h$. Let $V = U_1 \oplus \cdots \oplus U_m$ be a decomposition of V into a direct sum of irreducible representations. For $i, \dots, h$, take the direct sum of those of the $U_1, \dots, U_m$ which are isomorphic to W_i and denote this direct sum by V_i. Then we have $V = V_1 \oplus \cdots \oplus V_h$ and we call this the canonical decomposition.

One of the main things we will be exploring is decomposing the tensor product of two irreducible representations into irreducible representations, and in particular how this relates to objects called string cones.

4. Semisimple Lie Algebras

Lie Theory provides a wealth of interesting mathematics, and we will see that the representations of Lie groups and Lie algebras play a key role in understanding and developing string cones, which is our primary object of study.

Definition 4.1. Recall that a **Lie group** is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by

$$m(g, h) = gh, \qquad i(g) = g^{-1}$$

are booth smooth.

Examples 4.2. The typical first examples of a Lie group are $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$. Further, we have that $GL(V)$ when V is any real or complex vector space is also a Lie group. In light of this, we can identify $\mathbb{C}^*$ with $GL(1, \mathbb{C})$, making $\mathbb{C}^*$ another example. We have that the circle $S^1 \in \mathbb{C}^*$ is a Lie group, and since any product of Lie groups is a Lie group, the n-Torus $S^1 \times \ldots \times S^1$ is an n-dimensional Lie group.

The most important example, however, for us is SL_n, which we will be focusing on throughout the rest of the paper.

Definition 4.3. Let F be a field. A Lie algebra over F is an F-vector space L, together with a bilinear map called the Lie bracket,

$$L \times L \to L \qquad (x, y) \mapsto [x, y]$$

satisfying $[x, x] = 0$ and the Jacobi Identity.

Examples of Lie algebras are plentiful. Basic examples include $\mathbb{R}^3$ endowed with the cross product or any vector space where the Lie bracket is trivial. Lie algebras are useful because they help encapsulate the structure of Lie groups, removing enough data to more easily study them while still containing enough information to be useful; this Lie group-Lie algebra correspondence is nicely given by something known as the exponential map. We will be working with SL_n, and its corresponding Lie algebra is $\mathfrak{sl}_n$, given by the trace 0 matrices.

One of the major questions in Lie theory is the classification of simple Lie algebras. There are four families and five exceptions, completely described by the connected Dynkin diagrams. Type A_n for $n \geq 1$ corresponds to $\mathfrak{sl}_{n+1}(\mathbb{C})$, the special linear Lie algebra. Type B_n for $n \geq 2$ corresponds to $\mathfrak{so}_{2n+1}(\mathbb{C})$, the odd-dimensional special orthogonal Lie algebra. Type C_n for $n \geq 3$ corresponds to $\mathfrak{sp}_{2n}(\mathbb{C})$, the symplectic Lie algebra. Finally, type D_n for $n \geq 4$ corresponds to $\mathfrak{so}_{2n}(\mathbb{C})$, the even-dimensional special orthogonal Lie algebra. If n is small enough, we have in fact that $A_1 = B_1 = C_1$, $B_2 = C_2$,and $A_3 = D_3$, which is why there are restrictions on n in the above groups. Further, there are five exceptional simple complex Lie algebras: E_6, E_7, E_8, F_4, and G_2. These four classes and five exceptions are in fact the only simple Lie algebras over the complex numbers.

Now, if we have a subalgebra A of the Lie algebra $\mathfrak{gl}(V)$, then it would make sense to say that $v \in V$ is an eigenvector for A if it is an eigenvector for every element of A–in particular, if $a(v) \in \mathrm{Span}(V)$ for every $a \in V$. It is a fact that a set of diagonalizable matrices commutes if and only if the set is simultaneously diagonalizable. An equivalent statement is being able to find a basis for V of simultaneous eigenvectors.

However, we are still left with coming up with a way to generalize eigenvalues. In particular, if we considered the Lie subalgebra of $\mathfrak{gl}_n(F)$ of diagonal matrices $A = d_n(F)$, then it is clear that each e_i in the standard basis $\{e_1, \ldots, e_n\}$ of F_n is an eigenvector for A. However, if a is the diagonal matrix $(\alpha_1, \ldots, \alpha_n)$, then the eigenvalue of a on e_i is α_i, so it is clear that the eigenvalues on e_i vary as the diagonal matrix a runs through the elements of A. We can rectify this by defining a function $\lambda : A \to F$. In particular, the eigenspace is then

$$V\lambda := \{v \in V \mid a(v) = \lambda(a)v \text{ for all } a \in A\}.$$

It can further be shown that λ is linear so that $\lambda \in A^*$, the dual space of linear maps from A to F. We therefore make the following definition:

Definition 4.4. A **weight** for a Lie subalgebra A of $\mathfrak{gl}(V)$ is a linear map $\lambda : A \to F$ such that

$$V_\lambda := \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\}$$

is a non-zero subspace of V .

We now begin exploring representations of Lie algebras.

Definition 4.5. We define the **adjoint map** as

$$ad : L \to \mathfrak{gl}(L), \qquad (adx)y = [x, y]$$

It turns out that this is in fact a Lie homomorphism, so ad is a representation of L. We call this the **adjoint representation**.

The adjoint representation encapsulates much of the structure of L and turns out to play a central role in the representation theory of Lie algebras. We have a specific name for weights of the adjoint representation; we call them **roots**. We now see that these roots have a nice structure.

Definition 4.6. A subset R of a real inner-product space E is a **root system** if it satisfies the following axioms.

(1) R is finite, it spans E, and it does not contain 0.
(2) If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.
(3) If $\alpha \in R$, then the reflection $s\alpha$ permutes the elements of R.
(4) If $\alpha, \beta \in R$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Definition 4.7. The **Weyl Group** is the subgroup of the isometry group of the root system generated by the reflections through the hyperplanes orthogonal to the roots.

As the Weyl Group is generated by simple reflections, we have a notion of a length function which further allows us to talk about reduced words. The longest word is given by w_0, the standard notation. A reduced word for $w \in W$ is a sequence of indices $(i_1, \dots, i_l)$ that satisfies $w = s_{i_1} \cdots s_{i_l}$ and has the shortest possible length $l = l(w)$. The set of reduced words for w is denoted by $R(w)$.

5. String Cones

We can now shift our focus to the main objects of study, the string cone. If V_μ and V_λ are irreducible representations of a Lie algebra, then we can write

$$V_\lambda \otimes V_\mu \cong \bigoplus_\nu V_\nu^{c^\nu_{\lambda\mu}}$$

Berenstein-Zelevinsky compute the integers $c^\nu_{\lambda\mu}$ as the number of integral points inside certain polytopes arising from the string cone. The string cone thus gives a solution to this tensor product multiplicity problem.

If we restrict to the type A_n case, then [1] gives an explicit formula to compute the string cone. But we will need first to introduce some notation.

Notation 5.1. For any $i \in [1, r]$, let $u^{(i)}$ denote the minimal representative of the coset $W_{\hat{i}} s_i w_0$ in W, where $W_{\hat{i}}$ is the (maximal parabolic) subgroup in W generated by all s_j with $j \neq i$.

Theorem 5.2. *Let $i = (i_1, \ldots, i_m) \in R(w_0)$. For any $i \in [1, r]$ and any subword $(i_{k(1)}, \ldots, i_{k(p)})$ of i which is a reduced word for $u(i)$, all the points $(t_1, \ldots, t_m)$ in the string cone C_i satisfy the inequality*

$$\sum_{j=0}^{p} \sum_{k(j)<k<k(j+1)} (s_{i_{k(1)}} \cdots s_{i_{k(j)}})(\omega_i^\vee) \cdot t_k \geq 0 \tag{5.3}$$

(with the convention that $k(0) = 0$ and $k(p+1) = m+1$), and if $\mathfrak{g} = \mathfrak{sl}_{r+1}$, then C_i is the set of all $t \in \mathbb{R}^m$ satisfying the inequalities (5.3).

Example 5.4. Parsing this formula is a bit of a task, so it is useful to work through an example. If we have $\mathfrak{g} = \mathfrak{sl}_3$, then the longest word is $w_0 = (13)$ and we have the two reduced expressions $R(w_0) = \{s_1 s_2 s_1, s_2 s_1 s_2\}$. For the sake of example, we shall study the case when $\mathbf{i} = s_1 s_2 s_1$.

We need to calculate the $u^{(i)}$. We see that $u^{(1)}$, lies the coset $W_{\hat{1}} s_1 w_0 = W_{\hat{1}}(12)(13) = W_{\hat{1}}(132)$, whence the minimal element is $u^{(1)} = (12) = s_1$. Similarly, we have that $u^{(2)}$, lies the coset $W_{\hat{2}} s_2 w_0 = W_{\hat{2}}(23)(13) = W_{\hat{2}}(123)$, where we have $u^{(2)} = s_2$.

With the $u^{(i)}$ calculated, we can now calculate the string cone with the formula given in 5.3, where we use the formula each time each $u^{(i)}$ appears as a subword of w_0. In particular, we have,

$$\begin{aligned} u^{(1)} &: (\underline{1}, 2, 1), \quad (1, 2, \underline{1}) \\ u^{(2)} &: (1, \underline{2}, 1) \end{aligned}$$

Again, note that $\mathbf{i} = (i_1, i_2, i_3) = (s_1 s_2 s_1)$. When $i = 1$, we can realize $u^{(1)} = s_1$ as a subword of i in two different ways, underlined above. Namely, in the first case, we write the subword as $s_1 = (i_1) = (i_{k(1)})$ whence $k(1) = 1$. Then we can work through the sum:

When $j = 0$, we see that $k(0) = 0$ and $k(1) = 1$ so we have no summand.

When $j = 1$, we have $k(1) = 1$ and $k(2) = k(p+1) = 4$, so we look at when $k = 2$ and $k = 3$.

When $k = 2$, the term $(s_{i_{k(1)}} \cdots s_{i_{k(j)}})$ evaluates to $(s_{i_{k(1)}} \cdots s_{i_{k(1)}}) = s_{i_{k(1)}} = s_{i_1} = s_1$. Then s_1 acts on $\alpha_{i_2} = \alpha_2 = \chi_2 - \chi_3$ by $s_1(\chi_2 - \chi_3) = \chi_1 - \chi_3 = \alpha_1 + \alpha_2$. We further have $(\alpha_1 + \alpha_2)(\omega_1^\vee) = \langle (1, 0, 1), (1, 0, 0) \rangle = 1$, which gives us t_2.

When $k = 3$, the permutation term remains s_1 (note this does not depend on k). Then s_1 acts on $\alpha_{i_3} = \alpha_1 = \chi_1 - \chi_2$ by $s_1(\chi_1 - \chi_2) = \chi_2 - \chi_1 = -\alpha_1$. And $(-\alpha_1)(\omega_1^\vee) = \langle (-1, 1, 0), (1, 0, 0) \rangle = -1$, which gives us $-t_3$.

In short, when $i = 1$, the subword $(\underline{1}, 2, 1)$ yields the inequality $t_2 - t_3 \geq 0$. One can work out the remaining inequalities when $i = 1$ and the subword is $(1, 2, \underline{1})$ (one should get $t_1 \geq 0$) and when $i = 2$ and the subword is $(1, \underline{2}, 1)$ (one should get $t_3 \geq 0$). Putting everything together will give us the end result that $C_{\mathbf{i}}$ is a cone in $\mathbb{R}^3$ determined by

$$t_2 \geq t_3 \geq 0, \ t_1 \geq 0 \ .$$

Example 5.5. Another example is when we take $\mathfrak{g} = \mathfrak{sl}_4$, then as seen in [2], we have

$$\begin{aligned}
u^{(1)} \quad &: \quad (2,\underline{1},3,\underline{2},1,3) \to t_6 \geq 0;\\
u^{(2)} \quad &: \quad (\underline{2},\underline{1},\underline{3},2,1,3) \to t_4 - t_5 - t_6 \geq 0,\ (\underline{2},\underline{1},3,2,1,\underline{3}) \to t_3 - t_5 \geq 0,\\
&\quad\ \ (\underline{2},1,\underline{3},2,\underline{1},3) \to t_2 - t_6 \geq 0,\ (\underline{2},1,3,2,\underline{1},\underline{3}) \to t_2 + t_3 - t_4 \geq 0,\\
&\quad\ \ (2,1,3,\underline{2},\underline{1},\underline{3}) \to t_1 \geq 0;\\
u^{(3)} \quad &: \quad (2,1,\underline{3},\underline{2},1,3) \to t_5 \geq 0.
\end{aligned}$$

Therefore, $C_{\mathbf{i}}$ is a cone in $\mathbb{R}^6$ given by:

$$t_1 \geq 0,\ t_2 \geq t_6 \geq 0,\ t_3 \geq t_5 \geq 0,\ t_2 + t_3 \geq t_4 \geq t_5 + t_6\ .$$

6. Gelfand-Tsetlin Patterns

The simplest reduced decomposition of the longest element in $W = S_{n+1}$ is $w_{\text{std}} = (s_1)(s_2 s_1)(s_3 s_2 s_1) \cdots (s_n s_{n-1} \cdots s_1)$, where s_i denotes the transposition exchanging i with $i+1$. The string cone is defined by

$$t_1 \geq 0; \qquad t_2 \geq t_3 \geq 0; \qquad \ldots \qquad t_{(n-1)(n)/2+1} \geq \cdots \geq t_{n(n+1)/2} \geq 0$$

7. Preliminary Results

So far, we can compute C_{w_0} for any reduced word in SL_n for $n \leq 7$ efficiently, based on the formula given in 5.3. The codes outputs for $\mathfrak{sl}_5$ and $\mathfrak{sl}_6$ are given below, which confirms the results found in [2].

Example 7.1. The Gelfand-Tsetlin case for $\mathfrak{g} = \mathfrak{sl}_5$:

$$\begin{aligned}
u^{(1)} \quad &: \quad (\underline{1},\underline{2},1,\underline{3},2,1,4,3,2,1) \to t_7 - t_8 \geq 0\\
&\quad\ \ (\underline{1},\underline{2},1,3,2,1,4,\underline{3},2,1) \to t_4 - t_5 \geq 0\\
&\quad\ \ (\underline{1},2,1,3,\underline{2},1,4,\underline{3},2,1) \to t_2 - t_3 \geq 0\\
&\quad\ \ (1,2,\underline{1},3,\underline{2},1,4,\underline{3},2,1) \to t_1 \geq 0\\
u^{(2)} \quad &: \quad (1,\underline{2},1,\underline{3},2,\underline{1},\underline{4},3,\underline{2},1) \to t_3 \geq 0\\
&\quad\ \ (1,\underline{2},\underline{1},\underline{3},2,1,\underline{4},3,\underline{2},1) \to t_5 - t_6 \geq 0\\
&\quad\ \ (1,\underline{2},\underline{1},\underline{3},\underline{2},1,\underline{4},3,2,1) \to t_8 - t_9 \geq 0\\
u^{(3)} \quad &: \quad (1,2,1,\underline{3},\underline{2},1,\underline{4},\underline{3},2,\underline{1}) \to t_6 \geq 0\\
&\quad\ \ (1,2,1,\underline{3},\underline{2},\underline{1},\underline{4},\underline{3},2,1) \to t_9 - t_{10} \geq 0\\
u^{(4)} \quad &: \quad (1,2,1,3,2,1,\underline{4},\underline{3},\underline{2},1) \to t_{10} \geq 0
\end{aligned}$$

Therefore, $C_{\mathbf{i}}$ is a cone in $\mathbb{R}^{10}$ given by:

$$t_1 \geq 0, \quad t_2 \geq t_3 \geq 0, \quad t_4 \geq t_5 \geq t_6 \geq 0, \quad t_7 \geq t_8 \geq t_9 \geq t_{10} \geq 0\ .$$

Example 7.2. The Gelfand-Tsetlin case for $\mathfrak{g} = \mathfrak{sl}_6$:

$$
\begin{aligned}
u^{(1)} \quad &: \quad (\underline{1},\underline{2},1,\underline{3},2,1,\underline{4},3,2,1,5,4,3,2,1) \to t_{11} - t_{12} \geq 0 \\
& \quad (\underline{1},\underline{2},1,\underline{3},2,1,4,3,2,1,5,\underline{4},3,2,1) \to t_7 - t_8 \geq 0 \\
& \quad (\underline{1},\underline{2},1,3,2,1,4,\underline{3},2,1,5,\underline{4},3,2,1) \to t_4 - t_5 \geq 0 \\
& \quad (\underline{1},2,1,3,\underline{2},1,4,\underline{3},2,1,5,\underline{4},3,2,1) \to t_2 - t_3 \geq 0 \\
& \quad (1,2,\underline{1},3,\underline{2},1,4,\underline{3},2,1,5,\underline{4},3,2,1) \to t_1 \geq 0 \\
u^{(2)} \quad &: \quad (1,\underline{2},1,\underline{3},2,\underline{1},\underline{4},3,\underline{2},1,\underline{5},4,\underline{3},2,1) \to t_3 \geq 0 \\
& \quad (1,\underline{2},\underline{1},\underline{3},2,1,\underline{4},3,\underline{2},1,\underline{5},4,\underline{3},2,1) \to t_5 - t_6 \geq 0 \\
& \quad (1,\underline{2},\underline{1},\underline{3},\underline{2},1,\underline{4},3,2,1,\underline{5},4,\underline{3},2,1) \to t_8 - t_9 \geq 0 \\
& \quad (1,\underline{2},\underline{1},\underline{3},\underline{2},1,\underline{4},\underline{3},2,1,\underline{5},4,3,2,1) \to t_{12} - t_{13} \geq 0 \\
u^{(3)} \quad &: \quad (1,2,1,\underline{3},\underline{2},1,\underline{4},\underline{3},2,\underline{1},\underline{5},\underline{4},3,\underline{2},1) \to t_6 \geq 0 \\
& \quad (1,2,1,\underline{3},\underline{2},\underline{1},\underline{4},\underline{3},2,1,\underline{5},\underline{4},3,\underline{2},1) \to t_9 - t_{10} \geq 0 \\
& \quad (1,2,1,\underline{3},\underline{2},\underline{1},\underline{4},\underline{3},\underline{2},1,\underline{5},\underline{4},3,2,1) \to t_{13} - t_{14} \geq 0 \\
u^{(4)} \quad &: \quad (1,2,1,3,2,1,\underline{4},\underline{3},\underline{2},1,\underline{5},\underline{4},\underline{3},2,\underline{1}) \to t_{10} \geq 0 \\
& \quad (1,2,1,3,2,1,\underline{4},\underline{3},\underline{2},\underline{1},\underline{5},\underline{4},\underline{3},2,1) \to t_{14} - t_{15} \geq 0 \\
u^{(5)} \quad &: \quad (1,2,1,3,2,1,4,3,2,1,\underline{5},\underline{4},\underline{3},\underline{2},1) \to t_{15} \geq 0
\end{aligned}
$$

Therefore, $C_{\mathbf{i}}$ is a cone in $\mathbb{R}^{15}$ given by:

$$
\begin{aligned}
&t_1 \geq 0 \\
&t_2 \geq t_3 \geq 0, \\
&t_4 \geq t_5 \geq t_6 \geq 0, \\
&t_7 \geq t_8 \geq t_9 \geq t_{10} \geq 0, \\
&t_{11} \geq t_{12} \geq t_{13} \geq t_{14} \geq t_{15} \geq 0 \ .
\end{aligned}
$$

It is also interesting to study results when $\mathfrak{g}$ is not Gelfand-Tsetlin.

Example 7.3. When $\mathfrak{g} = \mathfrak{sl}_3$, $w_0 = s_1 s_3 s_2 s_3 s_1 s_2$:

$$
\begin{aligned}
u^{(1)} \quad &: \quad (\underline{1},3,\underline{2},3,1,2) \to t_4 - t_6 \geq 0; \\
& \quad (\underline{1},3,2,3,1,\underline{2}) \to t_3 - t_5 \geq 0; \\
& \quad (1,3,2,3,\underline{1},\underline{2}) \to t_1 \geq 0; \\
u^{(2)} \quad &: \quad (1,3,\underline{2},\underline{3},\underline{1},2) \to t_6 \geq 0; \\
u^{(3)} \quad &: \quad (1,\underline{3},\underline{2},3,1,2) \to t_5 - t_6 \geq 0; \\
& \quad (1,\underline{3},2,3,1,\underline{2}) \to t_3 - t_4 \geq 0; \\
& \quad (1,3,2,\underline{3},1,\underline{2}) \to t_2 \geq 0;
\end{aligned}
$$

Therefore, $C_{\mathbf{i}}$ is a cone in $\mathbb{R}^6$ given by:

$$
t_1 \geq 0, \quad t_2 \geq 0, \quad t_3 \geq \begin{Bmatrix} t_4 \\ t_5 \end{Bmatrix} \geq t_6 \geq 0 \ .
$$

8. Further Study

Further study is still required in order to turn these string cones into their corresponding polytopes and toric degenerations. As weve seen, from a reduced expression for the longest word w_0, we can determine the string cone C_{w_0} via the process above. The next step is finding the graded string cone $\mathcal{C}_{w_0} \subset \Lambda_{\mathbb{R}} \times \mathcal{C}_{w_0}$.

Theorem 8.1. *The graded string cone $\mathcal{C}_{w_0} \subset \Lambda_{\mathbb{R}} \times \mathbb{R}^N$ is the intersection of the preimage $\Lambda_{\mathbb{R}} \times C_{w_0}$ of the string cone $C_{w_0} \subset \mathbb{R}^N$ with the N half-spaces*

$$t_k \leq \langle \lambda, \alpha_{i_k}^\vee \rangle - \sum_{l=k+1}^{N} \langle \alpha_{i_l}, \alpha_{i_k}^\vee \rangle t_l, \quad k = 1, \ldots, N$$

Finally, from the graded string cone, we can compute the polytope $Q_{w_0}(\lambda)$.

Example 8.2. From the example in 7.3, the fan Σ_{w_0} consists of two maximal dimensional cones obtained by splitting $\Lambda_{\mathbb{R}}^+$ into two halves by the hyperplane $\langle \lambda, \alpha_1^\vee \rangle = \langle \lambda, \alpha_3^\vee \rangle$. For a regular weight λ, the polytope $Q_{w_0}(\lambda)$ has 38 or 44 vertices depending on which cone λ lies in.

Example 8.3. Again, as stated in [2], the polytopes $Q_{w_0}(\rho)$ are integral for $G = SL_n$, $n \leq 5$. In type A_3, they have 12 or 13 facets and 38, 40, or 42 vertices. In type A_4, the polytopes have from 20 to 27 facets and from 334 to 425 vertices.

Example 8.4. Alexeev and Brion showed that for a group of type A_n, $n \leq 4$, the polytopes $Q_{w_0}(\omega_i)$ for a fixed fundamental weight ω_i are isomorphic. In particular, the polytopes do not depend on w_0. The polytopes do differ, marked by two distinguished vertices (the highest and lowest weight vectors). We see this by looking at the tangent cones. For, in the A_3 case, the tangent cone at the origin of the polytope $Q_{w_0^{\text{std}}}(\omega_2)$ is non-simplicial. Though, for $w_0 = s_2 s_1 s_2 s_3 s_2 s_1$, the tangent cone of $Q_{w_0}(\omega_2)$ at the origin is simplicial.

Conjecture 8.5. *It is a conjecture by Alexeev and Brion that for G of type A_n and any reduced decomposition w_0, the polytope $Q_{w_0}(\lambda)$ is integral if and only if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all i.*

Finally, even further study is required in looking at G of different types (in particular, $\mathbb{C}^n$). Again, for this paper, we only looked at when G is of type A_n.

9. Explanation of the Code

We now take a look at the code that computes the string cone. There are two files, coneFunctions.sage and stringCone.sage, where the former consists of supporting functions, and stringCone.sage is the main file. The input to stringCone.sage is a number n, representing $\mathfrak{sl}_n$, and an expression for the longest word. The code is currently configured to output the inequalities of the cone corresponding to each $u^{(i)}$ as seen in Examples 7.2 and 7.3.

```
#File: coneFunctions.sage
#Author: Kyle Casey, Champ Davis, Mark Rychnovsky
#Date: July 2015

def find(s, ch):
    return [(i+1,) for i, ltr in enumerate(s) if ltr == ch]
```

```
def find_subword(word,subword):
    arr=[]
    for i in subword:
        arr.append(find(word,i))

    current=arr[0][:]
    temp=[]
    for k in range(1,len(arr)):
        for j in arr[k]:
            for i in current:
                if(i[-1]<j[-1]):
                    temp.append(i+j)
        current=temp[:]
        temp=[]
    return current

def find_subwords(word,subwords):
    arr=[]
    for i in subwords:
        arr.extend(find_subword(word,i))
    return arr

def k(j,t,n):
    if j==0:
        return 0
    elif j>len(t):
        return l+1
    else:
        return t[j-1]

def omega(i):
    om=[]
    for k in range(0,i):
        om.append(1)
    for k in range(i,n):
        om.append(0)
    return om

def alpha(i):
    al=[]
    for k in range(0,i-1):
        al.append(0)
    al.append(1)
    al.append(-1)
    for k in range(i+1,n):
        al.append(0)
    return al

def innerprod(a,b):
    inner=0
    for i in range(0,n):
```

```
        inner=inner+a[i]*b[i]
    return inner

def permute(h,t):
    if h.order()>1:
        perm=[]
        for i in range(0,n):
            perm.append(t[h.dict()[i+1]-1])
        return perm
    else:
        return t

def addlambdainequality(lamb,Z):
    sum=0
    for k in range(0,l):
        for i in range(k+1,l):
            sum=sum-innerprod(alpha(word[i]),omega(word[k]))*t[i+1]
        Z.append(innerprod(lamb,omega(word[k]))*t[0]+sum-t[k+1])
        sum=0
    return Z

def lambdainequality(lamb):
    Z=[]
    sum=0
    for k in range(0,l):
        for i in range(k+1,l):
            sum=sum-innerprod(alpha(word[i]),omega(word[k]))*t[i+1]
        Z.append(innerprod(lamb,omega(word[k]))*t[0]+sum-t[k+1])
        sum=0
    return Z

def Matrixlist(ineq):
    Mlist=[]
    for i in range(0,len(ineq)):
        Mlist.append([])
        for k in range(1,l+1):
            ev=[]
            for s in range(0,l+1):
                ev.append(-kronecker_delta(s,k))
            Mlist[i].append((ineq[i])(ev))
    return Mlist

def Vectorlist(ineq):
    Vlist=[]
    for i in range(0,len(ineq)):
        k=0
        ev=[]
        for s in range(0,l+1):
            ev.append(kronecker_delta(s,k))
        Vlist.append([(ineq[i])(ev)])
    return Vlist
```

```
#File: stringCone.sage
#Author: Kyle Casey, Champ Davis, Mark Rychnovsky
#Date: July 2015

load("coneFunctions.sage")
#Helper variables
n=4
number=0
#Expression for the Longest Word (e.g. Gelfand-Tsetlin)
longestword=[1,2,1,3,2,1]

G=SymmetricGroup(n)  #S_n
w0=G.w0 #getting w0 as type group element so we can get the reduced words
w0_words = G.w0.reduced_words() #the reduced words for w0
number=w0_words.index(longestword) #the index of the longest word for w0
gens=[]
W=[]

for i in range(1, n):
    gens.append(G.simple_reflection(i))

#First we find the W_i
W=[]
for i in gens:
    b=gens[:]
    b.remove(i)
    W.append(G.subgroup(b))

#Next we find the u^i
U=[]
i=1
for j in W:
    elmt = gens[i-1]*w0
    rightCosets=G.cosets(j,side="right")

    for sublist in rightCosets:
        if elmt in sublist:
            coset=sublist
            break
    u=min(coset)
    U.append(u.reduced_words())
    i+=1

word=list(w0_words[number])
l=len(word)


#Next we find the u^(i) as a subword of w_0
subwords=[]
for u in U:
    subwords.append(find_subwords(word,u)) #get permutation group element
        as a list, list of list of tuples of indices of u in word
```

```
#Now we can find the inequalities using the double sum formula
P=PolynomialRing(ZZ,l+1,"t")
t=P.gens()

inequalities=[]
ui=""
for i in range(1,n):
    ui=""
    ui+="u^{("+str(i)+")} & : &"
    for current_subword in subwords[i-1]:
        ul="\underline "
        ui+="("
        for iii in range(0,len(word)):
            for jjj in range (0,len(current_subword)):
                if(iii+1==current_subword[jjj]):
                    ui+=ul
            ui+=str(word[iii])+","
        ui+=") \\to "

        sum=0
        for j in range(0,len(current_subword)+1):
            perm_elmt=PermutationGroupElement(()) #product of s_i_j's
            for s in range(1,j+1):
                perm_elmt=perm_elmt*G.simple_reflection(word[k(s,current_subword,n)-1])
            for kk in
             range(k(j,current_subword,n)+1,k(j+1,current_subword,n)):
                sum=sum+innerprod(permute(perm_elmt,alpha(word[kk-1])),omega(i))*t[kk]
        ui+=str(sum)+" \\geq 0; \\\\ [.05in] \n& & "
        inequalities.append(sum)
    ui+="[.1in] \n"
    print ui
inequalities
```

A sample output for $n = 4$, longestword $= [1, 2, 1, 3, 2, 1]$ (pasted into latex):

$$\begin{aligned}
u^{(1)} &: & (\underline{1}, \underline{2}, 1, 3, 2, 1,) \to t4 - t5 \geq 0;\\
& & (\underline{1}, 2, 1, 3, \underline{2}, 1,) \to t2 - t3 \geq 0;\\
& & (1, 2, \underline{1}, 3, \underline{2}, 1,) \to t1 \geq 0;\\
u^{(2)} &: & (1, \underline{2}, 1, \underline{3}, 2, \underline{1},) \to t3 \geq 0;\\
& & (1, \underline{2}, \underline{1}, \underline{3}, 2, 1,) \to t5 - t6 \geq 0;\\
u^{(3)} &: & (1, 2, 1, \underline{3}, \underline{2}, 1,) \to t6 \geq 0;
\end{aligned}$$

And the list inequalities stores the expressions: [t4 - t5, t2 - t3, t1, t3, t5 - t6, t6]. This is precisely one of the examples seen above.

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References

[1] A. Berenstein and A. Zelevinsky. Tensor Product Multiplicities, Canonical Bases and Totally Positive Varieties. arXiv:math/9912012 [math.RT]
[2] V. Alexeev and M. Brion, Toric Degenerations of Spherical Varieties
[3] D. Cox, J. Little, and H. Schenck, Toric Varieties, 2009.
[4] K. Erdmann and M.J. Wildon, Intro to Lie Algebras, Springer-Verlag, London, 2006.
[5] B. Grunbaum, Convex Polytopes, Springer-Verlag, New York, 2003.
[6] J. Lee, Introduction to Smooth Manifolds, Springer-Verlag, New York, 2003.
[7] J-P. Serre, Linear Representations of Finite Groups, Springer-Verlag, New Y ork,1977.
[8] W. Fulton, J. Harris, Representation Theory: A First Course, Springer-Verlag, New York, 2004.