Partisan and Bipartisan Gerrymandering*

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Abstract

This paper analyzes the optimal partisan and bipartisan gerrymandering policies in a model with electoral competitions in policy positions and transfer promises. With complete freedom in redistricting, partisan gerrymandering policy generates the most one-sidedly biased district profile, while bipartisan gerrymandering generates the most polarized district profile. In contrast, with limited freedom in gerrymandering, both partisan and bipartisan gerrymandering tend to prescribe the same policy. Friedman and Holden (2009) find no significant empirical difference between bipartisan and partisan gerrymandering in explaining incumbent reelection rates. Our result suggests that gerrymanderers may not be as free in redistricting as popularly thought.

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1 Introduction

It is widely agreed that election competitiveness has decreased significantly in recent decades. For example, the reelection rate has increased from 91.82% in 1950 to 98.25% in 2004 (Friedman and Holden 2009). Also, 74 House seats were won by a margin less than 55% in 2000, but this number decreased to 24 in 2004 (Fiorina et al. 2011). A popular explanation for this in US politics is gerrymandering. Thanks to the advance of computing technology and comprehensive datasets like TIGER/Line Shapefiles, gerrymandering has become extremely sophisticated today.[1] Famous examples, including the 4th congressional district in Illinois and the 5th district of Florida, among others, have been noticed by the public. It is argued that the gerrymandering biased toward *incumbents*, i.e., *bipartisan gerrymandering*, has an effect on the decrease in competitiveness. Fiorina et al. (2011) state that "Many (not all) observers believe that the redistricting that occurred in 2001-2002 had a good bit to do with this more recent decline in competitive seats—the party behaved conservatively, concentrating on protecting their seats rather than attempting to capture those of the opposition." (see Fiorina et al. pp. 214-215).

During the same period, the US Congress has become quite polarized. The distribution of the House representatives' political positions was more concentrated at the center of political spectrum with considerable overlap between Republican and Democratic representatives' positions in the 1960s, while it became sharply twin-peaked without overlap in the 2000s.[2] Simultaneously, Fiorina et al. (2011) argue that US voters have not polarized so much during

[1] See Friedman and Holden (2009) and the references therein for details.

[2] It is now standard to use a one-dimensional scaling score (DW-Nominate procedure on economic liberal-conservative, Poole and Rosenthal, 1997) to measure representatives' political positions.

the same time period. These conflicting observations generate an obvious puzzle: How could the Congress polarize if voters didn't? They argue that this decrease in competitiveness from gerrymandering is one of the driving forces behind the recent political polarization in Congress (see also Gilroux, 2001).[3]

However, recent empirical studies show that the effects of gerrymandering may be insignificant. Friedman and Holden (2009) investigate whether or not the incumbent reelection rate depends on gerrymandering being partisan or bipartisan.[4] In partisan gerrymandering cases, the majority party may try to oust the opposing party's incumbents, and this may be reducing the incumbent reelection rate. In contrast, in bipartisan gerrymandering cases, both parties try to secure their incumbents' reelections, maximizing safe seats. Fiorina et al. (2011) illustrate how bipartisan gerrymandering can create noncompetitive districts under complete freedom in gerrymandering by a simple example (Fiorina et al. pp. 214-217). Interestingly, Friedman and Holden (2009) did not find significant differences between bipartisan and partisan gerrymandering on the effect on the incumbent reelection rate. As they mention, this result suggests that partisan gerrymandering may not be as effective as popularly thought. McCarty et al. (2006, 2009) document that the political polarization of the House of Representatives has increased in recent decades, using data

[3]Another possible explanation is that voters sorted out into Republican and Democratic parties by their political positions during the period, and that the parties' political positions were polarized in party members' preference aggregation. Levendusky (2009) suggests that party elites' polarization led voter sorting, although it is controversial how much mass polarization actually occurred by voter sorting.

[4]Redistricting in the US is usually conducted by state legislatures (partisan gerrymandering), but in Arizona, Hawaii, Idaho, Montana, New Jersey, and Washington it is conducted by bipartisan redistricting commissions. In California and Iowa, redistricting lines are drawn by nonpartisan redistricting committees.

on roll call votes, but they find only a minimal relation between polarization and gerrymandering.[5] Regarding the recent decline in the competitiveness of districts, Friedman and Holden (2009) investigate whether or not gerrymandering caused the rising incumbent reelection rate by using data up to 2004, finding evidence of the opposite effect, all else equal.[6]

Traditionally, the literature often discusses two tactics in partisan gerrymandering: one is to concentrate or "*pack*" those who support the opponent in losing districts, and the other is to evenly distribute or "*crack*" supporters in winning districts. Packing serves to waste the opponent party's strong supporters' votes, while cracking utilizes the votes of party supporters as effectively as possible. Owen and Grofman (1988) show that a pack-and-crack policy is optimal when a partisan gerrymanderer has limited freedom in redistricting.[7] In contrast, Friedman and Holden (2008) argue that advances in computing technologies and availability of big datasets allow gerrymanderers higher degrees of freedom in redistricting, and they obtained a very different optimal policy from *pack-and-crack*: the *slice-and-mix* policy, in which districts are created by first mixing the strongest opposition group of voters and the strongest supporter group, then mixing the second strongest opposition and supporting groups, and so on. This policy wastes opposition groups' votes, generating the most one-sided allocation from the most extreme to the most moderate districts.

[5]Krasa and Polborn (2015) argue that their answer may be incomplete if the political positions of district candidates are mutually interdependent.

[6]As an early evidence, Ferejohn (1977) finds little support for gerrymandering being the cause of declines in competitiveness of congressional districts from the mid-1960s to the 1980s.

[7]Owen and Grofman (1988) assume that the average of district median voter's position must stay constant in redistricting (a constant average constraint).

In this paper, we consider a two-party political competition model in which policy-motivated party leaders compete with their candidates' (one-dimentional) political positions and pork-barrel promises in each electoral district. We assume that there exist minimum units of indivisible localities with the same population, and that a gerrymanderer partitions the set of localities freely to create electoral districts. Each locality has a voter distribution, and we say that the gerrymanderer has more freedom in redistricting if the voter distribution is concentrated on a point in the political spectrum. We investigate the optimal gerrymandering policies within the same political competition model. With pork-barrel politics, the party leader understands that pork-barrel policies in competitive districts are costly, and therefore she has strong gerrymandering incentives to collect their supporters in the winning districts in order to avoid large pork-barrel promises.

In particular, we compare the optimal policies under partisan and bipartisan gerrymandering when the gerrymanderer(s) face different levels of freedom in redistricting. This has never been done in the literature. We show that the slice-and-mix policy is optimal for the party leaders in charge of gerrymandering when they can redistrict with complete freedom, but the resulting outcomes in partisan and bipartisan gerrymandering are very different: bipartisan gerrymandering results in most polarized electoral districts without leaving moderate and competitive ones, while partisan gerrymandering results in an one-sided allocation, leaving some competitive districts. In contrast, we obtain essentially the same optimal policy when they face the constraint in redistricting imposed by Owen and Grofman (1988) and voters and party leaders are more policy-sensitive (roughly speaking): a consecutive partition of localities stratified by limited freedom in redistricting, since each locality is composed of a spectrum of voters (slice-them-all). Given Friedman and

Holden's (2009) empirical finding on insignificant differences on the effect of district competitiveness between bipartisan and partisan gerrymandering, the results may suggest that despite recent advances in computing technologies and availability of comprehensive election data, gerrymanderers' freedom in redistricting may still be rather limited.

An additional finding of this paper for partisan gerrymandering case is that it matters whether a party leader in charge of redistricting is policy-motivated or not. Without policy-motivation, "pack-and-crack" is optimal when the freedom in redistricting is limited as Owen and Grofman (1988) has shown. In contrast, whith policy-motivation, "slice-them-all" tends to be optimal especially if leaders and voters are more policy-sensitive.

The rest of the paper is organized as follows. Section 2 discusses related literature. In Section 3, we start with analyzing political-position and pork-barrel competition and characterize the party leader's payoff from each winning district by the district median voter's position (Lemmas 2, 3, and 4). In Section 4, we investigate the optimal gerrymandering strategy when the party leader has complete freedom as in Friedman and Holden (2008), and show that their "*slice-and-mix*" is also an optimal strategy in partisan gerrymandering cases, generating the most one-sided allocation (Proposition 1). In contrast, in bipartisan gerrymandering cases, we obtain a rule that first partitions voters into two consecutive sets in their political positions, and both parties apply "*slice-and-mix*" to their groups. This policy generates the most polarized allocation (Proposition 2). In Section 5, we proceed to cases where the gerrymanderer's freedom is limited by indivisibility of localities. We also assume that each district has normally distributed voters to justify the constant-average constraint imposed by Owen and Grofman (1988). We show that the gerrymanderer optimally packs the opponent's supporters and slices her own supporters in order from the strongest to moderate when voters and party leaders are policy-sensitive, in the sense that their cost functions have positive third derivatives (Proposition 3). One of these optimal strategies is the one that slices the entire localities in order: "*slice-them-all.*"With bipartisan gerrymandering, the result is again "*slice-them-all*" under the same conditions, since both parties want to slice their supporters and to pack their opponents' (Proposition 4). The two parties' preferences totally coincide with each other under the conditions. Although it is hard to generalize it, an example shows that the positive third derivative conditions may not be essential to this slice-them-all result (Example 1). Section 6 concludes the study. All proofs are collected in Appendix A.

2 Related Literature

Our paper is related to three branches of literature. The first one is partisan and bipartisan gerrymandering literature. Introducing uncertainty in each district's median voter's position, Owen and Grofman (1988) consider the situation where a partisan gerrymanderer redesigns districts in order to maximize the expected number of seats. They assume that the uncertainty in the median voter's political position is local and is independent across districts when the objective is expected number of seats. Assuming that the average of the positions of district median voters must stay the same after redistricting (a *constant average constraint*), they show that the optimal strategy is "packing" the opponents in losing districts, and "cracking" the rest of voters evenly across the winning districts *with substantial margins*, so that the party can win districts even in the cases of negative shocks under both cases.[8,9] Friedman and Holden (2008), on the other hand, assume that a partisan gerrymanderer has *full freedom* in allocating population over a finite number of districts, and that she maximizes the expected number of seats when there is only valence uncertainty in median voters' utilities (thus, there is no uncertainty in the median voter's political position). In this idealized situation, they find that the optimal strategy is "*slice-and-mix*" which is similar to our optimal strategy under a different model. Thus, theoretically, the levels of freedom in gerrymandering can affect the optimal policy.

In bipartisan gerrymandering, Gul and Pesendorfer (2010) extend Owen and Grofman (1988) by introducing a continuum of districts, and voters' party affiliations. Here, bipartisan gerrymandering means that the two parties own their territories and redistrict exclusively within each territory. They assume that each party leader can redistrict her party's territory (the districts with her party's seats) independently, maximizing the probability of winning the majority of seats.[10] They show that the optimal policy is again a version of "pack-and-crack." However, these papers do not compare the optimal partisan and bipartisan gerrymandering policies. They also do not model spatial

[8] They also consider the case where the partisan gerrymanderer maximizes the probability to win a working majority of seats for her party by assuming that the uncertainty is global. They again get pack-and-crack policy as the optimal policy.

[9] The original "cracking" tactics create the maximum number of winning districts *with the smallest margins*. In the traditional literature, some argue that gerrymandering will increase political competition by this reason. In this paper, we use "cracking" tactics in the sense of Owen and Grofman (1988).

[10] They consider two feasibility constraints. The first is the constant mean of median voters' positions which is the same as the one in Owen and Grofman (1988). The second one is that the status quo needs to be a *mean-preserved spread* of a feasible redistricting plan.

competition in policy positions, and the elected representatives' positions are implicitly assumed to be the district median voters' positions (Downsian competition).

The second branch is the pork-barrel literature. Our model is most closely related to Lindbeck and Weibull (1987) and Dixit and Londregan (1996). The former introduces a two-party competition model in which (extreme) parties use pork-barrel policies to attract agents with heterogeneous policy preferences. The latter generalizes Lindbeck and Weibull (1987) to allow that parties have different abilities in practicing pork-barrel policies, and this difference determines whether the pork-barrel policy's target is swing voters or loyal supporters. Our model is different from theirs in that we introduce parties' platform decisions besides pork-barrel politics, and party leaders choose these two policies simultaneously.[11] Moreover, the political competition result is deterministic in our model, which is different from the setup with uncertainty in the literature. A similar political competition model has been used in the recent vote-buying literature, e.g., Dekel, Jackson, and Wolinsky (2008).

The third branch is normative gerrymandering literature. The focus is on how gerrymandering affects the relation between seats and the vote shares won by a party, the so-called "*seat-vote curve.*" Coate and Knight (2007) identify the social welfare optimal seat-vote curve and then the conditions under which the optimal curve can be implemented by a districting plan. With fixed and extreme parties' policy positions, they find that the optimal seat-vote is

[11] Dixit and Londregan (1998) propose a pork-barrel model with strategic ideological policy decision based on their previous work. However, the ideology policy in their paper is the equality-efficiency concern engendered by parties' pork-barrel strategies. Therefore, the ideology decision in their work is a consequence of pork-barrel politics, instead of an independent policy dimension.

biased toward the party with larger partisan population. However, Bracco (2013) shows that, when parties strategically choose their policy position, the direction of seat-vote curve bias should be the opposite. Besley and Preston (2007) construct a model similar to Coate and Knight (2007) and show the relation between the bias of seat-vote curve and parties' policy choices. They further empirically test the theory and the result shows that reducing the electoral bias can make parties strategy more moderate.

3 The Model

We consider a two-party (L and R) multidistrict model. There are many (possibly infinite) localities in the state, each of which is considered the minimal unit in redistricting (a locality cannot be divided into smaller groups in redistricting, e.g., a street block). We assume that there are $\mathcal{L}$ discrete localities, each of which has population $\frac{1}{\mathcal{L}}$. The state has K districts, and $\mathcal{L}$ is a multiple of K. To comply with the equal population requirement, the party in power needs to create those K districts by combining $\frac{\mathcal{L}}{K} = n$ localities in each one. Locality $\ell = 1, ..., \mathcal{L}$ has a voter distribution function $F_\ell : (-\infty, \infty) \to [0, 1]$, where $(-\infty, \infty)$ is the one-dimensional ideology (or political) spectrum and $F_\ell(\theta)$ is non-decreasing with $F_\ell(-\infty) = 0$ and $F_\ell(\infty) = 1$. Ideology $\theta < 0$ is regarded left, and $\theta > 0$ is right. With a slight abuse of notation, we denote the set of localities also by $\mathcal{L} \equiv \{1, ..., \mathcal{L}\}$. A *redistricting plan* $\pi = \{D^1, ..., D^K\}$ with $|D^k| = n$ for all $k = 1, ..., K$, is a partition of $\mathcal{L}$.[12] The gerrymandering party's leader chooses the optimal district partition π from the set of all possi-

[12] A partition π of $\mathcal{L}$ is a collection of subsets of $\mathcal{L}$, $\{D^1, ..., D^K\}$, such that $\cup_{k=1}^K D^k = \mathcal{L}$ and $D^k \cap D^{k'} = \emptyset$ for any distinct pair k and k'.

ble partitions Π.[13] In each district k, the voter distribution function F^k is an average of distribution functions of n localities: $F^k(\theta) = \frac{1}{n}\sum_{\ell \in D^k} F_\ell(\theta)$. District k's median voter is denoted by $x^k = x^k(D^k) \in (-\infty, \infty)$ with $F^k(x^k) = \frac{1}{2}$. We assume the uniqueness of x^k in each districting plan. Although x^k is solely determined by D^k, we can write $x^k = x^k(D^k(\pi)) = x^k(\pi)$ for all $k = 1, ..., K$ with a slight abuse of notation. Finally, let $F(\theta) = \frac{1}{\mathcal{L}}\sum_\ell F_\ell(\theta)$ be the state population distribution, and θ_m, *the state median voter*, be determined by $F(\theta_m) = \frac{1}{2}$.

We will consider two cases later: one case is with complete freedom in redistricting as in Friedman and Holden (2008), and the other is with limited ability in the line of Owen and Grofman (1988). Throughout the paper, we order localities by the political positions of the median voter.

We also introduce uncertainty in the position of median voter after redistricting is done. At each election time, the economic and social state at that moment and which party is in power affect voters' political positions in the same direction: i.e., the voter distribution is shifted by common shocks. Formally, let y be a realization of the uncertain shock term. The median voter of the actual election in district k is denoted by $\hat{x}^k = x^k + y$.[14] We assume that

[13] In reality, there are many restrictions on what can be done in a redistricting plan. For example, a district is required to be connected geographically. Despite the complication involved, our analysis can still be extended to the case with geographic restrictions by introducing the set of admissible partitions $\Pi^A \subseteq \Pi$ (see Puppe and Tasnadi, 2009)

[14] The results are not affected even if we assume that each district k has district-specific shocks drawn from G^k, since the party leader's payoff function is additive across districts (see below). To be specific, our results hold for the general case in which one consider location specific shocks $(y^1, ..., y^k)$ with p.d.f. $g(y^1, ..., y^k)$ and the realized district k median voter's position being $\hat{x}^k = x^k + y^k$. Our benchmark model describes the case that y^ks are prefect correlated. Another possible case is y^ks being i.i.d. and $g(y^1, ..., y^k) = g(y^1)g(y^2)...g(y^k)$.

y follows a probabilistic distribution function $G : [-\bar{y}, \bar{y}] \to [0, 1]$, where $\bar{y} > 0$ is the largest value of relative economic shock and $G(0) = \frac{1}{2}$. We assume that electoral competition occurs after y is realized: the resulting median voter's position after the shock realization is $\hat{x}^k$.

We model pork-barrel elections in a similar manner with Dixit and Londregan (1996). A type θ voter in district k evaluates party j according to the utility function with two arguments: one is the policy position of the candidate representing the corresponding party, $\beta_j^k \in \mathbb{R}$, and the other is the party's *pork-barrel* transfer $t_j^k \in \mathbb{R}_+$. We interpret this pork-barrel transfer as a *promise* of local public good provision (measured by the amount of monetary spending) in the case where the party's candidate is elected. Formally, a voter θ in district k evaluates party j's offer by

$$U_\theta(j) = t_j^k - c(|\theta - \beta_j^k|) \tag{1}$$

where $c(d) \geq 0$ is the *ideology cost function*, which is increasing in the distance between a candidate's position and her own position. We assume that $c(\cdot)$ is continuously differentiable, and satisfies $c(0) = 0$, $c'(0) = 0$, and $c'(d) > 0$ and $c''(d) > 0$ for all $d > 0$ (strictly increasing and strictly convex).

Therefore, voter θ votes for party L if and only if

$$U_\theta(L) - U_\theta(R) = [c(|\theta - \beta_R^k|) - c(|\theta - \beta_L^k|)] + t_L^k - t_R^k > 0 \tag{2}$$

Since the (after shock) median voter's type in district k is $\hat{x}^k = x^k + y$, given β_L^k, β_R^k, t_L^k and t_R^k, L wins in district k if and only if

$$U_{\hat{x}^k}(L) - U_{\hat{x}^k}(R) = [c(|\hat{x}^k - \beta_R^k|) - c(|\hat{x}^k - \beta_L^k|)] + t_L^k - t_R^k > 0 \tag{3}$$

Each party leader in the state (composed of these K districts) cares about (i) the influence or status within her party based on the number of winning districts in her state, (ii) the candidate's policy position in each district, and (iii) the district-specific pork-barrel spending. We assume that the party leader prefers to win a district with a candidate's position closer to her own ideal ideological position and a *less* pork-barrel promise. The former is regarded as the "policy-motivation" in the literature. By formulating the latter, we consider a situation where the leader bears some costs when implementing the promised local public spending, as in the example of the bargaining efforts needed to push for federal funding. To simplify the analysis, we assume that the negative utility by pork-barrel is measured by the amount of money promised. We denote the ideal political positions of the leaders of party L and R by θ_L and θ_R, respectively, with $\theta_L < \theta_R$. Without loss of generality, we set $\theta_L = -\theta_R$, but we will stick to notations θ_L and θ_R until the gerrymandering analysis starts to help the reader comprehend the model more easily. Formally, by winning in district k, party j's leader gets utility

$$V_j^k = Q_j - t_j^k - C(\left|\beta_j^k - \theta_j\right|),$$

where $Q_j > 0$ is the fixed payoff that party j's leader obtains from each winning district, and $C(d)$ is a *party leader's ideology cost function* with $C(0) = 0$, $C'(0) = 0$, $C'(d) > 0$ and $C''(d) > 0$ (strictly increasing and strictly convex). This cost function C can be different from the voter's cost function c. If the party leader loses in district k, she gets zero utility from the district. The national party elites are ultimately interested in the number of seats their party gets, so the number of seats a state party leader wins is important in recognizing her contribution to the national party. Also, since we are considering a state's gerrymandering problem, it is reasonable to assume that the benefit from winning a district does not depend on which district is won.

We introduce a *tie-breaking rule* in each district based on the relative levels of the state party leaders' utilities V_L^k and V_R^k. We assume that if two parties' offers are tied for the median voter $\hat{x}^k$ ($U_{\hat{x}^k}(L) = U_{\hat{x}^k}(R)$) while one party's leader gets strictly higher (indirect) utility than the other's, the median voter will vote for that party. That is,

Assumption 1. ***(Tie-Breaking)*** Given two parties' offers are such that $U_{\hat{x}^k}(L) = U_{\hat{x}^k}(R)$, L (R) wins if $V_L^k > V_R^k$ ($V_L^k < V_R^k$).

This assumption is justified by the fact that the higher utility is equivalent to the higher ability to provide a better offer to the median voter. In particular, consider the case in which two parties are tied and, say, $V_L^k > V_R^k = 0$, and party L has the ability to provide $\epsilon > 0$ more pork-barrel promise. Therefore, we break the tie by assuming the median voter prefers L, which is a standard assumption.

Our second assumption is a simple sufficient condition that assures interior solutions for both parties.

Assumption 2. ***(Relatively Strong Office Motivation)*** For all feasible $\hat{x}^k$, $Q_j \geq \min_{\beta}\{C(|\theta_j - \beta|) + c(|\beta - \hat{x}^k|)\}$ holds for $j = L, R$.

Notice that if the party leader gets 0 utility, she must offer pork-barrel promise equal to $Q_j - C(|\theta_j - \beta|)$. Therefore, the median voter get utility $U_{\hat{x}^k} = Q_j - C(|\theta_j - \beta|) - c(|\beta - \hat{x}^k|)$ if party j wins. This assumption means that the payoff from winning a district, Q_j, is large enough so that for any $\hat{x}^k$, both parties can offer the median voter positive utility, which is a sufficient condition for the candidate selection problem to have an interior solution. Note that the set of feasible $\hat{x}^k$ is not the entire real line. The model only allows bounded finite median voters' positions and $\bar{y}$ being also finite. Therefore, must exist a Q_j to satisfy this assumption. Moreover, the implication of this assumption is that it guarantees that in equilibrium both parties promise positive pork-barrel. We will see this more clearly in the next section.

The state redistricting may be decided by one or both parties. It is straightforward that, in the first case, one party leader chooses π. In the later one, we assume that K_L districts belong to L and the remaining $K_R = K - K_L$ districts belong to R. Without loss of generality, we assume L choosing $\{D^1, ..., D^{K_L}\}$ and R choosing $\{D^{K_L+1}, ..., D^K\}$. We will discuss the bipartisan case in details later.

The timing of the game is as follows:[15]

1. One party, say L, or both parties jointly choose a redistricting plan $\pi = (D^k)_{k=1}^K$, and thus a median voter vector $(x^1, ..., x^k, ..., x^K)$.

2. The common shock $y \in [-\bar{y}, \bar{y}]$ realizes.

3. Given the districting plan in stage 1 and the realized median voter $\hat{x}^k = x^k + y$ in stage 2, party leaders L and R simultaneously choose local policy positions and pork-barrel promises $(\beta_L^k, t_L^k)_{k=1}^K$ and $(\beta_R^k, t_R^k)_{k=1}^K$, respectively.

4. All voters vote sincerely (with our tie-breaking rule). The winning party is committed to its policy position and its pork-barrel promise in each district $k = 1, ..., K$. All payoffs are realized.

We will employ **weakly undominated subgame perfect Nash equilibrium** as the solution concept. We require that in stage 3, party leaders

[15] We can separate stage 3 into two substages: policy position choices followed by pork-barrel promises. If we do so, the loser of a district k will get zero payoff in every subgame, so it becomes indifferent among policy positions. Thus, we need equilibrium refinement to predict the same allocation. By assuming that the loser party chooses the policy position that minimizes the opponent party leader's payoff, we can obtain exactly the same allocation in SPNE.

play weakly undominated strategies so that the losing party leader does not make cheap promises to the district median voters.[16] We will call a weakly undominated subgame perfect Nash equilibrium simply an equilibrium.

3.1 Stage 3: Electoral Competition with Pork-Barrel Politics

We solve the equilibria of the game by backward induction. We start with stage 3, knowing that voters vote sincerely in stage 4. Notice that the key player is the median voter in the voting stage. Thus, when the leader of party L makes her policy decisions in district k, she at least needs to match R's offer in terms of median voter's utility in order to win. Without loss of generality, we consider the case that party L wins with the tie-breaking rule in Assumption 1. In this case, the leader of party L tries to offer the same utility to the median voter $\hat{x}^k$. Formally, the party leader's problem is described by

$$\begin{gathered}\max_{\beta_L^k, t_L^k}\{Q_L - t_L^k - C(\left|\theta_L - \beta_L^k\right|)\} \\ \text{subject to } t_L^k - c(|\hat{x}^k - \beta_L^k|) \geq \bar{U}_R^k,\ t_L^k \geq 0, \text{ and} \\ Q_L - t_L^k - C(\left|\theta_L - \beta_L^k\right|) \geq 0,\end{gathered} \tag{4}$$

[16] This game is the first price auction under complete information. In general, there is a continuum of pure strategy equilibria. The losing party does not suffer from cheap promise, since she gets zero utility in losing districts anyway. The winning party needs to match the offer as long as she can get a positive payoff by doing so. Demanding that players play weakly undominated strategies, we can eliminate these unreasonable equilibria. Another justification for this is to require mixed strategy equilibrium. There is a unique mixed strategy equilibrium in which the winning party plays a pure strategy while the losing party plays a mixed strategy equilibrium. The outcome of this mixed strategy equilibrium coincides with the weakly undominated Nash equilibrium in pure strategies.

where $\bar{U}_R^k$ is the median voter's utility level from R's offer. Notice that $t_L^k \geq 0$ and $Q_L - t_L^k - C(\left|\theta_L - \beta_L^k\right|) \geq 0$ may or may not be binding while $t_L^k - c(|\hat{x}^k - \beta_L^k|) \geq \bar{U}_R^k$ must be binding. The solution for this maximization problem is straightforward. *Define* $\hat{\beta}_j(\hat{x}^k, \theta_j)$ *by the following equation*

$$c'(|\hat{x}^k - \hat{\beta}_j(\hat{x}^k, \theta_j)|) = C'(\left|\theta_j - \hat{\beta}_j(\hat{x}^k, \theta_j)\right|). \tag{5}$$

Notice that (5) is simply the first-order condition of optimization problem (4) after substituting $t_L^k = c(|\hat{x}^k - \beta_L^k|) + \bar{U}_R^k$ into the objective function. Also, the optimal policy $\beta_L^{k^*} = \hat{\beta}_L(\hat{x}^k, \theta_L)$ when $-c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|) \leq \bar{U}_R^k$. That is, it is not enough for the winning party to win just by using the policy platform. In this case, it is clear that the optimal pork barrel promise is

$$t_L^{k^*}(\bar{U}_R^k) = \bar{U}_R^k + c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|) \tag{6}$$

Although it seems unclear at first that $-c(|\hat{x}^k - \hat{\beta}_L(\hat{x}^k, \theta_L)|) \leq \bar{U}_R^k$ holds or not, it turns out this condition always holds. This is because a similar optimization problem applies for the losing party and Assumption 2.

It is obvious that the winning party L's pork-barrel promise is related to what the losing party R proposes in equilibrium. In the proof of the following lemma, we can show that in equilibrium, R proposes the policy pair $(\beta_R^{k^*}, t_R^{k^*})$, which is the solution of the following problem

$$\max_{\beta_R^k, t_R^k} U_{\hat{x}^k}(R) = t_R^k - c(|\hat{x}^k - \beta_R^k|) \tag{7}$$
$$\text{subject to } t_R^k \geq 0 \text{ and } Q_R - t_R^k - C(\left|\theta_R - \beta_R^k\right|) \geq 0$$

That is, the losing party leader offers a policy position and pork-barrel promise that leave herself zero surplus in equilibrium. The intuition is straightforward. If the losing party does not offer the median voter the best one, then since the winning party will provide the median voter the same utility level, the losing one can always offer the median voter something better than her original offer and win the district. This cannot happen in equilibrium. Therefore, for the losing party R, the equilibrium strategy is $\beta_R^{k^*} = \hat{\beta}(\hat{x}^k, \theta_R)$ and $t_R^{k^*} = Q_R - C(|\theta_R - \beta_R^{k^*}|)$. We have the following result.

Lemma 1. *In stage 3 equilibrium, we have:*

1. *For the losing party j, the equilibrium strategy is $\beta_j^{k^*} = \hat{\beta}_j(\hat{x}^k, \theta_j)$ which lies in the interval $(\hat{x}^k, \theta_j)$ (or $(\theta_j, \hat{x}^k)$) and $t_j^{k^*} = Q_j - C(|\theta_j - \beta_j^{k^*}|)$*

2. *For the winning party i, the equilibrium strategy is $\beta_i^{k^*} = \hat{\beta}_i(\hat{x}^k, \theta_i)$, which lies in the interval $(\hat{x}^k, \theta_i)$ (or $(\theta_i, \hat{x}^k)$), and $t_i^{k^*} = Q_j - C(|\theta_j - \beta_j^{k^*}|) - c(|\hat{x}^k - \beta_j^{k^*}|) + c(|\hat{x}^k - \beta_i^{k^*}|)$.*

One thing left to decide is which party should be the winning party. We have the following results in stages 3 and 4.

Lemma 2. *In stage 3 equilibrium, we have*

1. *Party i's payoff in the kth district is*

$$\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j) \equiv \max\left\{(Q_i - Q_j) - \left(\mathcal{C}(|\theta_i - \hat{x}^k|) - \mathcal{C}(|\theta_j - \hat{x}^k|)\right), 0\right\}, \tag{8}$$

where $C(|\theta_i - \hat{x}^k|) \equiv C\left(\left|\theta_i - \hat{\beta}_i(\hat{x}^k, \theta_i)\right|\right) + c(|\hat{x}^k - \hat{\beta}_j(\hat{x}^k, \theta_j)|)$, and party i wins if and only if

$$Q_i - Q_j > \mathcal{C}(|\theta_i - \hat{x}^k|) - \mathcal{C}(|\theta_j - \hat{x}^k|).$$

2. *Irrespective of $\hat{x}^k \gtrless \theta_i$, we have $\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} = \frac{c_i''}{C_i'' + c_i''}$, where $c_i'' = c''(|\hat{x}^k - \beta_i^{k^*}(\hat{x}^k, \theta_i)|)$ and $C_i'' = C''\left(\left|\theta_i - \hat{\beta}_i(\hat{x}^k, \theta_i)\right|\right)$.*

Note that Lemma 2-1 implies that if $Q_i = Q_j$, then j wins if and only if

$$\left|\theta_i - \hat{x}^k\right| < \left|\theta_j - \hat{x}^k\right|.$$

Now, we turn to the properties of $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j)$. Recalling that we assume $\theta_L = -\theta_R$ without loss of generality, we can prove the following properties.

Lemma 3. *The following properties are satisfied for $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j)$:*

1. *The realized winning payoff for party L (R), $\tilde{V}_L^k$ ($\tilde{V}_R^k$) is decreasing (increasing) in $\hat{x}^k$ when $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j) > 0$.*
2. *The realized winning payoff for party i, $\tilde{V}_i^k$, is strictly convex in $\hat{x}^k$ for $\hat{x}^k$ when $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j) > 0$, if $C'''(\cdot) > 0$ and $c'''(\cdot) > 0$, and $Q_L = Q_R$.*

Using $\tilde{V}_i^k(\hat{x}^k, \theta_i, \theta_j)$, when party i wins in district k, the expected payoff from district k for party leader i is written as:

$$E\tilde{V}_i^k(x^k, \theta_i, \theta_j) = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_i^k(x^k + y, \theta_i, \theta_j) g(y) dy$$

Note that due to the additive separability of the payoff function, party leader i's expected payoff under partition π (district median profile $\left(x^k(\pi)\right)_{k=1}^K$) is written as

$$\begin{aligned} E\tilde{V}_i(\pi, \theta_i, \theta_j) &\equiv \int_{-\bar{y}}^{\bar{y}} \sum_{k=1}^K \tilde{V}_i^k(x^k(\pi) + y, \theta_i, \theta_j) g(y) dy \\ &= \sum_{k=1}^K \int_{-\bar{y}}^{\bar{y}} \tilde{V}_i^k(x^k(\pi) + y, \theta_i, \theta_j) g(y) dy \\ &= \sum_{k=1}^K E\tilde{V}_i^k(x^k(\pi), \theta_i, \theta_j) \end{aligned}$$

The next lemma is in preparation of the Stage 1 analysis.

Lemma 4. *The following properties are satisfied for* $E\tilde{V}_i^k(x^k, \theta_i, \theta_j)$ *and* $E\tilde{V}_i(x, \theta_i, \theta_j)$*:*

1. *The expected winning payoff for party* L *(*R*) in district* k*,* $E\tilde{V}_L^k$ *(*$E\tilde{V}_R^k$*) is decreasing (increasing) in* x^k *unless the winning probability for party* L *(*R*) in district is zero.*

2. *The expected winning payoff for party* i *in district* k*,* $E\tilde{V}_i^k$ *is strictly convex in* x^k*, if* $C'''(\cdot) > 0$ *and* $c'''(\cdot) > 0$*, and* $Q_L = Q_R$ *unless the winning probability for party* i *in district is zero.*

3. *The expected winning payoff for party* L *(*R*),* $E\tilde{V}_L$ *(*$E\tilde{V}_R$*) is decreasing (increasing) in* x^k *if party* L *(*R*) has a nonzero chance to win in any district.*

4. *The expected winning payoff for party* i*,* $E\tilde{V}_i$ *is strictly convex, if* $C'''(\cdot) > 0$ *and* $c'''(\cdot) > 0$*, and* $Q_L = Q_R$*, if party* i *has a nonzero chance to win in any district.*

We are now ready to discuss the setup of partisan and bipartisan gerrymandering problems.

3.2 The Partisan Gerrymandering Problem

Without loss of generality, we formalize *the partisan gerrymandering party leader's optimization problem* as the case where $K_L = K$ and L is in charge of redistricting. Lemma 2 shows that $x^k = x^k(\pi)$ is the sufficient statistic to determine the outcome of the kth district. Notice that the indirect utility of L, $\tilde{V}_L^k(\hat{x}^k, \theta_L, \theta_R)$, is relevant only when party L wins in district k. The choice of $\pi = \left(D^1, ..., D^K\right)$ affects the party leader L's payoff $E\tilde{V}_L$ through $\left(x^1(\pi), ..., x^K(\pi)\right)$ represented by its indirect utility $\tilde{V}_L^k(x^k(\pi)+y, \theta_L, \theta_R)$ conditional on L winning, where $x^k(\pi)$ is a function of D^k only.

From now on, we suppress θ_L and θ_R in indirect utility $\tilde{V}_L^k$, $E\tilde{V}_L^k$, and $E\tilde{V}_L$. We can rewrite the party leader L's gerrymandering choice to be the result of the following maximization problem

$$\pi^* \in \arg\max_{\pi\in\Pi} E\tilde{V}_L(\pi)$$

The SPNE of this game is $(\pi^*, (\beta_L^{k^*})_{k=1}^K, (\beta_R^{k^*})_{k=1}^K, (t_L^{k^*})_{k=1}^K, (t_R^{k^*})_{k=1}^K)$.

3.3 Bipartisan Gerrymandering Problem

Since bipartisan gerrymandering requires negotiation between the two parties, there can be many possible formulations. As mentioned before, one way is to assume that each party has preexisting "territory" as in Gul and Pesendorfer (2010). In our context, we can assume that, before redistricting, party L and R rearrange localities that belong to $\{1, ..., K_L\}$ and $\{K_L+1, ..., K\}$ by negotiating which localities belong to their own territory.

Given the above formulation, it may be beneficial for both parties to swap some of the localities in their territories, if the original allocation of localities in each district is arbitrary. If localities are ordered one-dimensionally as we assume in this paper, then there is always a chance to Pareto-improve the welfare by swapping localities, unless territories are consecutive due to the monotonicity in Lemma 4. In this case, leftmost nK_L localities go to party L, while rightmost $n(K-K_L)$ localities go to party R. This locality allocation is the unique Pareto-efficient one in the negotiation before redistricting.

For the complete freedom case we discuss in the next section, we can partition voters by some point $\bar{\theta}$, i.e., party L can take population to the left of $\bar{\theta}$, while party R can take population to the right of $\bar{\theta}$. It might not be the case that $KF(\bar{\theta})$ is an integer. However, it is reasonable to assume that party L and R create $K_L = \langle K \times F(\bar{\theta}) \rangle$ and $K_R = \langle K \times (1 - F(\bar{\theta})) \rangle$ districts, respectively, where $\langle \bullet \rangle$ denotes the nearest integer of $\bullet$. Some examples of $\bar{\theta}$ are (a) $F(\bar{\theta})$ being the vote share for L from the previous election, or (b) $\bar{\theta} = \theta_m$ from the recent census data. In both cases, the party controls a majority of districts if the whole population is biased toward it in the available data.

4 Gerrymandering with Complete Freedom

As a limit case, let us consider the *ideal situation* for the gerrymanderer (Friedman and Holden, 2008): there is a large number of infinitesimal localities with politically homogeneous population: for all position $x \in (-\infty, \infty)$, there are localities ℓs with $F_\ell(x - \delta) = 0$ and $F_\ell(x + \delta) = 1$ for a small $\delta > 0$. That is, the gerrymanderer can freely create any kind of population distributions for K districts as long as they sum up to the total population distribution. We ask what strategy the gerrymanderer should take. By Lemma 4, she is better off by making the (ex ante) median voter's allocation as far from the other party's leader's position as possible. This strategy increases the winning payoff and the probability of winning the district. Thus, the gerrymanderer tries to create the furthest district structure from the opponent party leader's position.

4.1 Partisan Gerrymandering

In partisan gerrymandering cases, the party leader in charge of gerrymandering will try to make district medians as far away as possible from the other party leader's position.[17] Without loss of generality, we assume that party L is in charge of gerrymandering. To create the most extreme district, x^1 should satisfy $F(x^1) = \frac{1}{2K}$ (x^1 is the median voter of the district: the most extreme district achievable with population $\frac{1}{K}$). Although the remaining population to the right of x^1 can be anything in district 1, wasting the other party's strong supporters by combining them is a good idea, since it would make the remaining population lean more toward her position. Thus, she will create district 1 by combining sets $\left\{\theta \leq \underline{\theta}^1 : F(\underline{\theta}^1) = \frac{1}{2K} + \frac{\epsilon}{K}\right\}$ and $\left\{\theta \geq \bar{\theta}^1 : 1 - F(\bar{\theta}^1) = \frac{1}{2K} - \frac{\epsilon}{K}\right\}$ where $\epsilon > 0$ is arbitrarily small. In district 1, the (ex ante) median voter would be x_L^{*1} defined by $F(x_L^{*1}) = \frac{1}{2K}$. Similarly, she can create districts $2, ..., K$ sequentially. Let $\underline{\theta}^k$ be such that $F(\underline{\theta}^k) = \frac{k}{2K} + \frac{k\epsilon}{K}$ for all $k = 1, ..., K$, and let $\bar{\theta}^k$ be such that $1 - F(\bar{\theta}^k) = \frac{k}{2K} - \frac{k\epsilon}{K}$. For small enough $\epsilon > 0$, we have

$$-\infty = \underline{\theta}^0 < \underline{\theta}^1 < ... < \underline{\theta}^K = \bar{\theta}^K < ... < \bar{\theta}^1 < \bar{\theta}^0 = \infty.$$

We call this redistricting plan a **party-L-slice-and-mix** policy, which is proposed in Friedman and Holden (2008). Under the slice-and-mix policy, the resulting district median voter allocation is $x_L^* \equiv (x_L^{*1}, ..., x_L^{*K})$ with x_L^{*k} is such that $F(x_L^{*k}) = \frac{k}{2K}$ for each $k = 1, ..., K$, with ϵ close to zero ($\lim_{\epsilon \to 0}(\underline{\theta}^1, ..., \underline{\theta}^K) = (x_L^{*1}, ..., x_L^{*K}) = x_L^*$). We will show that this is the optimal policy for party

[17] As long as there are positive winning probabilities in all districts (if $\bar{y}$ is large enough), this is true. If not, party L's leader may need to create unwinnable districts, but she would be indifferent as to how to draw lines for these districts. But the slice-and-mix below is one of the optimal strategies even in that case.

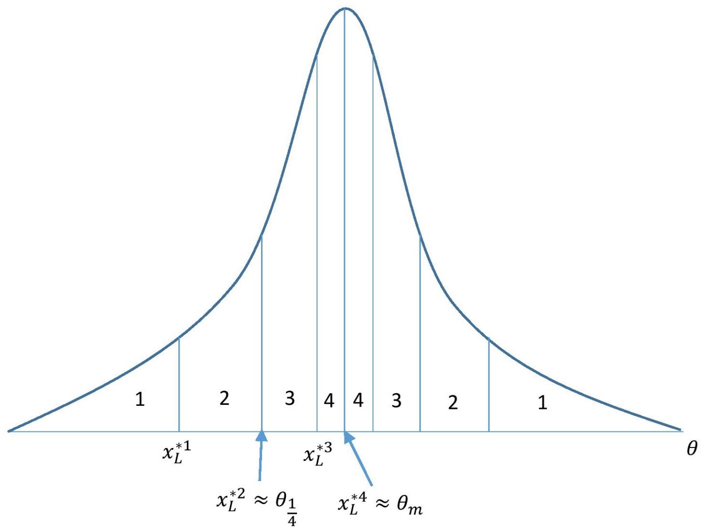

Figure 1: Party-L-slice-and-mix when $K = 4$.

L leader. Symmetrically, we can define a **party-R-slice-and-mix** where the resulting district median voter allocation is $x_R^* \equiv (x_R^{*K}, ..., x_R^{*1})$, with x_R^{*k} such that $1 - F(x_R^{*k}) = \frac{k}{2K}$ for each $k = 1, ..., K$, with ϵ close to zero ($\lim_{\epsilon \to 0}(\bar{\theta}^K, ..., \bar{\theta}^1) = (x_R^{*K}, ..., x_R^{*1}) = x_R^*$). Figure 1 is an example of party-L-slice-and-mix strategy when $K = 4$. District $k = 1, ..., 4$ is composed of two slices numbered by k. District median voter allocation is $x_L^* \equiv (x_L^{*1}, ..., x_L^{*4})$.

The following result is straightforward by noticing that in order for x^k to be the median voter in district $k = 1, ..., K$, x^k must satisfy $F(x^k) \geq \frac{k}{2K}$ and $1 - F(x^k) \geq \frac{k}{2K}$.

Lemma 5. *There is no median voter allocation $x = (x^1, ..., x^K)$ with $x^1 \leq x^2 \leq ... \leq x^K$ such that $x^k < x_L^{*k}$ for any $k = 1, ..., K$. Symmetrically, there is no median voter allocation $x = (x^1, ..., x^K)$ with $x^1 \geq x^2 \geq ... \geq x^K$ such that $x^k > x_R^{*k}$ for any $k = 1, ..., K$.*

Clearly, these district median voter allocations x_L^* and x_R^* are the most biased district median voter allocations toward left and right, respectively. Under x_L^*, redistricting the first and the second districts does not make two districts with intermediate medians. With this lemma and Lemma 4-1, we have the following result.

Proposition 1. *Suppose that the gerrymanderer can create districts with complete freedom and that party L (R) is in charge of gerrymandering. Then the party-L (R)-slice-and-mix policy is an optimal gerrymandering policy. The resulting district median voter allocation in district k is approximately x_L^{*k} (x_R^{*k}).*

Another interesting observation from this proposition is that even when $Q_L = Q_R$, if party L is the majority party in terms of the state population (That is $\theta_m < 0$ where $F(\theta_m) = \frac{1}{2}$), then it can win all seats with a probability of 50% or higher ($x^K < 0$). Also, one can observe that the median of x^k's is around $\theta_{\frac{1}{4}}$ where $F(\theta_{\frac{1}{4}}) = \frac{1}{4}$. Therefore, complete freedom in gerrymandering means the minority's impact on the election will be completely diluted. However, one party monopolize all districts is rare in US politics, partly because of the presence of majority-minority district requirement (see Shotts, 2001).[18] The majority-minority requirement forces the gerrymanderer to seek the second-best districting plan as a result even when she has complete freedom. It is worthwhile to note that the slice-and-mix strategy is identical to the optimal policy analyzed in Friedman and Holden. Both papers share the features that (i) the party leader prefers a more extreme median voter's

[18]In fact, even though either one of the two parties must be the majority in a state, the majority party usually does not win all districts. This can be attributed to Section 2 of the Voting Rights Act (accompanied by other United States Supreme Court cases), which essentially prevents the minority votes from being diluted in the voting process similar to our slice-and-mix strategy.

position than a moderate one, and (ii) complete freedom in gerrymandering unlike the constrained problem in Owen and Grofman (1988) and in the basic model of Gul and Pesendorfer (2010).[19] However, there are big differences between our paper and Friedman and Holden. Our model is based on competitions with political positions as well as transfer promises, while Friedman and Holden have neither element in their model. Nonetheless, we can say that the above two common conditions are the keys for getting the same results.

4.2 Bipartisan Gerrymandering

Suppose the preexisting territory is K_L and $K_R = K - K_L$: i.e., party L takes localities with population in $(-\infty, \bar{\theta})$ and party R takes localities with population in $(\bar{\theta}, \infty)$ where $F(\bar{\theta}) = \frac{K_L}{K}$.[20] By applying the same method as in the previous section, let $\bar{\theta}_L^0 = \bar{\theta}$ and $\underline{\theta}_L^k$ be such that $F(\underline{\theta}_L^k) = \frac{k}{2K} + \frac{k\epsilon}{K}$, and let $\bar{\theta}_L^k$ be such that $F(\bar{\theta}) - F(\bar{\theta}_L^k) = \frac{k}{2K} - \frac{k\epsilon}{K}$ for $k = 1, ..., K_L$. However, the support for L's territory is now $(-\infty, \bar{\theta}]$. Similarly, let $\underline{\theta}_R^0 = \bar{\theta}$ and $\underline{\theta}_R^k$ be such that $1 - F(\underline{\theta}_R^k) = \frac{k}{2K} - \frac{k\epsilon}{K}$, and let $\bar{\theta}_R^k$ be such that $1 - F(\bar{\theta}_R^k) = \frac{k}{2K} + \frac{k\epsilon}{K}$ for $k = K_L + 1, ..., K$. Party R's territory has support $(\bar{\theta}, \infty)$. We call this bipartisan policy (K_L, K_R)**-bipartisan-slice-and-mix policy**, and the resulting median voter profile is $(x_L^{*1}, ..., x_L^{*K_L}, x_R^{*K_L+1}, ..., x_R^{*K})$. By Lemma 5 again, $(x_R^{*K_L+1}, ..., x_R^{*K})$ is the K_R right-most median voter profile, and $(x_L^{*1}, ..., x_L^{*K_L})$ is the K_L left-most median voter profile, with small enough ϵ. Figure 2 is an example of (K_L, K_R)-bipartisan-slice-and-mix policy when $K_L = K_R = 2$ and $\bar{\theta} = \theta_m$. In this case, both parties use the slice-and-mix to create (x_L^{*1}, x_L^{*2}) and (x_R^{*3}, x_R^{4}).

[19]Gul and Pesendorfer (2010) also include aggregate uncertainty, generalizing Owen and Grofman (1988).

[20]To avoid roundup, we choose $\bar{\theta}$ such that $KF(\bar{\theta})$ is an integer. However, $\bar{\theta}$ can be a general one.

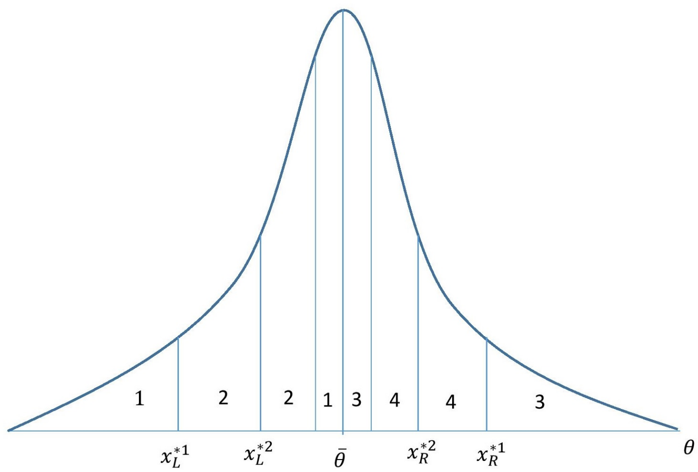

Figure 2: (K_L, K_R)-slice-and-mix when $K_L = K_R = 2$.

Thus, this is one of the most polarized district median voter allocation, and is very different from partisan gerrymandering median voter allocation, which has some more competitive districts. If uncertainty $\bar{y}$ is small, then there may not be any uncertainty in district elections under bipartisan gerrymandering.

Proposition 2. *Suppose that the gerrymanderer can create districts with complete freedom and that bipartisan gerrymandering takes place with party line $\bar{\theta}$. Then the (K_L, K_R)-bipartisan-slice-and-mix policy is an optimal gerrymandering policy. The resulting district median voter allocation is approximately $(x_L^{*k})_{k=1}^{K_L}$ and $(x_R^{*k})_{k=K_L+1}^{K}$.*

5 Gerrymandering with Limited Freedom

In this section, we will explore how the"slice-and-mix" result would be modified if we drop the "complete freedom" in gerrymandering. In the spirit of

Owen and Grofman (1988) and Gul and Pesendorfer (2010), we say a gerrymandering problem is subject to a *constant-average-constraint* if the resulting $(x^1(\pi), ..., x^K(\pi))$ satisfying

$$\frac{\sum_{k=1}^{K} x^k(\pi)}{K} = \bar{\mu} \tag{9}$$

for all $\pi \in \Pi$ and some fixed $\bar{\mu}$. Owen and Grofman (1988) analyzed the optimal partisan gerrymandering policy by imposing the same constraint. They obtained the famous pack-and-crack result when the office-motivated party leader maximizes the number of seats under this constraint.

To apply the above constraint to our locality setup, we will focus on the case where the political position is normally distributed in all localities. With normality, any feasibility redistricting plan satisfies exactly this constraint (8) (the proof is obvious by noting that the median is equivalent to the mean under normality).

Lemma 6. *Suppose that the voter distribution in each locality is normally distributed, i.e., $F_\ell \sim N(\mu_\ell, \sigma_\ell)$ for each $\ell \in L$. Then, the median of district k is*

$$x^k(\pi) = \frac{1}{n} \sum_{\ell \in D^k(\pi)} \mu_\ell.$$

Moreover, for all $\pi \in \Pi$, $\frac{\sum_{k=1}^{K} x^k(\pi)}{K} = \theta_m = \bar{\mu}$.

Therefore, under the normal distribution assumption, we focus on two redistricting plans, say, π and π', where the difference between two plans is due to swapping the sets of localities S and T between districts $\hat{k}$ and $\tilde{k}$. Formally,

$$\Delta = x^{\hat{k}}(\pi') - x^{\hat{k}}(\pi) = \frac{\sum_{\ell \in T} \mu_\ell - \sum_{\ell \in S} \mu_\ell}{n} = x^{\tilde{k}}(\pi) - x^{\tilde{k}}(\pi'),$$

and $x^k(\pi) = x^k(\pi')$ for all $k \neq \tilde{k}$ and $\hat{k}$. If $|x^{\tilde{k}}(\pi) - x^{\hat{k}}(\pi)| > |x^{\tilde{k}}(\pi') - x^{\hat{k}}(\pi')|$, π' is more centered relative to π. In this case, we say π' is *cracking supporters relative to* π. Otherwise, we say π' is *slicing supporters*.

Which plan should the party leader choose between π and π'? The answer depends on the curvature of $E\tilde{V}_i$. It is obvious that if $E\tilde{V}_i$ is a convex function in the ex ante median voter's position x^k, the party leader would prefer a slicing strategy. As we have seen in Lemma 4-3, if the third derivatives of cost functions are positive, we have convex expected payoff functions.

We are ready to characterize the optimal partisan gerrymandering policy under the constant average constraint. Remember that we order localities by their means. That is, $\ell < \ell'$ means $\mu_\ell \leq \mu'_\ell$. Let the median voter in the most possible extreme right district be μ_T. Suppose that $\mu_T - \bar{y} > 0$, that is, there exists some unwinnable districts for L if R's supporters are grouped together. We consider a redistricting plan that "slices" ordered localities from the left to the right. Formally, let $\bar{x}^k = \frac{1}{n}\sum_{\ell=n(k-1)+1}^{nk} \mu_\ell - \bar{y} < 0$ for all $k = 1, 2, ..., K'$, where K' is such that for all districts $k > K'$, there is absolutely no chance for party L to win. When $K' \geq K$, we call the allocation $(\bar{x}^k)_{k=1}^K$ a **slice-them-all** gerrymandering policy. If $K' < K$, then for those unwinnable districts $\{K' + 1, ..., K\}$, L's party leader should *pack* those most opposing localities with $\ell \geq K'n + 1$ into them. The reason is that, otherwise, the winning payoff in winnable district increases by switching those strong opposing localities into unwinnable districts. If $K' < K$, then how L packs those strong opposing localities does not matter, but $(\bar{x}^k)_{k=1}^K$ is one of the optimal policy for party L. This policy is optimal since the gerrymanderer always prefers a slicing swap and all other redistricting plans can be transformed into the slice-them-all policy by a series of slicing swaps.

Proposition 3. *Suppose that the voter distribution is normal in each locality and $Q_L = Q_R$. In addition, suppose that $C'''(\cdot) \geq 0$ and $c'''(\cdot) \geq 0$ hold. Then, the optimal partisan gerrymandering policy is slice-them-all $(\bar{x}^k)_{k=1}^{K'}$ with packing in the unwinnable districts. In particular, $(\bar{x}^k)_{k=1}^{K}$ is one of the optimal partisan gerrymandering policy. If $K' = K$, the unique optimal policy is slice-them-all $(\bar{x}^k)_{k=1}^{K}$.*

Thus, cracking is not necessarily a good strategy unlike in Owen and Grofman (1988). The difference between the current paper and theirs is that our party leaders are also policy-motivated.[21] What about the case where $C'''(\cdot) \geq 0$ and $c'''(\cdot) \geq 0$ do not hold? Actually, we can show that $\tilde{V}_i^k$ is concave if $C'''(\cdot) \leq 0$ and $c'''(\cdot) \leq 0$, so it appears that pack-and-crack is the way to go. Indeed, it is true for the deterministic case ($\bar{y} = 0$) or the cases where $\bar{y}$ is small enough. However, if $\bar{y}$ is large, even if the third derivatives are negative, $\tilde{V}_L^k$ can be convex as is seen in the following example (see Appendix B).

Example 1. We introduce a convenient special ideology cost function such that both voters' and party leaders' cost functions have common constant elasticity. Let $C(d) = a^C d^\gamma$ and $c(d) = a^c d^\gamma$, where $\gamma > 1$, $a^C > 0$, and $a^c > 0$ are parameters. In this case, both party leaders and voters have the same elasticity that is constant γ. Thus, we have the following convenient formula. Denote $A = A(a^C, a^c) = a^C \left(\frac{\alpha}{1+\alpha}\right)^\gamma + a^c \left(\frac{1}{1+\alpha}\right)^\gamma > 0$ where $\alpha = \left(\frac{a^C}{a^c}\right)^{\frac{1}{\gamma-1}}$. . We can choose a^C and a^c to set $A = 1$ for each γ: then we have $\mathcal{C}(d) = Ad^\gamma = d^\gamma$.

[21]Without the policy motivation, the payoff function is only related to winning probability, pack-and-crack is optimal under a mild assumption on g. See also Gul and Pesendorfer (2010).

In this case, $\tilde{V}_L^k$ is concave (convex) in $\hat{x}^k$ if $\gamma \leq 2$ ($\gamma \geq 2$). Suppose that $\theta_L = -1$, $\theta_R = 1$ (thus L wins if and only if $\hat{x}^k < 0$), and $g(y) = \frac{1}{2\bar{y}}$ if and only if $y \in [-\bar{y}, \bar{y}]$ (uniform distribution). Also, suppose that all possible x^k are in $[-1, 1]$ and $\left(\frac{Q}{A}\right)^{\frac{1}{\gamma}} \geq 2 + \bar{y}$ holds to assure Assumption 2. If $\bar{y} > 1$, there is always a chance to win the election: we have $x^k - \bar{y} < \theta_L$ and $x^k + \bar{y} > 0$. In this case,

$$E\tilde{V}_L^{k\prime\prime} = \frac{\gamma}{2\bar{y}} \left[(1 - x^k + \bar{y})^{\gamma-1} - ((1 - x^k + \bar{y}) - 2)^{\gamma-1} + 2\right] > 0$$

since $\gamma < 2$, $(1-x^k+\bar{y})^{\gamma-1} - ((1-x^k+\bar{y})-2)^{\gamma-1} > -2$ holds. Thus, the expected utility is convex in x^k, despite the fact that $\mathcal{C}'''(d) < 0$ holds. This example shows that even without positive third derivatives, the slice-and-mix strategy and the slice-them-all strategy are optimal in the complete freedom case and in the constrained case with the constant average constraint, respectively.□

How about bipartisan gerrymandering? The result is the same, since both parties want slice-them-all anyway. If both party leaders adopt slice-them-all, it does not matter whether the slicing is from one end (partisan gerrymandering) or both ends (bipartisan). This observation shows that if the gerrymandering problem has the constant average constraint, then bipartisan gerrymandering does not create a more polarized allocation than partisan gerrymandering, and incumbents' reelection rates would be the same.

Proposition 4. *Suppose that the voter distribution is normal in each locality and $Q_L = Q_R$. In addition, suppose that $C'''(\cdot) \geq 0$ and $c'''(\cdot) \geq 0$ hold. Then, the optimal bipartisan gerrymandering policy is slice-them-all $(\bar{x}^k)_{k=1}^K$ which is identical to the partisan policy.*

The constant average constraint forbids a gerrymanderer from diluting supporters of the other party by mixing in his own supporters. Notice that while

L gerrymanderer can pull the median of district medians to $\theta_{\frac{1}{4}}$ in the complete freedom case, the median of medians has to remain as θ_m, the population median, when the constant average constraint applies. Friedman and Holden (2009) interpret their results as a possible consequence of the Voting Rights Act of 1982, which significantly limits the gerrymanderer's ability to dilute votes.

6 Conclusion

In this paper, we propose a gerrymandering model with endogenous candidates' political positions, in which two parties compete in their positions and pork-barrel politics. The model's tractability allows us to analyze partisan and bipartisan gerrymandering under different constraints.

We find that, under the complete freedom case (Friedman and Holden, 2008), the partisan and bipartisan gerrymandering plans are very different. Under partisan gerrymandering, the gerrymanderer creates the most biased district structure and completely dilutes the opponent's supporters. On the other hand, in the bipartisan case, gerrymanderers create the most polarized districts.

The difference between partisan and bipartisan gerrymandering disappears when we add the extra constraint that requires that the mean of median voters in all districts remain constant (Owen and Grofman, 1988). The optimal plan under positive third derivatives in cost functions becomes what we call *slice-them-all* in both situations. That is, the gerrymanderers simply group their own supporters to form districts according to the avidity of supportiveness. This result is based on the fact that the constant average constraint forces the party leader to choose between having one extreme supporting and one relatively neutral districts or having two moderate supporting ones. However, since the party leader is policy-motivated and has to consider uncertainties, she prefers the former, which saves her more pork-barrel and ideological costs. Our Example 1 suggests that the positive third derivative condition may be weakened significantly for the same result when the shock is large enough to provide a winning chance for both parties in every district.

Another explanation for nonsignificant difference in competitiveness between partisan and bipartisan gerrymandering is the fact that redistricting takes place every ten years based on census data, and district population profiles can change significantly. If there is a risk for some demographic change in districts, then it is too risky to use extremely elaborate slice-and-mix strategy even if the gerrymanderers have complete freedom in redistricting. This is because a district median voter profile can change dramatically by demographic changes. Thus, the gerrymanderer may try to mix a smaller and less extreme opponent group with a larger strong supporter group, which may make the difference between partisan and bipartisan gerrymandering less significant.

There are some potentially interesting yet difficult extensions. First, one may want to introduce uncertainty in election results (e.g., uncertainty in median voter's position after policy proposal) into our model. If uncertainty is infinitesimal, e.g., the gerrymander can only observe that the median voter's position belongs to the interval $[\hat{x}^k - \epsilon, \hat{x}^k + \epsilon]$ for ϵ being a (small) preference perturbation, and if the gerrymanderer has complete freedom in redistricting, the slice-and-mix strategy may still be optimal à la Friedman and Holden (2008). However, with significant uncertainty in median voters' positions, as Gul and Pesendorfer (2010), we do not know what can happen.

Second, in this paper we concentrated on one type of pork-barrel politics: candidates' "promise" transfer contingent on their winning of the districts (the first price auction). These kinds of promises are different from campaign expenditures. In the latter case, even if a candidate loses in a district, the spent campaign expenditure will not come back (an all-pay auction). In some circumstances, such a model may be more realistic if there is uncertainty in election results. However, introducing uncertainty in election results is not trivial, as we mentioned before. These issues are left for future research.

Appendix A: Proofs

Proof of Lemma 1. As we mentioned in the text, we first prove that in equilibrium, the loser party j proposes the policy pair $(\beta_j^{k^*}, t_j^{k^*})$, which is the solution of the problem in equation (7). To prove this, first note that the non-negativity constraint of t_j^k is not needed by Assumption 2. There are three cases: if j loses with $Q_j - t_j^{k^*} - C(|\theta_j - \beta_j^{k^*}|) > 0$ and its offer gives the median voter utility equal to $\bar{U}$ in equilibrium, it must be that i wins with positive indirect utility and also provides the median voter with the utility level $\bar{U}$. However, this means that j can win the election by providing, say, ϵ more pork-barrel promise. This contradicts the equilibrium condition. The second case is that $Q - t_j^{k^*} - C(|\theta_j - \beta_j^{k^*}|) = 0$ but $U_{x^k}(j)$ is not maximized. In this case, there must exist some points (t', β') that satisfy $Q - t' - C(|\theta_j - \beta'|) = 0$ but the pair provides the median voter strictly higher utility. Then any point on the segment connecting (t', β') and $(t_j^{k^*}, \beta_j^{k^*})$ is strictly better off for both j and the median voter x^k by the strict convexity of the preferences. Again, this contradicts the equilibrium condition. Thus, $\beta_j^{k^*} = \hat{\beta}_j(\hat{x}_k, \theta_j)$ must hold. The third case, $Q - t_j^{k^*} - C(|\theta_j - \beta_j^{k^*}|) < 0$, cannot happen, since the strategy that generates a negative payoff is a weakly dominated strategy for party j's leader. Thus, $t_j^{k*} = Q_j - C(|\theta_j - \beta_j(\hat{x}^k, \theta_j)|)$ holds.

The policy pair provides the median voter with the utility $\bar{U}_R^{k^*} = Q_R - C(\left|\theta_R - \beta_R^{k^*}\right|) - c(|\hat{x}^k - \beta_R^{k^*}|)$. Substituting this $\bar{U}_R^{k^*}$ into equation (6), we obtain $t_L^{k^*} = Q_R - C(\left|\theta_R - \beta_R^{k^*}\right|) - c(|\hat{x}^k - \beta_R^{k^*}|) + c(|\hat{x}^k - \beta_L^{k^*}|)$.□

Proof of Lemma 2. First, we show Lemma 2-1. By Lemma 1, the losing party always proposes the best offer by depleting all her surplus. Therefore, the party that can *potentially* provide the median voter with a higher utility level is the winner. Notice that party j's pork-barrel promise is bounded above by party j leader's payoff evaluated at β_j^{k*} (otherwise, the leader gets a negative utility):

$$Q_j - C(\left|\theta_j - \beta_j^{k^*}\right|).$$

Substituting this into the median voter's utility, we obtain

$$W_j^k = Q_j - C(\left|\theta_j - \beta_j^{k^*}\right|) - c(|\hat{x}^k - \beta_j^{k^*}|),$$

and similarly, for party i,

$$W_i^k = Q_i - C(\left|\theta_i - \beta_i^{k^*}\right|) - c(|\hat{x}^k - \beta_i^{k^*}|),$$

where W_j^k and W_i^k are the (potential) maximum utilities that the median voter gets from the corresponding party's offer. Therefore, party i wins in the third stage if and only if

$$\begin{aligned} &Q_i - Q_j > \\ &\quad \left[c(|\hat{x}^k - \beta_i^{k^*}|) + C(\left|\theta_i - \beta_i^{k^*}\right|)\right] - \left[c(|\hat{x}^k - \beta_j^{k^*}|) + C(\left|\theta_j - \beta_j^{k^*}\right|)\right], \end{aligned} \tag{10}$$

Now, we only need to prove Lemma 2-2. We consider two cases: **(Case-1)** $\hat{x}^k > \theta_i$, and **(Case-2)** $\hat{x}^k < \theta_i$.

(Case-1): In this case, $\hat{\beta}_i = \hat{\beta}(\hat{x}^k, \theta_i)$ is determined implicitly by the first-order condition

$$C'(\hat{\beta}_i - \theta_i) = c'(\hat{x}^k - \hat{\beta}_i)$$

Totally differentiating with respect to $\hat{x}^k$ and $\hat{\beta}_i$, we obtain

$$(C'' + c'')d\hat{\beta}_i = c''d\hat{x}^k$$

(Case-2): In this case, $\hat{\beta}_i = \hat{\beta}(\hat{x}^k, \theta_i)$ is determined implicitly by the first-order condition

$$C'(\theta_i - \hat{\beta}_i) = c'(\hat{\beta}_i - \hat{x}^k)$$

Totally differentiating with respect to $\hat{x}^k$ and $\hat{\beta}_i$, we obtain

$$(C'' + c'')d\hat{\beta}_i = c''d\hat{x}^k$$

Thus, either way, we get the same condition. We have completed the proof.□

Proof of Lemma 3. We will focus on the case of $i = L$. When $i = R$, we can apply the same procedure. We will first show the following claim.

Claim. $\mathcal{C}_i' = c_i'$ when $\theta_i < \hat{x}^k$, $\mathcal{C}_i' = -c_i'$ when $\theta_i > \hat{x}^k$ and $\mathcal{C}_i'' = \frac{c_i'' C_i''}{c_i'' + C_i''}$, where $\mathcal{C}_i = \mathcal{C}(\left|\hat{x}^k - \theta_i\right|)$, $c_i = c\left(\left|\hat{x}^k - \beta\left(\hat{x}^k, \theta_i\right)\right|\right)$, and $C_i = C\left(\left|\beta\left(\hat{x}^k, \theta_i\right) - \theta_i\right|\right)$.

Proof of Claim. So, there are two cases: **(Case-a)** $\theta_i < \hat{x}^k$, and **(Case-b)** $\theta_i > \hat{x}^k$.

(Case-a): Taking the first derivative, we have

$$\mathcal{C}'(\hat{x}^k - \theta_i) = C'(\hat{\beta}_i - \theta_i)\frac{\partial \hat{\beta}_i}{\partial \hat{x}^k} + c'(\hat{x}^k - \hat{\beta}_i)(1 - \frac{\partial \hat{\beta}_i}{\partial \hat{x}^k}) = c'(\hat{x}^k - \hat{\beta}_i),$$

Here, we used the first-order condition $C' = c'$, which must hold at the optimum. Taking the second-order derivative, we have

$$\begin{aligned}
\mathcal{C}''(\hat{x}^k - \theta_i) &= c''(\hat{x}^k - \hat{\beta}_i)(1 - \frac{\partial \hat{\beta}_i}{\partial \hat{x}^k}) \\
&= c''(\hat{x}^k - \hat{\beta}_i)\left(1 - \frac{c''(\hat{x}^k - \hat{\beta}_i)}{c''(\hat{x}^k - \hat{\beta}_i) + C''(\hat{\beta}_i - \theta_i)}\right) \\
&= \frac{c''(\hat{x}^k - \hat{\beta}_i)C''(\hat{\beta}_i - \theta_i)}{c''(\hat{x}^k - \hat{\beta}_i) + C''(\hat{\beta}_i - \theta_i)}
\end{aligned}$$

(Case-b): Taking the first-order derivative, we have

$$\mathcal{C}'(\theta_i - \hat{x}^k) = -C'(\theta_i - \hat{\beta}_i)\frac{\partial\hat{\beta}_i}{\partial\hat{x}^k} + c'(\hat{\beta}_i - \hat{x}^k)(\frac{\partial\hat{\beta}_i}{\partial\hat{x}^k} - 1) = -c'(\hat{\beta}_i - \hat{x}^k),$$

Taking the second-order derivative, we have

$$\begin{aligned}\mathcal{C}''(\theta_i - \hat{x}^k) &= -c''(\hat{\beta}_i - \hat{x}^k)(\frac{\partial\hat{\beta}_i}{\partial\hat{x}^k} - 1)\\ &= c''(\hat{\beta}_i - \hat{x}^k)\left(1 - \frac{c''(\hat{x}^k - \hat{\beta}_i)}{c''(\hat{x}^k - \hat{\beta}_i) + C''(\theta_i - \hat{\beta}_i)}\right)\\ &= \frac{c''(\hat{\beta}_i - \hat{x}^k)C''(\theta_i - \hat{\beta}_i)}{c''(\hat{\beta}_i - \hat{x}^k) + C''(\theta_i - \hat{\beta}_i)}\end{aligned}$$

We have completed the proof of the Claim.■

We start with Lemma 3-1. First, we consider **(Case-1):** $\hat{x}^k \in (\theta_L, \theta_R)$, then $\tilde{V}_L^k = (Q_L - Q_R) - \big(\mathcal{C}(x^k + y - \theta_L) - \mathcal{C}(\theta_R - x^k - y)\big)$. Thus, we have

$$\frac{d\tilde{V}_L^k}{dx^k} = -\big(\mathcal{C}'(\hat{x}^k - \theta_L) + \mathcal{C}'(\theta_R - \hat{x}^k)\big) < 0.$$

This implies that $\tilde{V}_L^k$ is decreasing in $\hat{x}^k$. In the case of $\tilde{V}_R^k$, $\frac{d\tilde{V}_R^k}{dx^k} > 0$ and $\tilde{V}_R^k$ is increasing in $\hat{x}^k$.

There are two more cases: **(Case-2)** $\hat{x}^k < \theta_L$, and **(Case-3)** $\hat{x}^k > \theta_R$.
(Case-2): $\frac{d\tilde{V}_L^k}{dx^k} = \mathcal{C}'(\theta_L - \hat{x}^k) - \mathcal{C}'(\theta_R - \hat{x}^k) < 0$, since $\mathcal{C}''(d) > 0$. Thus, $\tilde{V}_L^k$ is decreasing in $\hat{x}^k$.
(Case-3): $\frac{d\tilde{V}_L^k}{dx^k} = -\mathcal{C}'(\hat{x}^k - \theta_L) + \mathcal{C}'(\hat{x}^k - \theta_R) < 0$, since $\mathcal{C}''(d) > 0$. Thus, $\tilde{V}_L^k$ is decreasing in $\hat{x}^k$.

For the convexity, again we have three cases: **(Case-1)** $\hat{x}^k \in (\theta_L, \theta_R)$, **(Case-2)** $\hat{x}^k < \theta_L$, and **(Case-3)** $\hat{x}^k > \theta_R$. In each case, we have the same second derivatives:
(Case-1): $\frac{d^2\tilde{V}_L^k}{d(x^k)^2} = -\mathcal{C}''(\hat{x}^k - \theta_L) + \mathcal{C}''(\theta_R - \hat{x}^k)$.

(Case-2): $\frac{d\tilde{V}_L^k}{dx^k} = \mathcal{C}'(\theta_L - \hat{x}^k) - \mathcal{C}'(\theta_R - \hat{x}^k)$ and $\frac{d^2\tilde{V}_L^k}{d(x^k)^2} = -\mathcal{C}''(\theta_L - \hat{x}^k) + \mathcal{C}''(\theta_R - \hat{x}^k)$.

(Case-3): $\frac{d\tilde{V}_L^k}{dx^k} = -\mathcal{C}'(\hat{x}^k - \theta_L) + \mathcal{C}'(\hat{x}^k - \theta_R)$ and $\frac{d^2\tilde{V}_L^k}{d(x^k)^2} = -\mathcal{C}''(\hat{x}^k - \theta_L) + \mathcal{C}''(\hat{x}^k - \theta_R)$.

Therefore, in all cases, $\frac{d^2\tilde{V}_L^k}{d(x^k)^2} = -\mathcal{C}''_L + \mathcal{C}''_R$, so we have:

$$
\begin{aligned}
\frac{d^2\tilde{V}_L^k}{d(x^k)^2} &= -\mathcal{C}''_L + \mathcal{C}''_R \\
&= -\frac{c''_L C''_L}{c''_L + C''_L} + \frac{c''_R C''_R}{c''_R + C''_R} \\
&= \frac{-c''_L C''_L\left(c''_R + C''_R\right) + c''_R C''_R\left(c''_L + C''_L\right)}{\left(c''_L + C''_L\right)\left(c''_R + C''_R\right)} \\
&= \frac{C''_L C''_R\left(c''_R - c''_L\right) + c''_L c''_R\left(C''_R - C''_L\right)}{\left(c''_L + C''_L\right)\left(c''_R + C''_R\right)}
\end{aligned}
$$

Thus, if $c''_R \geq c''_L$ and $C''_R \geq C''_L$ then $\frac{d^2\tilde{V}_L^k}{d(x^k)^2} \geq 0$. Since $Q_L = Q_R$, if L wins, then $\hat{x}^k - \theta_L < \theta_R - \hat{x}^k$. Thus, if $c''' > 0$ and $C''' > 0$ then we have $c''_R \geq c''_L$ and $C''_R \geq C''_L$.$\square$

Proof of Lemma 4. We will focus on the case of $i = L$. When $i = R$, we can apply the same procedure. Let's start with Lemma 4-1. Consider the case where $x^k \pm \bar{y} \in (\theta_L, \theta_R)$. There are two subcases: **(Case 1)** is the case where L wins with certainty ($\tilde{V}_L^k(x^k + \bar{y}, \theta_L, \theta_R) \geq 0$), and **(Case 2)** is the one where L may lose depending on the realization of y ($\tilde{V}_L^k(x^k + \bar{y}, \theta_L, \theta_R) < 0$).

(Case 1): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R)g(y)dy$. Thus,

$$
\frac{dE\tilde{V}_L^k}{dx^k} = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy < 0
$$

(Case 2): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R)g(y)dy$, where $\tilde{V}_L^k(\bar{x}, \theta_L, \theta_R) = 0$. That is, if $x^k + y > \bar{x}$, then party L loses. (Note that $\bar{x}$ is solely determined by the value of $Q_L - Q_R$: $\frac{d\bar{x}}{d(Q_L - Q_R)} > 0$. If $Q_L = Q_R$, then $\bar{x} = 0$ holds, since $\theta_L = -\theta_R$.) Differentiating this with respect to x^k, we have

$$\begin{aligned}\frac{dE\tilde{V}_L^k}{dx^k} &= \tilde{V}_L^k(\bar{x}, \theta_L, \theta_R) + \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \\ &= \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy < 0\end{aligned}$$

Thus, we have completed the proof of Lemma 4-1.

For Lemma 4-2, we classify four cases:

(Case a: $x^k - \bar{y} \geq \theta_L$ and $x^k + \bar{y} \leq \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. Thus,

$$\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} = \int_{-\bar{y}}^{\bar{y}} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy$$

Case b: $x^k - \bar{y} \geq \theta_L$ and $x^k + \bar{y} > \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R) g(y) dy$. That is, if $x^k + y > \bar{x} = 0$, then party L loses. Differentiating this with respect to x^k, we have

$$\begin{aligned}\frac{dE\tilde{V}_L^k}{dx^k} &= -\tilde{V}_L^k(0, \theta_L, \theta_R) + \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy \\ &= \int_{-\bar{y}}^{\bar{x}-x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R) g(y) dy\end{aligned}$$

Thus, the second-order derivative is

$$\frac{d^2 E\tilde{V}_L^k}{d(x^k)^2} = -\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) + \int_{-\bar{y}}^{-x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) g(y) dy$$

From Lemma 3-2, we know $\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) < 0$ and $\tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) > 0$. Thus, $E\tilde{V}_L^k$ is convex.

(Case c: $x^k - \bar{y} < \theta_L$ and $x^k + \bar{y} \le \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{y}} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R)g(y)dy$. Differentiating this with respect to x^k, we obtain

$$\frac{dE\tilde{V}_L^k}{dx^k} = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{y}} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy$$

The second-order derivative is

$$\frac{d^2E\tilde{V}_L^k}{d(x^k)^2} = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{y}} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R)g(y)dy$$

From Lemma 3-2, we know $\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) < 0$ and $\tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) > 0$. Thus, $E\tilde{V}_L^k$ is convex.

(Case d: $x^k - \bar{y} < \theta_L$ and $x^k + \bar{y} > \bar{x}$): In this case, $E\tilde{V}_L^k = \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{x} - x^k} \tilde{V}_L^k(x^k + y, \theta_L, \theta_R)g(y)dy$. Differentiating this with respect to x^k, we obtain

$$\begin{aligned}\frac{dE\tilde{V}_L^k}{dx^k} &= \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{x} - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy \\ &\quad - \tilde{V}_L^k(0, \theta_L, \theta_R)g(-x^k) \\ &= \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{x} - x^k} \tilde{V}_L^{k\prime}(x^k + y, \theta_L, \theta_R)g(y)dy\end{aligned}$$

The second-order derivative is

$$\begin{aligned}\frac{d^2E\tilde{V}_L^k}{d(x^k)^2} &= \int_{-\bar{y}}^{\theta_L - x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R)g(y)dy + \int_{\theta_L - x^k}^{\bar{x} - x^k} \tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R)g(y)dy \\ &\quad - \tilde{V}_L^{k\prime}(0, \theta_L, \theta_R)g(-x^k)\end{aligned}$$

From Lemma 3-2, we know $\tilde{V}_L^{k\prime}(0, \theta_L, \theta_R) < 0$ and $\tilde{V}_L^{k\prime\prime}(x^k + y, \theta_L, \theta_R) > 0$ when $c''(d) > 0$ and $C''(d) > 0$. Thus, $E\tilde{V}_L^k$ is convex. We have completed the proof of Lemma 4-2.

For Lemma 4-3 and Lemma 4-4, first observe that,

$$E\tilde{V}_i(\pi) \equiv \int_{-\bar{y}}^{\bar{y}} \sum_{k=1}^{K} \tilde{V}_i^k(x^k(\pi) + y, \theta_i, \theta_j)g(y)dy$$

For any $k \neq k'$, $\frac{\partial^2 E\tilde{V}_i}{\partial x^k \partial x^{k\prime}} = 0$. The Hessian matrix of $E\tilde{V}_L$ has 0s on non-diagonal parts and negative terms on the diagonal due to Lemma 4-2. Therefore, the Hessian matrix is negative semidefinite and $E\tilde{V}$ are convex function in $(x^k)_{k=1}^K$ (we cannot say "strictly convex" since there are losing districts).

Also notice that the proof of Lemma 4-3 and 4-4 hold even when we consider the general case $\hat{x}^k = x^k + y^k$, where y^k is the district specific shock and y^ks have joint distribution $g(y^1, ..., y^k)$. The proof above works for the sepcial case when y^ks are perfect correlated. □

Proof of Lemma 5. Note that $F(x_L^{*k}) = \frac{k}{2K}$. Thus, to achieve x_L^{*k} as the median voter of the kth district, we need to use all voters to the left of x_L^{*k}. This is true for all $k = 1, ..., K$. Thus, x_L^* is the leftmost median voter allocation in lexicographic order. We can prove the statement for x_R^* by a symmetric argument.□

Appendix B: Constant Elasticity Example

In this appendix, we elaborate on the calculation involved in Example 1. Let $C(d) = a^C d^\gamma$ and $c(d) = a^c d^\gamma$, where $\gamma > 1$, $a^C > 0$, and $a^c > 0$ are parameters. In this case both party leaders and voters have the same elasticity that is constant γ. In this case, we have the following convenient formula. Denote $A = A(a^C, a^c) = a^C \left(\frac{\alpha}{1+\alpha}\right)^\gamma + a^c \left(\frac{1}{1+\alpha}\right)^\gamma > 0$ where $\alpha = \left(\frac{a^C}{a^c}\right)^{\frac{1}{\gamma-1}}$. Suppose that $\left(\frac{Q}{A}\right)^{\frac{1}{\gamma}} \geq 2 + \bar{y}$ holds to assure Assumption 2. Normalizing $A = 1$, we have $\mathcal{C}(d) = Ad^\gamma = d^\gamma$. In this case, $\tilde{V}_L^k$ is concave (convex) in $\hat{x}^k$ if $\gamma \leq 2$ ($\gamma \geq 2$).

$$E\tilde{V}_L^k = \int_{\theta_L - x^k}^{-x^k} \left(\mathcal{C}(\theta_R - x^k - y) - \mathcal{C}(x^k + y - \theta_L)\right) g(y)dy$$
$$+ \int_{-\bar{y}}^{\theta_L - x^k} \left(\mathcal{C}(\theta_R - x^k - y) - \mathcal{C}(\theta_L - x^k - y)\right) g(y)dy$$

$$E\tilde{V}_L^{k\prime} = \int_{\theta_L - x^k}^{-x^k} \left(-\mathcal{C}'(\theta_R - x^k - y) - \mathcal{C}'(x^k + y - \theta_L)\right) g(y)dy$$
$$+ \int_{-\bar{y}}^{\theta_L - x^k} \left(-\mathcal{C}'(\theta_R - x^k - y) + \mathcal{C}'(\theta_L - x^k - y)\right) g(y)dy$$

$$E\tilde{V}_L^{k\prime\prime} = \int_{\theta_L - x^k}^{-x^k} \left(\mathcal{C}''(\theta_R - x^k - y) - \mathcal{C}''(x^k + y - \theta_L)\right) g(y)dy$$
$$+ \int_{-\bar{y}}^{\theta_L - x^k} \left(\mathcal{C}''(\theta_R - x^k - y) - \mathcal{C}''(\theta_L - x^k - y)\right) g(y)dy$$
$$+ (\mathcal{C}'(\theta_R) + \mathcal{C}'(\theta_L))g(-x^k)$$

Suppose that $\mathcal{C}(d) = d^\gamma$ ($\gamma > 1$), $\theta_L = -1$, $\theta_R = 1$ (thus $\bar{x} = 0$), and $g(y) = \frac{1}{2\bar{y}}$ if and only if $y \in [-\bar{y}, \bar{y}]$. If $\bar{y} \geq 2$ and $x^k \in [-1, 1]$ for all possible x^k, Case-d in the proof of Lemma 4 applies. In this case, we have

$$
\begin{aligned}
E\tilde{V}_L^{k\prime\prime} &= \gamma(\gamma-1)\int_{\theta_L-x^k}^{-x^k}\left((\theta_R-x^k-y)^{\gamma-2}-(x^k+y-\theta_L)^{\gamma-2}\right)\frac{1}{2\bar{y}}dy \\
&+\gamma(\gamma-1)\int_{-\bar{y}}^{\theta_L-x^k}\left((\theta_R-x^k-y)^{\gamma-2}-(\theta_L-x^k-y)^{\gamma-2}\right)\frac{1}{2\bar{y}}dy \\
&+\frac{1}{2\bar{y}}\times 2\gamma\theta_R^{\gamma-1} \\
&= \frac{\gamma}{2\bar{y}}\left[-(\theta_R-x^k-y)^{\gamma-1}-(x^k+y-\theta_L)^{\gamma-1}\right]_{\theta_L-x^k}^{-x^k} \\
&+\frac{\gamma}{2\bar{y}}\left[-(\theta_R-x^k-y)^{\gamma-1}+(\theta_L-x^k-y)^{\gamma-1}\right]_{-\bar{y}}^{\theta_L-x^k}+\frac{\gamma}{\bar{y}}\theta_R^{\gamma-1} \\
&= \frac{\gamma}{2\bar{y}}[-(\theta_R)^{\gamma-1}+(\theta_R-x^k+\bar{y})^{\gamma-1}-(-\theta_L)^{\gamma-1}+0+0 \\
&-(\theta_L-x^k+\bar{y})^{\gamma-1}+2\theta_R^{\gamma-1}] \\
&= \frac{\gamma}{2\bar{y}}\left[(1-x^k+\bar{y})^{\gamma-1}-(-1-x^k+\bar{y})^{\gamma-1}+2\right] \\
&= \frac{\gamma}{2\bar{y}}\left[(1-x^k+\bar{y})^{\gamma-1}-((1-x^k+\bar{y})-2)^{\gamma-1}+2\right]
\end{aligned}
$$

When $\gamma < 2$, $(1-x^k+\bar{y})^{\gamma-1}-((1-x^k+\bar{y})-2)^{\gamma-1} > -2$ holds. Thus, $E\tilde{V}_L^{k\prime\prime} > 0$ holds as long as Case 4 holds ($\bar{y} \geq 1$: there is a chance to win district k for any x^k). That is, the expected utility is convex in x^k, although $\mathcal{C}'''(d) < 0$ holds. So, even without positive third derivatives, the slice-and-mix strategy and the slice-them-all strategy are optimal gerrymandering policies in the constant average constraint case, respectively.□

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