On the Nature and Stability of Sentiments

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Abstract

We show that non-trivial aggregate fluctuations may originate with vanishingly-small common shocks to either information or fundamentals. These "sentiment" fluctuations can be driven by self-fulfilling variation in either first-order beliefs (as in Benhabib et al., 2015) or higher-order beliefs (as in Angeletos and La'O, 2013), due to an endogenous signal structure. We analyze out-of-equilibrium best-response functions in the underlying coordination game to study whether sentiment equilibria are stable outcomes of a convergent process. We find that limiting sentiment equilibria are generally unattainable under both higher-order belief and adaptive learning dynamics, whereas equilibria without sentiment shocks show strong stability properties. Away from the limit case, however, multiple noisy rational expectations equilibria may be stable.

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1 Introduction

Macroeconomists have long sought to model the ways in which changes in expectations can affect aggregate outcomes without corresponding changes in underlying economic fundamentals. A recent and influential branch of literature focuses on models in which some equilibria reflect responses to extrinsic "sentiment" shocks. As in a correlated equilibrium (Aumann, 1987), imperfectly correlated signals can sustain equilibria in which outcomes depend on random realizations that are not payoff relevant. This new way of thinking about "animal spirits" circumvents well-known difficulties with traditional sunspot models, whose existence typically relies on non-convexity of the payoff structure.[1] These non-convexities have proven problematic for the out-of-equilibrium convergence of agents' actions under higher-order belief and adaptive learning dynamics.[2]

In this paper, we examine the sources of sentiment equilibria that arise with endogenous asymmetric information, and study the problem of agents who must coordinate on these equilibria starting from outside equilibrium. To demonstrate our basic insights, we build on the core version of the model presented by Benhabib et al. (2015). Our innovation is to incorporate a common noise within the endogenous signal structure that they consider. This innovation is key in characterizing the individual best response function in the underlying coordination game implied by dispersed information. With endogenous signals, the usual definition of a rational expectations equilibrium is that of a Nash equilibrium in which responding to the signal according to equilibrium prescriptions is a best action if all others do the same. The existence of a dispersed information rational expectations equilibrium does not, however, guarantee that agents will be able to coordinate on it; perfectly valid rational expectations equilibria may generate divergent local learning dynamics. The characterization of individual best response functions helps to establish conditions for the emergence of

[1] See Azariadis (1981); Cass and Shell (1983); Benhabib and Farmer (1994) among others.

[2] See Guesnerie (2005) and Evans and Honkapohja (2001). A notable exception is Woodford (1990) who shows the existence of adaptively learneable sunspots (see for a comprehensive discussionEvans and McGough (2011)). Examples of stability under higher order belief dynamics are found by Desgranges and Negroni (2003).

sentiment equilibria, and ultimately to elucidate the stability properties of the equilibria we find.

Our first result is to show that the sentiment equilibria uncovered by Benhabib et al. (2015) emerge as noisy rational expectations equilibria in the limit of vanishing variance of a common noise in the endogenous signal. When common noise is small enough, the equilibrium with sentiment fluctuations is always accompanied by a equilibrium with no such fluctuations.[3] Thus, both the sentiment and sentiment-free equilibria of Benhabib et al. (2015) can be seen as the limiting case of arbitrarily small noise in the endogenous signal. In equilibrium, endogenous signals can (but need not) deliver a multiplier effect on common noise that grows unboundedly as the noise goes to zero, up to the point where arbitrarily small amount of common error may serve to coordinate non-trivial fluctuations in a rational expectations equilibrium.

The first result suggests that sentiment-like fluctuations may be driven by small common errors but, in our second result, we show that the same characterization of sentiment equilibria as limit outcomes can also be obtained when the common component of the signal is of a fundamental nature, rather than noise. This sheds doubt about the nature of such equilibria as they can well arise in the limit of no correlation of idiosyncratic fundamentals under the same information structure. Whether originating in common noise or fundamentals, aggregate fluctuations in these limiting cases will remain unexplained in the eyes of an econometrician, who can only observe measurable characteristics of fundamentals, and thus will attribute such fluctuations to changes in "sentiments".

Next we show that, with a modest modification, the endogenous signal structure can also deliver sentiment equilibria that satisfy the first-order-belief irrelevance property of Angeletos and La'O (2013). There exists an equilibrium where a vanishing common shock (be it noise or fundamental) affects aggregate actions while leaving first-order beliefs unchanged, but also another equilibrium in which the small common shock has no effect on either beliefs or actions. Provided the common component in the signal is noise and its variance is sufficiently

[3]Benhabib et al. (2015) call the fluctuation-free equilibrium the *fundamental* equilibrium. To avoid confusion, we call the equilibrium without fluctuations induced by sentiments the *sentiment-free* equilibrium.

large, however, both first-order-belief irrelevance and the uniqueness of equilibrium obtain as in Angeletos and La'O (2013).

We finally examine the out-of-equilibrium properties of the rational expectations equilibria we have found, both in and out of the limit. We consider two equilibrium selection techniques, higher-order belief stability and adaptive learnability, that have gained prominence in both microeconomics and other areas of macroeconomics. The first assumes that agents behave like game-theorists who try to rationalize the response of others exploiting common knowledge of the model and rationality; the latter assumes agents act like econometricians who learn about the endogenous precision of the signal by regressing past observations as the economy is repeated through time. We show that the limiting sentiment equilibria do not survive either of these tests while the limiting sentiment-free equilibria do. Outside of the limit of vanishing common noise, however, multiple equilibria can be locally unique rationalizable outcomes, including an equilibrium with a large, albeit finite, multiplier on common noise.

Our learnability results contrast with the original stability analysis in Benhabib et al. (2015). In their approach, agents treat the signal as exogenous, conjecturing a common precision and then updating dynamically. In our characterization instead, agents' learning incorporates the endogenous relationship between the signal precision and the average action and this endogeneity generates a coordination issue not contemplated by Benhabib et al. (2015). The differences in our results suggest that apparently small differences in the micro foundations of sentiment-like fluctuations can lead to very different conclusions about their stability, and thus deserve close attention in this literature.

In addition to the sunspot and sentiment literature cited above, this paper belongs to a long literature that studies coordination games with incomplete information. Angeletos and Pavan (2007) characterize equilibria and the welfare consequences of exogenous signal structures. Amador and Weill (2010), Vives (2012), and Manzano and Vives (2011), among others, consider imperfect information models in which the endogeneity of signals plays an important role, including in generating multiple equilibria. Gaballo (2015) shows that information transmitted by prices can originate learnable dispersed-information equilibria in the limit of zero cross-sectional variance of fundamentals, where a non-learnable perfect-information equilibrium also exists. Recent work by Bergemann and Morris (2013) characterizes the full-set of incomplete information equilibria of similar coordination games, although it does not study their out-of-equilibrium properties. The studies of Hassan and Mertens (2011, 2014) have shown that arbitrarily small deviations from rational expectations can generate non-trivial aggregate consequences, in a manner that resembles the multiplier effect we demonstrate.

2 Noisy Multiple Equilibria

This section introduces the simple abstract version of the model of sentiment-driven multiplicity, as presented in Benhabib et al. (2015). Our innovation is to incorporate a common noise term within the endogenous signal structure that they consider. This innovation allows us to derive a well-defined best response function in the resulting dispersed information game and to characterize cases of both unique and multiple equilibria.

2.1 A Reduced-Form Model

A continuum of expected utility maximizing agents, indexed by $i \in (0,1)$, have utility $-(y_i - \epsilon_i)^2$, where $y_i \in \Re$ denotes the action of agent type i and $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ is an exogenous *idiosyncratic* fundamental, assumed to be iid across agents. Before choosing her action, agent i receives a signal s_i of the fundamental. The optimal action of agent i is given by

$$y_i = \mathrm{E}\,[\epsilon_i |\, s_i]. \tag{1}$$

Following the analogy to Benhabib et al. (2015), we refer to the aggregate action is given by $y = \int y_i di$ as aggregate output.

Notice that we have specified equation (1) so as to eliminate all interdependency in agents' payoffs. This has the advantage of focusing our analysis on interactions which occur because of the endogenous nature of information. All of the basic insights that follow do not depend on complementarity (positive or negative) in actions.

The signal that agent i receives is a linear combination of her own idiosyncratic state and a potentially noisy indicator of the aggregate action in the economy:

$$s_i = \lambda \epsilon_i + (1 - \lambda)(y + \zeta). \tag{2}$$

Here, $\lambda \in (0, 1)$ denotes the signal weight on the private fundamental and $\zeta \sim N\left(0, \sigma_\zeta^2\right)$ is an error term that is common across agents. Benhabib et al. (2015) microfound their signal structure as arising from a survey in which consumers must forecast their own demand for a firm i's goods. The signal structure here could be rationalized similarly, with the addition of a correlated error term among consumers' forecasts. The choice to explicitly model this error term is, conceptually, our only departure from Benhabib et al. (2015).[4]

2.2 Endogenous Information and Optimal Weight

Given her signal, s_i, which depends on the aggregate action, agent i must infer her own private state. The key feature of the resulting signal extraction problem is that the precision of the signal depends on the nature of average action across the population and, in particular, on the average reaction to the same signal. This is a typical property of endogenous signals. An equilibrium is therefore a situation in which the individual reaction to the signal is consistent with its actual precision, i.e. is an optimal response to the average reaction of others, which each individual takes as given.

We now turn to the task of characterizing the equilibria of the economy. Since we assume that all stochastic elements are normal, the optimal forecasting strategy is linear. As a consequence the individual action is linear in s_i and can be written as

$$y_i = a_i \left[\lambda \epsilon_i + (1 - \lambda)(y + \zeta)\right], \tag{3}$$

where a_i is the coefficient measuring the strength of the reaction of agent i to the signal she receives s_i. Since the signal is ex-ante identical for all agents, each uses a similar strategy,

[4] Adding additional *idiosyncratic* noise to the signal, s_i, does not qualitatively change any of our results.

and we can recover the average action by integrating across agents:

$$y = a\,(1-\lambda)\,(y+\zeta)\,, \tag{4}$$

with $a \equiv \int a_i di$ denoting the average weight applied on the signal.[5] Solving the expression above for the average action yields

$$y = \frac{a\,(1-\lambda)}{1-a\,(1-\lambda)}\zeta, \tag{5}$$

which is a non-linear function of the average weight placed on the signal by agents. Importantly, this function features a singularity at the point $1/(1-\lambda)$. When $a < 1/(1-\lambda)$ the average action is positively correlated with the common noise, whereas the opposite holds when $a > 1/(1-\lambda)$.

The variance of the aggregate action is then given by

$$\sigma_y^2(a) = \left(\frac{a\,(1-\lambda)}{1-a\,(1-\lambda)}\right)^2 \sigma^2. \tag{6}$$

where σ_y^2 and $\sigma^2 \equiv \sigma_\zeta^2/\sigma_\epsilon^2$ are the variance of the aggregate action and the variance of the common noise respectively, both being normalized by the variance of the idiosyncratic fundamental. For a given σ^2, the volatility of the average action is increasing in the average weight with $a < 1/(1-\lambda)$ and conversely when $a > 1/(1-\lambda)$.

Substituting the average action in (5) into the signal described in equation (2), we get an expression for the agent's signal exclusively in terms of the idiosyncratic and common shocks, as governed by the average response a:

$$s_i = \lambda\epsilon_i + \frac{1-\lambda}{1-a\,(1-\lambda)}\zeta. \tag{7}$$

Notice that the closer the average weight is to the value $1/(1-\lambda)$ the lower is the precision of the signal with regard to ϵ_i, and that the effect of deviations of a is symmetric. Since the average weight a determines the precision of the signal, rational expectations requires that

[5]Keep in mind that a_i and s_i do not covary as the former is a strategy fixed prior to the realization of uncertainty.

agent's own weighting of the signal is also function of a. We are now ready to compute such an optimal response.

Taking the average weight as given, it is straightforward to work out an expression for the optimal individual weight a_i such that $\mathrm{E}[s_i(\epsilon_i - a_i s_i)] = 0$, i.e. the covariance between signal and forecast error is zero in expectation. This condition implies that the information is used optimally. The best individual weight is given by

$$a_i(a) = \frac{\lambda\left(1 - a\left(1-\lambda\right)\right)^2}{\lambda^2\left(1 - a\left(1-\lambda\right)\right)^2 + \sigma^2\left(1-\lambda\right)^2}, \tag{8}$$

which is a function of the average weight.

Given the linear-quadratic environment, we can interpret $a_i(a)$ in a game-theoretic fashion as an individual best reply to the profile of others' actions summarized by a sufficient statistic a. To be precise, every a_i is associated to one and only one contingent strategy that provides for an action $y_i = a_i s_i$ contingent to the realization of s_i, which identifies a set of states of the world indistinguishable to the agent i.

2.3 Noisy Equilibria

Given that agents face an informational structure with the same stochastic properties, an equilibrium has to be symmetric. This last requirement completes our notion of equilibrium which is formally stated below.

Definition 1. *A noisy rational expectations equilibrium (REE) is characterized by a value $\hat{a}$ such that $a_i\left(\hat{a}\right) = \hat{a}$ for each i.*

Our game-theoretic interpretation of the optimal coefficient makes clear the equivalence between a rational expectations equilibrium and a Nash equilibrium: none has any individual incentive to deviate when everybody else conforms to the equilibrium prescriptions.

An equilibrium of the model is given as a fixed-point of the individual best weight mapping. In practice, there are as many equilibria as intersections between $a_i(a)$ with the bisector. Figure 1 illustrates the equilibrium condition graphically. The fixed-point relation delivers a cubic equation, which may have one or three real roots. To the aim of the paper,

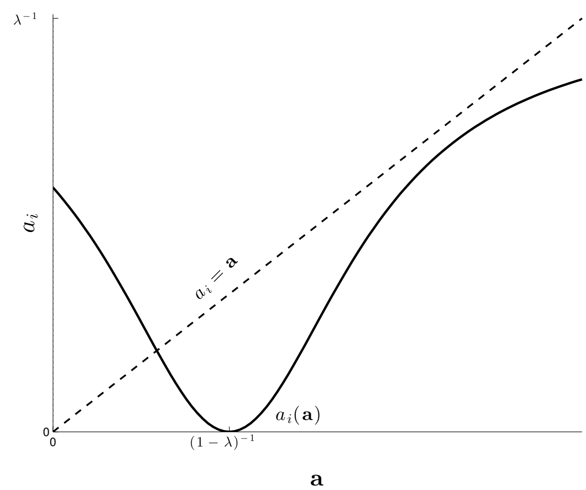

Figure 1: When common noise is large enough, a single Nash equilibrium a_- exists.

it is useful to provide a taxonomy of the equilibria in terms of the local properties of the individual best weight function.

Definition 2. *An equilibrium $\hat{a}$ exhibits* ***complementarity in information*** *if the optimal individual weight is marginally increasing in the average weight, i.e. $a_i'(\hat{a}) > 0$. Alternatively, agents face* ***substitutability in information*** *if $a_i'(\hat{a}) < 0$.*

An equilibrium featuring complementarity (substitutability) in information is such that a marginal increase in the average weight would imply a higher (lower) individual average weight. This property is key to the comprehension of the local dynamics behind the possibility that agents can actually converge to the equilibrium from an out-of-equilibrium initial condition.

We can now characterize the equilibria in our economy. Proposition 1 establishes that when the endogenous signal weights idiosyncratic conditions strongly-enough, a unique noisy equilibrium exists. Moreover, in the limit of small variance in the common noise term, this equilibrium converges to a point with zero aggregate fluctuations.

Proposition 1. *Suppose that $\lambda \geq 1/2$. For any for any σ^2, there exists a unique equilibrium characterized by a_u. Moreover, $\lim_{\sigma^2 \to 0} a_u = \lambda^{-1}$ and $\lim_{\sigma^2 \to 0} \sigma_y^2(a_u) = 0$.*

Proof. Given in appendix. ∎

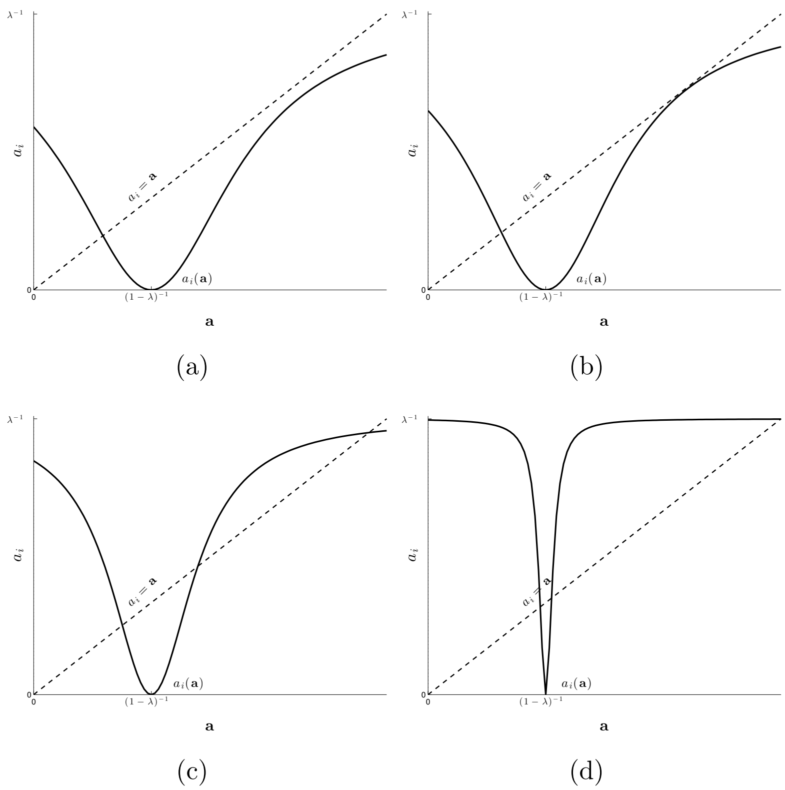

Figure 2: Uniqueness and multiplicity for different variances of common noise.

Proposition 2 proves instead that, when the aggregate component has relatively high weight in the signal, the model may exhibit a multiplicity. In particular, there are three equilibria whenever $\lambda < 1/2$ and the variance of the aggregate error is small enough. For the remainder of this section we maintain the assumption that $\lambda < 1/2$, whenever not specified otherwise.

Proposition 2. *Suppose that $\lambda < 1/2$. For any for any σ^2, there exists a low equilibrium $a_- \in (0, (1-\lambda)^{-1})$. Moreover, there exists a threshold $\bar{\sigma}^2$ such that, for any $\sigma^2 \in (0, \bar{\sigma}^2)$ a medium equilibrium $a_\circ$ and a high equilibrium a_+ exist in the range $((1-\lambda)^{-1}, \lambda^{-1})$.*

Proof. Given in appendix. ∎

A few observations are warranted. First, the equilibrium a_- is characterized by strategic substitutability in information, while the equilibria $a_\circ$ and a_+ are characterized by information complementarity. To see this compute, compute the derivative of a_i with respect to

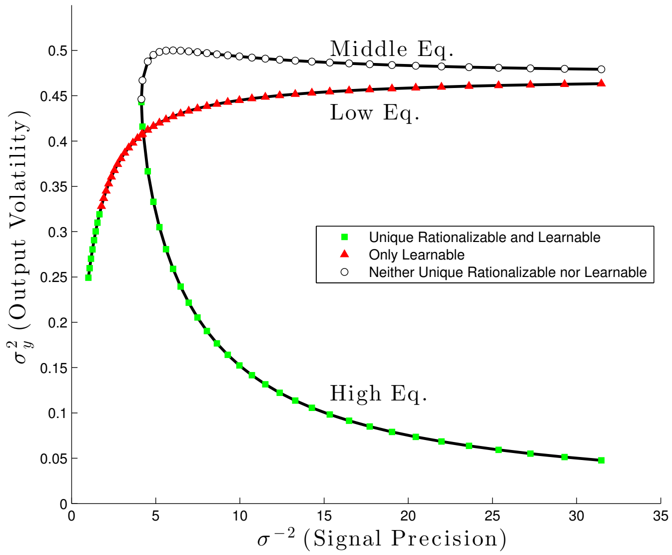

Figure 3: Volatilities and equilibrium stability approaching the limit.

a:

$$a_i'(a) = -\frac{2\lambda (1-\lambda)^3 (1 - a(1-\lambda))\, \sigma^2}{\left((1-\lambda)^2 \sigma^2 + \lambda^2 (1-(1-\lambda)a)^2\right)^2}. \tag{9}$$

The denominator of this expression is always positive, which implies that that $a_i'(a)$ is positive whenever $a > 1/(1-\lambda)$, and negative otherwise.

While an analytical characterization of the three equilibria is possible, the expressions themselves are rather complicated. The properties of these equilibria are of interest, however, and proposition 3 establishes an important relationship between the aggregate variances implied by different equilibria.

Proposition 3. *For any given σ^2 such that a multiplicity exists, it is always the case* $\sigma_y^2(a_\circ) \geq \sigma_y^2(a_+)$.

Proof. Given in appendix. ∎

Panel (b) of figure 2 shows the emergency of multiplicity, as the $a_i(a)$ function becomes tangent with the bisector. At this moment, the variances of output in the two equilibria $a_\circ$ and a_+ coincide. Figure 3 shows how these aggregate variance evolve apart as the size of the common error shrinks.

3 Sentiments in the Limit: Noise or Fundamentals?

This section characterizes the sentiment equilibria uncovered in Benhabib et al. (2015) as limit noisy equilibria emerging with a zero-variance of the common noise in the signal. Nevertheless, we will demonstrate that an equivalent characterization is possible when the common component in the signal has a fundamental nature, i.e. it is a correlated component of fundamentals with vanishing variance. This result leaves open the issue concerning the nature of such equilibria.

3.1 Sentiment Equilibria as Limit *Noisy* Equilibria

We want to study the different equilibria in the important limit that σ^2 goes to zero. That is, we want to consider the case in which the measurement error in the aggregate action (or sentiment) goes to zero. The analysis of Benhabib et al. (2015), in contrast, occurs at the limit point rather than approaching it. We show here that considering the limit of our model sheds considerable light on the equilibria they study.

The panels of figure 2 show that, as σ^2 becomes small, the low and medium equilibria converge to a single equilibrium outcome with sizable aggregate volatility, while high equilibria converges to a point with no aggregate fluctuations. The following proposition establishes the result formally.

Proposition 4. *Suppose $\lambda < 1/2$. In the limit $\sigma^2 \to 0$, the low and middle equilibria converge to the same limit "sentiment" equilibrium which is characterized by $a_{\circ,-} = (1-\lambda)^{-1}$ and exhibits sizable aggregate volatility*

$$\lim_{\sigma^2 \to 0,} \sigma_y^2(a_{\circ,-}) = \frac{\lambda(1-2\lambda)}{(1-\lambda)^2}, \tag{10}$$

while the high equilibrium converges to the "sentiment-free" equilibrium which is characterized by $a_+ = \lambda^{-1}$ and exhibits no aggregate fluctuations.

Proof. Given in appendix. ∎

One implication of this analysis is that the addition of small amount of *aggregate* noise in the signal can introduce an additional equilibria. A previous literature has demonstrated cases where adding idiosyncratic noise to signals can either eliminate (Morris and Shin, 1998) or generate additional equilibria (Gaballo, 2015). But this is the first time it has been observed, to our knowledge, that adding aggregate noise can cause equilibria to proliferate.

3.2 Sentiment Equilibria as Limit *Fundamental* Equilibria

It turns out that assuming that the aggregate term in the signal given by (2) is noise is not essential to generating sentiment-like equilibria. To prove that, consider a version of the model in which agents have utility function $-(y_i - \epsilon_i - \zeta)^2$ where $\zeta \sim N\left(0, \sigma_\zeta^2\right)$ now represents a common exogenous *fundamental.* The individual optimal action

$$y_i^f = \mathrm{E}\left[\epsilon_i + \zeta \right| s_i], \tag{11}$$

is, as before, conditional to the signal

$$s_i^f = \lambda(\epsilon_i + \zeta) + (1 - \lambda)\, y^f, \tag{12}$$

that, in this case, does not embed any noise. Nonetheless, correlated fundamentals generate confusion between the idiosyncratic and common component of the signal. As before, the signal embeds the average action y^f which makes endogenous the precision of the signal to the average weight. Following the analysis of the earlier section, the realization of the signal can be rewritten as

$$s_i^f = \lambda\epsilon_i + \frac{\lambda}{1 - a(1 - \lambda)}\zeta. \tag{13}$$

where a represents the average weight placed on the signal by other agents. The variance of the average action y^f is now given by

$$\sigma_{y^f}^2 = \left(\frac{\lambda a}{1 - a(1 - \lambda)}\right)^2 \sigma^2, \tag{14}$$

which is slightly different from (6). As before, agents take the average weight as given, when setting their optimal individual weight a_i^f such that $\mathrm{E}[s_i(\epsilon_i + \zeta - a_i^f s_i^f)] = 0$, which implies the best individual weight

$$a_i^f(a) = \frac{1}{\lambda}\left(\frac{(1 - a(1 - \lambda))^2 + (1 - a(1 - \lambda))\,\sigma^2}{(1 - a(1 - \lambda))^2 + \sigma^2}\right). \tag{15}$$

While the best response function in equation (15) is slightly different than that of equation (8) for the exogenous noise case, we can prove that the characterization of the limit equilibria is identical.

Proposition 5. *Suppose $\lambda < 1/2$. In the limit $\sigma^2 \to 0$, there exist a limit "sentiment" equilibrium for $a = (1-\lambda)^{-1}$ which exhibits sizable aggregate volatility*

$$\lim_{\sigma^2 \to 0,} \sigma^2_{y^f}((1-\lambda)^{-1}) = \frac{\lambda(1-2\lambda)}{(1-\lambda)^2}, \tag{16}$$

and a limit "sentiment-free" equilibrium for $a = \lambda^{-1}$ which exhibits no aggregate fluctuations.

Proof. Given in appendix. ∎

In general, it is easy to show that propositions 1 through 4 follow identically, and their proofs in parallel with only the obvious algebraic substitutions. Most importantly, the best response function $a_i^f(a)$ converges in the limit to $a_i(a)$, the best-response function of the model with common noise. In other words, the limiting equilibria exhibit the same degree of complementarity in information irrespective of their characterization. As a consequence, the stability properties of limit equilibria that we will study in section 5 will be identical regardless of the "source" of the sentiment shock.

The observation that sentiment-like fluctuations might be driven by imperceptible changes in fundamentals is new, and suggests that maintaining a strict dichotomy between "sentiment" fluctuations and "fundamental" fluctuations may be misleading. Since endogenous signal structures can generate strong multiplier effects on small shocks, they deliver fluctuations that can effectively span a continuum from purely fundamental-driven to purely sentiment-driven. Of course, this possibility do not preclude the existence of fluctuations that originate from truly payoff-irrelevant shocks, but the possibility of fundamental-based sentiments may appeal to those who find such fluctuations implausible.

4 Second-order Sentiments

In this section we show how to exploit the structure explored above to build up sentiment equilibria which integrate the features of Benhabib et al. (2015) and Angeletos and La'O

(2013). In particular, the original payoff can be modified so that we can obtain equilibria where common noise perturbs the average action and second order beliefs, without affecting first-order beliefs. Still, the variance of the common noise governs the existence and potential multiplicity of limit equilibria.

4.1 Introducing Strategic Interactions

Angeletos and La'O (2013) propose an alternative model of sentiment-driven fluctuations. Their model of sentiments is distinctive because, for them, sentiment shocks cause aggregate fluctuations without any effect on first-order beliefs; i.e. sentiment fluctuations are driven entirely by a complementarity in actions and agents expectations regarding other agents. For their leading case, the authors use an exogenous information structure in which errors about higher-order beliefs are *assumed* to be correlated via a non-trivial common shock to expectations.

To introduce second-order sentiment fluctuations, we must also introduce strategic interactions among agents. The structure that we adopt is motivated by the random matching environment of Angeletos and La'O (2013), but is simplified to maintain tractability. The economy consists of a continuum of islands indexed by $i \in (0, 1)$, each populated by two agent types. Type one agent on island i has utility function $-(d_i - \epsilon_i)^2$ where $d_i \in \Re$ denotes her individual action and $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ is an island-specific fundamental iid across islands. Agent type one on island i chooses her action based on an information set that contains only the private signal $x_i = \epsilon_i + \eta_i$, where $\eta_i \sim N(0, \sigma_\eta^2)$. The action of type one agents is therefore given by

$$d_i = E[\epsilon_i | x_i]. \tag{17}$$

Type two agent on island i has utility function $-(y_i - \epsilon_i - d_i)^2$, that is she fixes her action

$$y_i = E^i \left[\epsilon_i + d_i\right]. \tag{18}$$

in $\Re$, to track the island specific state, but also respond to their beliefs about the actions of type one agents. Type two agents directly observe the island-specific fundamental ϵ_i but not the action of the agent type one in her own island. Nevertheless, agents of type two observe a signal that, in analogy to the endogenous signal of section 2, mixes a noisy measure of the aggregate action with island-specific action of type one agents:

$$s_i = \lambda d_i + (1-\lambda)(y+\zeta). \tag{19}$$

Notice that this information structure no longer commingles idiosyncratic fundamentals with an endogenous aggregate, but instead combines the idiosyncratic action of type-one agents on island i with the average action. In this case uncertainty is driven only by the inability of type two agents to disentangle the situation of the aggregate economy from the departure of beliefs of type one agents from the true fundamental ϵ_i.

4.2 Characterization of Equilibria

Here we show that our endogenous information structure with an arbitrarily small common noise can lead to equilibria with exactly the sort of fluctuations modeled by Angeletos and La'O (2013) using exogenous information.

Let $\gamma = \sigma_\epsilon^2/(\sigma_\epsilon^2 + \sigma_\eta^2)$ be the optimal inference coefficient for type one agents. Then their action, denoted d_i, is just

$$d_i = \gamma x_i = \gamma \epsilon_i + \gamma \eta_i. \tag{20}$$

Since type two agents directly observe ϵ_i, the signal s_i is informationally-equivalent to observing

$$s_i = s_i - \lambda\gamma\epsilon_i = \lambda\gamma\eta_i + (1-\lambda)(y+\zeta). \tag{21}$$

The linear strategy of a type two agent can then characterized by the coefficient a_i according to

$$y_i = (1+\gamma)\epsilon_i + a_i s_i. \tag{22}$$

Given symmetry, we have an expression for the aggregate action analogous to equation (5)

$$y = \frac{a(1-\lambda)}{1-a(1-\lambda)}. \tag{23}$$

and the signal becomes

$$s_i = \lambda\gamma\eta_i + \frac{1-\lambda}{1-a(1-\lambda)}\zeta. \tag{24}$$

The type two agents seeks to forecast $\gamma\eta_i$, and the optimal forecast given s_i is given by

$$a_i(a) = \frac{\lambda\left(1-a\left(1-\lambda\right)\right)^2}{\lambda^2\left(1-a\left(1-\lambda\right)\right)^2+\sigma^2\left(1-\lambda\right)^2}. \tag{25}$$

Equation (25) thus represents the best-response function of an individual agent i, given the average action a, and it can be analyzed in the exactly the same way as before.

Definition 3. *A higher-order noisy REE is noisy first-order equilibrium where common noise does not affect first-order beliefs on fundamentals.*

The key difference is that now the first order beliefs of both agents type one and two about the fundamental ϵ_i are orthogonal to the impact of ζ. The following proposition follows directly from the equivalence established above.

Proposition 6. *The set of higher-order noisy equilibria is characterized by the same $\hat{a}$ values and exhibit the same qualitative properties of the noisy equilibria uncovered in section 2.*

Proof. The result follows from a comparison of equations (8) and (25) ∎

With endogenous information the equilibria with sentiment fluctuations need not be unique. In particular, uniqueness is recovered with sufficiently large common noise, while a sentiment-free equilibrium always exists when common noise approaches zero.

When a unique equilibrium obtains, this has the same feature of the equilibrium characterized in Angeletos and La'O (2013). Nevertheless, our formalization gives an explicit interpretation to the sentiment as a common error in the measurement of the average action. Moreover we also have shown the possibility that a sentiment equilibrium can coexist with a sentiment-free equilibrium, as in Benhabib et al. (2015), with sentiments concerning only higher-order beliefs as in Angeletos and La'O (2013).

In section 3.2, we extended the our framework to deliver sentiments based fundamentals as well as noise. The analysis also applies here, with one caveat: when the common variation in the signal is both non-trivial and of fundamental nature, then first-order expectations move in every equilibrium. Common noise instead, regardless of its variance, leaves first-order expectations unchanged. In the limiting case, however, the noisy and fundamental equilibria once again converge and all have the first-order expectation neutrality property of Angeletos and La'O (2013).

5 Stability of Sentiments

This sections exploits the characterization of the individual best response function in the game implied by dispersed information to examine the stability of sentiment equilibria under two popular out-of-equilibrium beliefs dynamics: rationalizability and adaptive learning. We will show that only the sentiment-free limit equilibrium exhibits strong stability properties whereas sentiment-like equilibria are generally excluded by these tests.

5.1 Higher-Order Belief Dynamics

Any REE can be seen as a Nash equilibrium of a coordination game where holding an expectation that a certain REE will emerge is the "rational" expectation *if and only if* all other hold the same. More generally an individual "rational" expectation is a function of given state of others beliefs.

In our case a "rational"expectation is characterized by a mapping $a_i(a) : \Re \rightarrow \Re$ which associates to any value of the average weight a an individual best weight $a_i(a)$. A REE is an equilibrium weight $\hat{a}$ such that $a_i(\hat{a}) = \hat{a}$ for each $i \in (0, 1)$. That $\hat{a}$ reflect the precision of the endogenous signal at the equilibrium. But how can people achieve common knowledge that others will conform to equilibrium prescription so that $\hat{a}$ is actually the correct weight?

This is an old question on the epistemic foundations of Nash equilibrium with an important tradition in decision theory. A widely accepted concept is the one of rationalizable set (Bernheim, 1984; Pearce, 1984) defined as the profile set surviving to iterated deletion of never best replies. This criterion exploits implications from common knowledge of rationality in the model.

Guesnerie (1992) introduces the rationalizability argument to macroeconomics in the context of complete information competitive economies. Here we adapt Guesnerie's original setup in a dispersed information model focusing on the best-expectation coordination game entailed by the maps $\{a_i(a)\}_{i\in(0,1)}$. In contrast to the original Guesnerie setting, here agents agree on the unconditional expectation of their idiosyncratic fundamental which is exogenous to the average behavior. Nevertheless, they are uncertain about the precision of the information they are looking at. So nothing in the model guarantees that agents use the same conditional distribution to forecast their idiosyncratic fundamental. In the following we will check whether the assumption of common knowledge is sufficient to restrict the agents' strategic space to the REE prescriptions.

Initially we will take a local point of view. Suppose it is common knowledge that the individual weights on the signal lie in a neighborhood $\mathcal{F}(\hat{a})$ of $\hat{a}$, is this a sufficient condition for convergence in higher-order beliefs to the $\hat{a}$? The process of iterated deletion of never best replies works as follows. Let index by τ the iterative round of deletion. If $a_{i,0} \in \mathcal{F}(\hat{a})$ for each i then $a_0 \in \mathcal{F}(\hat{a})$. Nevertheless, the latter implies that a second order beliefs is justified for which $a_{i,1} = a_i(a_0)$ for each i, so that $a_{i,1} \in a_i(\mathcal{F}(\hat{a}))$. As a consequence $a_1 \in a_i(\mathcal{F}(\hat{a}))$. One can iterate the argument showing that $a_{i,\tau} \in a_i^{\tau}(\mathcal{F}(\hat{a}))$. Hence we have the following.

Definition 4. *A REE $\hat{a}$ is a locally unique rationalizable outcome if and only if there exists a neighborhood $\mathcal{F}(\hat{a})$ of $\hat{a}$ such that* $\lim_{\tau\to\infty} a_i^{\tau}(\mathcal{F}(\hat{a})) = \hat{a}$.

When a REE is a locally unique rationalizable outcome we can conclude that the equilibrium is stable to a sufficiently small higher-order beliefs perturbation (or is eductively stable in Guesnerie's language). In other words, the equilibrium is robust to beliefs that others could locally deviate from it, as agents conclude that no rational conjecture can sustain such a deviation.

A global qualification of the higher-order belief stability criterion obtains when the best response function entails a contraction for each point of the domain of a, that is when $\lim_{\tau\to\infty} a_i^{\tau}(\Re) = \hat{a}$. When an equilibrium is the globally unique rationalizable outcome, then this is the only profile of strategies that rational agents will play. In this sense the theory provides a complete out-of-equilibrium belief dynamics converging to the unique equilibrium.

On the other hand, notice that uniqueness of a REE is not sufficient to guarantee stability to even arbitrarily small perturbation to higher-order beliefs.

Belief convergence requires that $a_i(a)$ entails a contracting map. For a locally rationalizable REE a necessary and sufficient condition is $|a_i'(\hat{a})| < 1$. The proposition below states the result.

Proposition 7. *The low equilibrium is a locally unique rationalizable equilibrium provided σ is large enough. Whenever the middle and the high equilibria exists the latter is always a locally unique rationalizable equilibrium, whereas the former is never. In the limit of $\sigma \to 0$ the middle and the low equilibria are never stable under higher-order beliefs dynamics.*

Proof. Given in appendix. ∎

One can easily show that $a_\circ$ is never a locally unique rationalizable outcome from qualitative properties associated to the equilibria. First, $a_i(1-\lambda)^{-1} = 0$ lies below the forty-five degrees lines. Second, for $a > (1-\lambda)^{-1}$ the the best weight function is always monotonically increasing. This two observations taken jointly require $a_i'(a_\circ) > 1$, and thus are sufficient to claim that whenever the middle equilibrium $a_\circ$ exists, then it is not a locally unique rationalizable outcome.

A second easy result is that whenever a_+ exists distinct from $a_\circ$, it is always a locally unique rationalizable outcome since the first derivative at this equilibrium has to be bounded in $(0, 1)$ to meet the 45 degrees line. In knife-edge case that $a_\circ = a_+$ the fix point map is tangent to the bisector, meaning $a_i'(a_+) = 1$, which does not satisfy the condition for rationalizability.

To establish the convergence properties of a_-, one needs to check that there is a threshold $\underline{\sigma}$ such that for any $\sigma \in (0, \underline{\sigma})$ this equilibrium is not locally rationalizable, whereas it is otherwise. That this is the case is clear from figure 2; the detailed proof is postponed to the appendix.

To give an intuition notice that in the case of the two limit equilibrium outcome we have $\lim_{\sigma\to 0} a_- = \lim_{\sigma\to 0} a_\circ = (1-\lambda)^{-1}$ for which the derivative obtains as $\lim_{\sigma\to 0} a_i' = \pm\infty$. On the other hand $a_i'(a_-)$ increases in σ with 0 as upper bound so that there exists a $\underline{\sigma}$ such that for any $\sigma > \underline{\sigma}$ the low a_- is always rationalizable. Therefore there could be a multiplicity (two) of rationalizable REE for intermediate values of $\sigma \in (\underline{\sigma}, \bar{\sigma})$. To understand if this is the case we perform the following numerical analysis.

Figure 3 illustrates the size of output volatility generated by the three equilibria (whenever they exist) as a function of the inverse of σ. For σ high enough only the low equilibrium exists. The output volatility generated at that equilibrium is monotonically decreasing in σ. The low equilibrium is locally unique rationalizable outcome provided σ is sufficiently large. With sufficiently low σ, the middle and the high equilibrium exist too. The latter is always a locally unique rationalizable outcome, whereas the former is never.

Notice that in the example illustrated in figure 3 there is no region in which multiple locally unique rationalizable outcomes exist. Moreover, there exists a region in which the low equilibrium is the only equilibrium, but it is not a locally unique rationalizable outcome. Finally, only for sufficiently small σ can a globally unique rationalizable outcome arise, originating in the “low” equilibrium.

In figure 4, we show through numerical investigation the relation that for sufficiently low values of λ there is a region in which two equilibrium, the high and the low, emerge as locally unique rationalizable outcomes. Nevertheless this would not arise in the limit of infinite precision where only the high equilibrium remains locally unique rationalizable equilibrium.

5.2 Adaptive Learning

To address the question of learnability of the rational expectation equilibria we have analyzed, we now suppose that agents behave like econometricians, rather than game-theorists. That is, agents individually set their weights consistently with data generated by possibly out-of-equilibrium replications of the signal extraction problem, without internalizing the effect of the ongoing process of learning in the economy. In practice, at time t they set a weight $a_{i,t}$ which is estimated from the sample distribution of signals collected from past repetition of the signal extraction problem. If agents are close to a locally adaptively stable equilibrium,

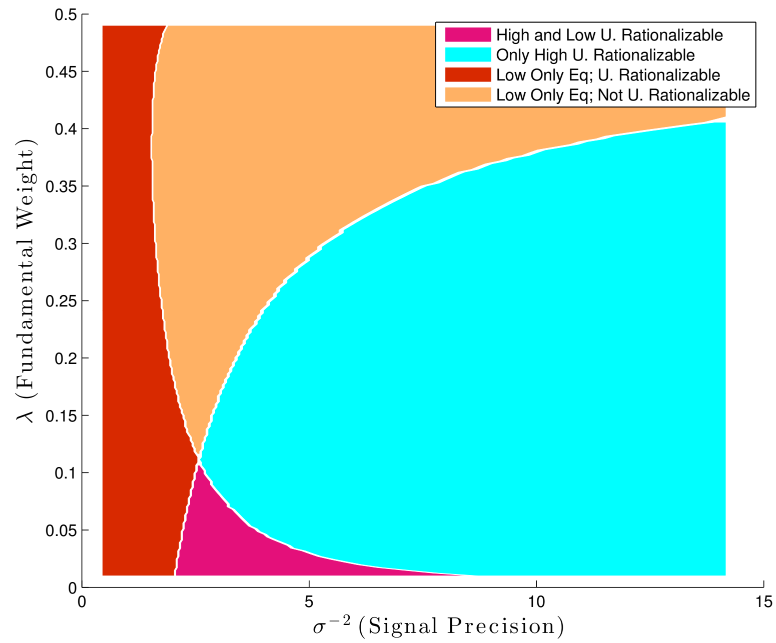

Figure 4: Stability properties for differing signal weights.

this implies that once estimates are close enough to the equilibrium values they will almost surely converge to the equilibrium.

It has been shown that the asymptotic behavior of statistical learning algorithms can be studied by stochastic approximation techniques (for details refer to Marcet and Sargent (1989b,a) and Evans and Honkapohja (2001). To see how this works in our context, consider the case that agents learn about the optimal weight according to an optimal adaptive learning scheme:

$$a_{i,t} = a_{i,t-1} + \gamma_t \; S_{i,t-1}^{-1} \; s_{i,t} \left(\eta_{i,t} - a_{i,t-1} s_{i,t}\right) \tag{26a}$$

$$S_{i,t} = S_{i,t-1} + \gamma_{t+1} \left(s_{i,t}^2 - S_{i,t-1}\right), \tag{26b}$$

where γ_t is a decreasing gain with $\sum \gamma_t = \infty$ and $\sum \gamma_t^2 = 0$, and matrix $S_{i,t}$ is the estimate variance of the signal rewritten with a convenient time index. The following formally define adaptive stability.

Definition 5. *A REE $\hat{a}$ is a locally learnable equilibrium if and only if there exists a neighborhood $\digamma(\hat{a})$ of $\hat{a}$ such that, given an initial estimate $a_{i,0} \in \digamma(\hat{a})$, it is $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$.*

Adaptive learning provides an out-of-equilibrium dynamics which can explain how agents can (or fail to) converge to REE equilibrium. The collective use of statistical techniques, although it does not account for the fully fledged effect of collective learning, can entail a situation in which agents' estimates about the precision of the signal are correct. Such a convergence point is necessarily a REE.

The global qualification of the learnability criterion is obtained when convergence occurs almost surely irrespective of any initial condition, that is $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$ for any $a_{i,0} \in \Re$. Notice that, differently from the rationalizability, there could exists a unique globally learnable equilibrium despite the existence of multiple rational expectation equilibria. This is because the stochasticity of the learning process will always displace estimates temporary away from equilibrium values. Nevertheless, if there only exists one REE, then if it is learnable it has to be globally learnable.

To check local learnability of the REE, suppose we are already close to the rest point of the system. That is, consider the case $\int \lim_{t\to\infty} a_{i,t}\ di = \hat{a}$ where $\hat{a}$ is one among the equilibrium points $\{a_-, a_\circ, a_+\}$ and so

$$\lim_{t\to\infty} S_{i,t} = \sigma_s^2(\hat{a}) = \lambda^2\sigma_\epsilon^2 + \frac{(1-\lambda)^2}{(1-\hat{a}(1-\lambda))^2}\sigma_\zeta^2. \tag{27}$$

According to stochastic approximation theory, we can write the associated ODE governing the stability around the equilibria as

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \int \lim_{t\to\infty} \mathrm{E}\left[S_{i,t-1}^{-1} s_{i,t}\left(\epsilon_{i,t} - a_{i,t-1}s_{i,t}\right)\right] di = \\
&= \sigma_s^2(\hat{a})^{-1} \int \mathrm{E}\left[s_{i,t}\left(\epsilon_{i,t} - a_{i,t-1}s_{i,t}\right)\right] di = \\
&= \sigma_s^2(\hat{a})^{-1}\left(\lambda\sigma_\eta^2 - a_{i,t-1}\left(\lambda^2\sigma_\epsilon^2 + \frac{(1-\lambda)^2}{(1-a_{t-1}(1-\lambda))^2}\sigma_\zeta^2\right)\right) = \\
&= a_i(a) - a.
\end{aligned}$$

For asymptotic local stability to hold, the eigenvalues of the Jacobian of $a_i(a)$ calculated at the equilibrium have to lie inside the unit circle. The relevant condition for stability is therefore $a_i'(a) < 1$. The result is stated by the following proposition.

Proposition 8. *Whenever the middle and high equilibria exist, the latter is locally learnable, whereas the former it is not. The low equilibrium is always locally learnable, except in the limit of $\sigma \to 0$, and it is globally learnable provided σ is large enough.*

Proof. Given in appendix. ∎

In practice, we have proved that $a_i(a)$ corresponds, at least locally around the equilibrium, to the dynamic map called "projected T-map" in the adaptive learning literature. The T-map is a correspondence between the parameters used to calibrate the individual forecasting rule and the ones that would be optimal given observed data. It is an useful tool to recover information on the local out-of-equilibrium dynamics when expectations are formed recursively as information is gathered through time.

Referring to figure 2, the slope of the curves at the intersection of the bisector features the stable or unstable nature of the equilibrium. In particular, notice that the middle equilibrium defines two distinct basins of attraction for the learnable equilibria. As σ decreases the basin of attraction of the high equilibrium shrinks from below. This means that estimates are more and more likely to converge to the high equilibrium, the sentiment-free one, as σ gets smaller. At the limit $\sigma \to 0$, the low equilibrium is no longer learnable from above, meaning that for any estimate a larger than a_-, no matter how close to a_-, is fated to trigger convergence to the high equilibrium. This would suggest that although sufficiently negative shock to the estimates can lead to a persistent deviation in the lower basin of attraction of the low "sentiment" equilibrium, long run-convergence can only obtain at the high sentiment-free equilibrium.

6 Conclusion

Endogenous structures of asymmetric information can deliver strong multipliers on common disturbances and thus offer a potential foundation for a variety of sentiment-like phenomena. The multiplicity of equilibria implied by such signal structures also implies that they need not do so. Here we have demonstrated that a single analysis can address such fluctuations whether they originate in common noise or common fundamentals, and regardless of whether they impact first-order or higher-order expectations. Examination of out-of-equilibrium properties suggest that the limiting cases of pure sentiment shocks are generally not stable. However, away from the limit, the large informational multipliers of endogenous signals may be more robust to expectational perturbations.

Appendix: proofs

Proposition 1. To prove uniqueness, observe that the function $a_i(a)$ is continuous, bounded above by λ^{-1}, and monotonically decreasing in the range $(-\infty, (1-\lambda)^{-1})$. From $\lambda \geq 1/2$, we have $(1-\lambda)^{-1} > \lambda^{-1}$. Thus $a_i(a)$ intersects the 45 degree line a single time.

To prove the limiting statements, consider any point $a_\delta = \frac{1-\delta}{1-\lambda}$ such that $\delta > 0$. Then, we have

$$a_i(a_\delta) = \frac{\lambda\delta^2}{\lambda^2\delta^2 + \sigma^2(1-\lambda)^2.} \tag{28}$$

Since $\lim_{\sigma^2 \to 0} a_i(a_\delta) = \frac{1}{\lambda}$ for any δ, the unique equilibrium must converge to the same point. That the variance of this equilibrium approaches zero follows from equation (5). ∎

Proposition 2. To prove the existence of a_-, notice that $\lim_{a \to -\infty} a_i = \lambda^{-1}$ and $a_i((1-\lambda)^{-1}) = 0$. By continuity, an equilibrium $a_- \in (0, (1-\lambda)^{-1})$ must always exist. Moreover a_- must be monotonically decreasing in σ^2 as a_i is monotonically decreasing in σ^2.

We now assess the conditions under which additional equilibria may also exist. Because $\lim_{a\to\infty} a_i = \lambda^{-1}$, the existence of a second equilibria (crossing the 45 degree line in figure 1) implies the existence of a third. Thus, we need to to check whether or not the difference $a_i(a) - a$ is positive anywhere in the range $a > (1-\lambda)^{-1}$. Such a difference is positive if and only if

$$\Phi(\sigma) \equiv \lambda \left(1 - a(1-\lambda)\right)^2 (1 - \lambda a) - a(1-\lambda)^2 \sigma^2 > 0 \tag{29}$$

which requires $a < \lambda^{-1}$ as a necessary condition. Therefore, if two other equilibria exist they must lie in $((1-\lambda)^{-1}, \lambda^{-1})$. Fixing $a \in ((1-\lambda)^{-1}, \lambda^{-1})$, $\lim_{\sigma\to 0} \Phi(\sigma)$ is clearly positive, implying that there always exists a threshold $\bar{\sigma}$ such that two equilibria $a_+, a_\circ \in ((1-\lambda)^{-1}, \lambda^{-1})$ exist with $a_+ \geq a_\circ$ for $\sigma^2 \in (0, \bar{\sigma}^2)$. ∎

Proposition 3. To prove the proposition is enough to look at the derivative

$$\frac{\partial \sigma_y^2(a)}{\partial a} = \frac{2a(1-\lambda)^2 \sigma^2}{(1-(1-\lambda)a)^3} \tag{30}$$

which is negative with $a > (1-\lambda)^{-1}$. This implies that $\sigma_y^2(\hat{a})$ is decreasing in the distance $|\hat{a} - (1-\lambda)^{-1}|$. In other words the closer the $\hat{a}$ to $(1-\lambda)^{-1}$, the higher the $\sigma_y^2(\hat{a})$. ∎

Proposition 4. Recall the monotonicity of $a_i(a)$ on the range $(0, (1-\lambda)^{-1}$. Following the logic of proposition 1, for any point a_δ in that range, $\lim_{\sigma^2\to 0} a_i(a_\delta) = \lambda^{-1}$, while $a_i((1-\lambda)^{-1}) = 0$. Thus, the intersection defining a_- must approach $(1-\lambda)^{-1}$. Analogous argument for point just to the right of $(1-\lambda)^{-1}$ establishes that a_- converges to the same value. Finally, the bounded monotonic behavior of $a_i(a)$ establishes that for the high equilibrium $\lim_{\sigma^2\to 0} a_+ = \lambda^{-1}$.

The output variance of the high equilibrium in the limit $\sigma \to 0$ is zero follows from equation (6). The limiting variance of the other limit equilibrium equilibria can be established by noticing that (8) implies

$$\frac{\sigma^2}{(1-a(1-\lambda))^2} = \frac{\lambda(1-a\lambda)}{(1-\lambda)} \tag{31}$$

that plugged in (6) gives 10 for $a \to (1-\lambda)^{-1}$. ∎

Proposition 5. We can prove that a sentiment-free equilibrium with no aggregate variance exists for $a = \lambda^{-1}$ by simple substitution in (15) and (16). The limiting variance of the other limit equilibrium at the singularity $a \to (1-\lambda)^{-1}$ can be established by noticing that (15) implies

$$\frac{\sigma^2}{(1-a(1-\lambda))^2} = \frac{1-a\lambda}{a\lambda} + \frac{1-a(1-\lambda)}{a\lambda}\frac{\sigma^2}{(1-a(1-\lambda))^2},$$

which gives

$$\frac{\sigma^2}{(1-a(1-\lambda))^2} = -\frac{1-a\lambda}{1-a}.$$

that plugged in (14) gives (16) for $a \to (1-\lambda)^{-1}$. The derivative of the best reply function is

$$a_i^f(a)' = -\frac{\sigma^2(\lambda-1)}{\lambda}\left(\frac{1-(1-\lambda)^2a^2+\sigma^2}{\lambda((1-a(1-\lambda))^2+\sigma^2)^2}\right), \tag{32}$$

which in the limit $\sigma \to 0$ is zero for any $a \in (0, \lambda^{-1})$, but it goes to $\pm\infty$ for $a \to (1-\lambda)^{-1}_{\pm}$. ∎

Proposition 7. The derivative of the $a_i(a)$ is given by (9) which is positive whenever $a > 1/(1-\lambda)^{-1}$. Given that $a_i((1-\lambda)^{-1}) = 0$ then necessarily $a_i'(a_\circ) > 1$ and $a_i'(a_+) \in (0,1)$. Concerning the stability of a_- notice that $\lim_{\sigma\to\infty} a_i'(a_-) = 0$ and

$$\lim_{\sigma^2\to 0, a\to(1-\lambda)^{-1}_{\pm}} a_i'(a) = \frac{2\lambda\left(1-\lambda\right)^3\left(a\left(1-\lambda\right)-1\right)^{-1}\sigma}{\left(\left(1-\lambda\right)^2+\sigma_y^2\lambda^2\right)^2} = \pm\infty$$

where we used (10), so by continuity the thesis is proved. ∎

Proposition 8. The derivative $a_i'(a)$ at the three equilibria has been already studied. We know that $a_i'(a_+) \in (0,1)$, $a_i'(a_-) < 0$ and $a_i'(a_\circ) > 1$. Nevertheless at the limit $\sigma \to 0$ where $a_+ = a_\circ$ coincide there is no neighborhood to qualify a_+ a locally learnable REE. ∎

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