Designing Fair Tiebreak Mechanisms: The Case of FIFA Penalty Shootouts*

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Abstract

In the current FIFA penalty shootout mechanism, a coin toss decides which team will kick first. Empirical evidence suggests that the team taking the first kick has a higher probability to win a shootout. We design *sequentially fair* shootout mechanisms such that in all symmetric Markov-perfect equilibria each of the skill-balanced teams has exactly 50% chance to win whenever the score is tied at any round. Consistent with empirical evidence, we show that the current mechanism is not sequentially fair and characterize all sequentially fair mechanisms. Taking additional desirable properties into consideration, we propose and uniquely characterize a practical mechanism.

Keywords: Fairness, mechanism design, soccer, penalty shootouts, market design, axiomatic approach

JEL Codes: D63, C79, D47

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1 Introduction

Soccer is not only the leading sport in the world in terms of fan base,[1] revenue,[2] and the number of players in organized leagues,[3] it also has a profound - albeit at times negative - impact on countries and on ordinary people's daily lives. Following a World Cup elimination match between Honduras and El Salvador, soccer was blamed for instigating a 100-hour war that took place in 1969 between these two neighboring countries with devastating consequences.[4] In addition, Edmans et al (2007) report that significant market declines following soccer losses, especially after important losses such as those in World Cup matches. Thus, soccer is a social and economic phenomenon throughout the world, especially with its major national- or club-level tournaments, such as the World Cup, the European Cup, and the Champions League. Consequently, special attention is paid to its match-deciding penalty shootouts in elimination tournaments, which happen relatively frequently: Looking up the FIFA website reveals that 19.1% of the 136 World Cup knockout matches since 1974 have been decided by penalty shootouts.

Penalty shootouts currently constitute the only way to determine the winning team when the score is tied in major soccer elimination tournament matches after the regular 90-minute period and the 30-minute extra time, i.e., the overtime. It is customary to use tiebreak mechanisms in many other sports as well to determine the eventual winner when the regular match ends with a tie, e.g., tennis, ice hockey, field hockey, water polo, handball, cricket, and rugby. As will be made clear below, the current penalty shootout mechanism is deemed problematic by many. Any proposed changes to the current system should be practical, transparent, and minimal for higher probability of adoption. Currently in a shootout, each team takes five penalty kicks from the penalty mark in alternating order, and the order of the kicks is decided by the referee's initial coin toss such that the team that wins the coin toss gets to kick first in each round. This mechanism has been used since 1970 with a recent minor tweak according to which since 2003 the team that wins the toss decides which team kicks first. If the shootout score is tied after each team takes five penalty kicks, sudden-

[1]FIFA World magazine reported that "46.4 percent of the global population, saw at least one minute of in-home television coverage of [the 2010 FIFA World Cup in South Africa], representing an eight percent rise on figures recorded during the 2006 FIFA World Cup in Germany."

[2]Soccer accounts for 43% of the sports industry's annual revenue, which is estimated to exceed $600 billion as of 2009 (see Zygband and Collignon, 2011). In addition, prominent soccer teams easily compare to major conglomerates. Forbes reports that Real Madrid posted a revenue of $650 million during the 2011-12 season and is worth $3.3 billion (see Ozanian, 2013).

[3]FIFA reported that 265 million people played soccer in organized leagues in 2007 worldwide, which is more than any other sport (http://www.fifa.com/worldfootball/bigcount/).

[4]It "left several thousand dead on both sides [and] turned 100,000 people into homeless and jobless refuges" (Durham, 1979).

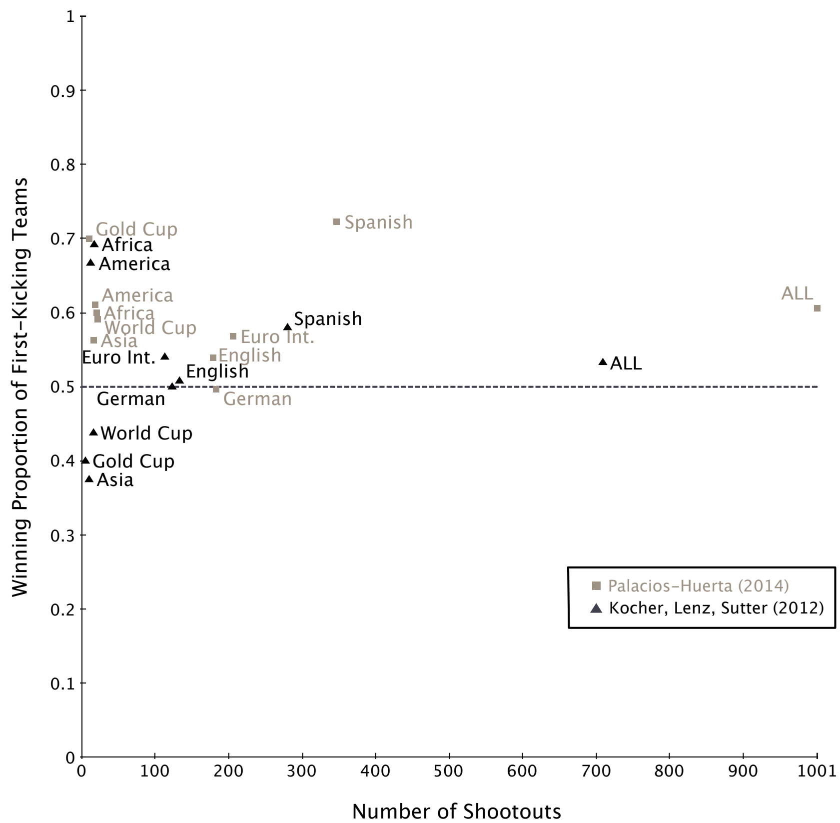

Figure 1: Empirical Evidence from Table 5.1 in Palacios-Huerta (2014) and Table 1 in Kocher, Lenz, Sutter (2012): The winning proportions of first-kicking teams are given on the vertical axis while the numbers of shootouts in the considered championships are given on the horizontal axis. Euro int refers to combined proportion for all European international championships such as European Championship, Champions League, Cup Winners Cup, and UEFA Cup.

death rounds are reached, which go on until the tie is broken such that the kicking order is preserved or "fixed" in these extra rounds.

The current fixed-order shootout mechanism is perceived to be "unfair" by soccer professionals. Apesteguia and Palacios-Huerta (2010) found that "in each and every case that [they] were able to observe with just one exception, the winner of the coin toss chose to kick first." Further, in a survey of more than 240 players and coaches in the professional and amateur leagues in Spain reported by the latter study, almost 96% indicated that they would choose to kick first after winning the coin toss. So there is a perceived first-mover advantage among soccer professionals.

Empirical evidence seems to support this perception (Figure 1). More specifically, Apesteguia and Palacios-Huerta (2010), using a dataset of 269 shootouts extending until

2008, found that the first kicking team wins significantly more often around with 60.5% probability. Kocher, Lenz, and Sutter (2012) showed that this probability is only 53% in a dataset extending until 2003 with 540 shootouts (including all shootouts the previous study used until 2003 as a subset as well).[5] Palacios-Huerta (2014), using a larger dataset (including the datasets from the previous two studies as subsets) extending until 2012 with 1001 shootouts), reported that the first-mover advantage is recovered with almost 60% probability. The common theme in these studies is that while an overall second-mover advantage definitely does not exist with increasing data size, a case can be made for a first-mover advantage.

Another observation made by these empirical studies is that the degree of how much kicking order matters may differ across different soccer competitions/traditions. For example, in all the studies mentioned above, although kicking order does not matter for the German national cups, the Spanish national cup shootouts notoriously favor first-kicking teams significantly. On the other hand, in English national cups, the first-kicking team has only a slight advantage. These three countries represent different soccer styles and player characteristics, and their datasets consist of hundreds of shootouts. Therefore, another important empirical conclusion is that in different tournament/country environments the current shootout mechanism leads to different focal outcomes in terms of first- and second-moving teams' winning chances.

Given the above empirical evidence, this paper models soccer shootouts as a mechanism design problem with a fairness desideratum in mind. Shootouts tend to be shorter and more structured than a regular match and can be modeled similarly to dynamic versions of contests. We introduce such a model in which the kickers care not only care about their team's winning the shootout but also about the outcome and even the quality of the penalty shot they personally took. Given that fairness, rather than revenue maximization and even efficiency, is the main desideratum of the design of tiebreaks, the first important question we address is what fairness means in this real-world environment, where an unbiased coin-toss determines the first-kicking team. We tackle this question from two different angles: Whenever the score is tied after any round in a penalty shootout, having two teams (1) that

[5] Although this result is not statistically significant with a p-value of 0.12 at 10% level, Figure 1 in Kocher et al. (2012) shows that the lowest this probability gets is 51.4%, which takes place during the period 1992–2003. On the other hand, at recent intervals (i.e., after 1999), the probability approaches 60%. The statistical insignificance seems to come from the fact that they only consider the first 33 years of the shootout practice. The justification stems from the fact that there was a small change in the shootout rules in 2003. Starting from 2003, the initial coin toss has been used to determine who will pick to go first or second. While before, the coin-toss was used to determine who will go first. We are aware of only very few instances since 2003, where the coin-toss winner chose to go second. On the other hand, Palacios-Huerta (2014), which covers the period 1970–2012, shows a significant shift to first-mover advantage in the years since 2003, and hence, for the entire period.

are totally balanced in terms of their players' shootout abilities, a *sequentially fair* outcome should mean that each team is expected to win the shootout with 50% probability, and (2) where one team has higher-ability kickers than those of the other team, then the higher-ability team should have a higher probability of winning. These two parts lead to nothing but the age-old Aristotelian Justice principle, which rests on a two-part criterion: equals should to be treated equally and unequals unequally (cf. Aristotle, 1999).[6]

Complying with the above empirical findings, we first show that kicking order matters to each team's chance of winning the shootout in that the current fixed-order mechanism (1) is not sequentially fair and (2) can lead to many equilibria, with different winning probabilities for first- and second-kicking teams. We show that it is possible to devise a forward-looking equilibrium refinement – similar in vein to the Intuitive Criterion of Cho and Kreps (1987) – to get rid of the multiplicity of equilibria, so that the first-kicking team wins more often. Any potential equilibrium candidate with the second-mover advantage does not survive this refinement.

We then ask whether it is possible to devise a shootout mechanism that is sequentially fair. In our characterization of sequentially fair mechanisms in regular rounds, we find that there is only one *exogenous mechanism*, namely the random-order mechanism – in which the kicking order before any round is determined by an unbiased coin flip – that is sequentially fair. Exogenous mechanisms, which, like the current fixed-order mechanism, have a predetermined, random, or fixed kicking-order pattern by teams are otherwise not sequentially fair. Some examples of sequentially unfair exogenous mechanisms are thus the fixed-order mechanism and the one in which the kicking order reverses in each round, namely different versions of the alternating-order mechanism.

We then identify a continuum of other sequentially fair mechanisms in regular rounds, which turn out to be *endogenous*. Any sequentially fair endogenous mechanism has the following structure: when the score is tied at the end of any round, the kicking order can be determined arbitrarily (randomly or not), while when score is not tied at the end of any round, the team that is ahead goes first with a predetermined probability (as a function of the score and the round number in the next round). That is to say, if one team or the other leads at the end of any round, the same probability applies to either team for kicking first. For example, the mechanism can deem the team that is ahead after Round 1 to go first in Round 2 with some probability $\alpha \in [0, 1]$. Thus, there is a continuum of sequentially fair mechanisms. We also show that sequentially fair mechanisms satisfy the second part of the Aristotelian Justice principle: among two teams with unequal kicking skills, the better team

[6] Observe that the use of a coin toss as a simple tiebreak mechanism, as FIFA did before 1970 (see Section 3), would be sequentially fair and satisfy the first part of our (Aristotelian) justice criterion (equals), but it would fail its second part (unequals). Thus, our justice criterion rules this simple tiebreak mechanism out.

will have a higher probability of winning the shootout at symmetric equilibria.

Because of the continuum of the sequentially fair mechanisms in regular rounds, one needs to resort to criteria other than sequential fairness to refine the set of mechanisms that can be deemed desirable. Among the sequentially fair mechanisms, goal probabilities may differ; some may involve coin tosses or a kicking-order switch at every round while others may minimize these tosses or switches. In Section 8, where we adopt the approach of market design, we discuss the relative merits of different ex post fair mechanisms in terms of these additional practical criteria. We show that there is a class of sequentially fair mechanisms satisfying either the *dominance* property (an efficiency property on goal production and exerting maximum effort) or the *instant rectifiability* property, which we term the *behind-first mechanisms*, such that the team that is behind in score after a round always kicks first in the next round, but if the score is tied after any round, then any random or fixed exogenous or endogenous order is admissible at the next round.

Note that sequentially fair mechanisms, including behind-first mechanisms, leave unspecified how one should choose which team would kick first when the score is tied, which would be a major issue especially during the sudden-death rounds of a shootout. To find a plausible remedy for this vagueness, from a practical point of view, we consider the *alternating-order mechanism*, which is sequentially fair in the sudden-death rounds but not in regular rounds of a shootout. Although there are many other ways to obtain sequential fairness in sudden-death rounds, this mechanism provides a *sudden-death equality of opportunity* to both teams in addition to its simplicity. We conclude by using this approach to combine a behind-first mechanism with an alternating-order structure in a *simple* (and *stationary*) manner to obtain a practical shootout scheme. Our simplicity axiom minimizes the patterns of how kicking order changes across rounds while keeping the probability positive that either team will kick first ex ante. We uniquely characterize the mechanisms satisfying sequential fairness and dominance together with two axioms, namely, simplicity and sudden-death equality of opportunity: the team that is behind in score after a round always kicks first in the next round, but if the score is tied after any round, then the team that kicked second at that round kicks first in the next round. We refer to this mechanism the *alternating-order behind-first mechanism*.

2 Other Related Literature

Apart from the papers mentioned in the Introduction, our paper is also related to the following strands of research. In the first strand, Chiappori et al. (2002) studied soccer penalty kicks both theoretically and empirically to test mixed strategies, while Palacios-Huerta (2003) did so with a much more empirical focus. Both papers considered regular penalty kicks during matches rather than penalty kicks in shootouts. Bar-Eli et al. (2007), after studying mostly regular penalty kicks and some shootout penalty kicks, observed that goalies almost always jump right or left even though it would also be optimal for goalies to stay in the goal's center with some probability.[7]

In the second strand of literature, Carrillo (2007) considers having the penalty shootout in soccer before overtime, where the shootout outcome counts only if the tie is preserved during extra time. He finds that during overtime, this rule promotes offense (defense) for the team that loses (wins) the shootout. He also provides conditions under which this rule would dominate the current mechanism in terms of inducing more offensive play. Lenten et al. (2013) provide empirical (simulation) support for Carillo's (2007) proposal.

The third strand focuses on topics of economic design of sports contests, such as the optimal number of entrants/teams in a race/league, the optimal structure of prizes (revenue sharing) for a tournament (league), and so on. See Szymanski (2003) for a review of this literature.

Before we finish this section, we need to make two important remarks. First, a precursor of our concept of sequential fairness can be found in Che and Hendershott (2008), who use a static one-shot version of it, ex post fairness.[8] Finally, in a general sense, our paper belongs to market design, a relatively new field seeking to provide practical solutions mainly to resource-allocation problems in which monetary transfers are not allowed simply because such transfers would draw legal and ethical objections (e.g., public school slots and human kidneys are not allowed to be traded for money). Penalty shootouts are also in that category since the right to kick first in a round cannot be assigned via monetary transfers.[9]

[7]Both studies find that kickers kick it to the middle relatively rarely and that "goalies almost never stay in the middle." Chiappori et al. also note that kickers in regular penalty kicks do not kick to the middle unless their team's score advantage is already large enough.

[8]In the National Football League (NFL), matches that end in a tie are determined by a sudden-death-like overtime. The initial coin toss yields a significant advantage to the team that wins it and the outcome fails to be ex post fair. Che and Hendershott (2008, 2009) propose "auctioning off" or "dividing-and-choosing" the starting possession to restore ex post fairness.

[9]Clearly, in the absence of monetary transfers, efficiency and fairness need to be achieved through other means. Nevertheless, market design enjoyed impressive success in particular in organizing markets such as the one between medical interns and residents, in assigning students to public schools or to courses at a given university, in allocating housing to immigrants and dorm rooms to college students, and in creating paired kidney exchanges between kidney donors with medical incompatibilities and transplant patients (see, e.g., Abdulkadiroğlu and Sönmez, 2013; Che, 2010; Nobel Prize Organization, 2012; Sönmez and Ünver, 2011, 2013).

3 Background on Soccer and Shootouts

Until 1970, elimination matches that were tied after extra time were either decided by a coin toss or replayed in two days if it was a finals match. Finally, the events in the 1968 European football championship led FIFA in 1970 to try penalty shootouts instead.[10] Given that the current shootout mechanism is no panacea, FIFA experimented with the Golden Goal and the Silver Goal between 1993 and 2004.[11] It was hoped that these parallel measures would produce more offensive play during overtime, and thus would effectively reduce the number of penalty shootouts. However, they in fact led to defensive play to maintain the status-quo score, and were eventually abandoned.

3.1 Unpredictability of Penalty Kicks

The soccer players who take the penalty kicks in shootouts are typically among the most skilled and elite professionals in the world, while the task they have to perform is a relatively easy but a risky one. It involves hitting a spot with the ball from 12 yards (approximately 11 meters) at a sufficiently high speed to elude a high-caliber goalie who is scrambling to protect an eight-yard-wide goal. Thus, each such kick involves an element of risk and can turn out to be costly for the kicker, especially if the miss is unambiguously his fault. The following quote by Italy's Roberto Baggio provides strong implications about plausible assumptions regarding players' preferences and the various basic physical aspects of a penalty kick (see the next subsection):[12]

> As for the penalty, I don't want to brag but I've only ever missed a couple of penalties in my career. And they were because the goalkeeper saved them not because I shot wide. That's just so you understand that there is no easy explanation for what happened at Pasadena. When I went up to the spot I was pretty lucid, as much as one can be in that kind of situation. I knew [the Brazilian goalie] Taffarel always dived so I decided to shoot for the middle, about halfway

[10] A semifinal match was decided by a coin flip. The final match also ended in a tie after over time. But because of the growing public outrage of soccer fans and since there was no further match left in the tournament, authorities decided to replay this match in two days.

[11] The Golden Goal, introduced in 1993, means that the match ends instantly after the first goal during extra time, and the team that scores it wins the match. The Silver Goal was announced in 2002 and ended in 2004. With the Silver Goal, in overtime the team leading after the first fifteen-minute half would win, but the game would no longer stop the instant a team scored.

[12] Baggio had a stellar career and his five goals in the tournament helped Italy to reach the final's match of the 1994 World Cup against Brazil. With the shootout score at 3-2, as the last kicker in regular rounds Baggio had to score to keep Italy's chances alive. He aimed for the middle but the ball sailed over the crossbar. The quote is from Baggio's (2001) autobiography *Una Porta Nel Cielo*.

up, so he couldn't get it with his feet. It was an intelligent decision because Taffarel did go to his left, and he would never have got to the shot I planned. Unfortunately, and I don't know how, the ball went up three meters and flew over the crossbar. ... I failed that time. Period. And it affected me for years. It is the worst moment of my career. I still dream about it. If I could erase a moment from my career, it would be that one.

3.2 Kickers' Preferences Over Outcomes and Physical Aspects of Penalty Kicks

Clearly, the outcomes of players' kicks pertain to their teams as well as to themselves. From the team's perspective a goal is preferred to a non-goal, and clearly there is no difference at all between a saved kick and a kick that misses the goal. From Baggio's quote, we also infer that, from a player's perspective, while scoring a goal is the best outcome and the goalie's save has to some extent a face-saving value, missing the penalty kick can be a devastating outcome for a kicker. Thus, a kick can be extremely costly for the kicker if his kick misses the goal, in which case the entire blame can be assigned to him unambiguously. One cannot, however, posit whether a player's individual utility from his kick or his collective utility from his team's winning the shootout should outweigh one another. For example, a player can be somewhat happy and heartbroken at the same time (1) if he scored his penalty kick while his team lost the shootout, or (2) if he missed his penalty kick outright while his team won the shootout.

We also infer from Baggio's quote (as well as from other studies mentioned before) that goalies typically feel the need to dive. This is because, at the optimal speed-accuracy combinations of world-class kickers, the kicked ball typically takes around 0.3 seconds to reach the goal line (see, e.g., Harford, 2006; Chiappori et al., 2002; Palacios-Huerta, 2003), which is less than the total of (1) roughly 0.2 seconds' reaction time of the goalie to clearly recognize the kick direction of the ball first, plus (2) the time during his dive to reach the expected arrival spot of the ball before it reaches the goal plane. Hence, a goalie cannot afford to wait until he clearly observes the kick direction: to prevent a goal with non-trivial probability, he must commit to pick a side to dive - or alternatively to stay in the middle[13] at the time the ball is kicked. For that reason, for a significant portion of penalty kicks, the goalie and the ball end up in opposite corners of the goal. Even when the goalie dives in the correct direction, he cannot save a goal with 100% chance, since he must also be able to reach the

[13] As Baggio's quote also indicates, a shot aimed at the middle may be missed outright or may hit the feet or the legs of the diving goalie that cover part of the middle; thus, the shot can be saved even if the goalie dives.

ball.

4 Model

Two soccer teams, which we refer to as Teams 1 ($T1$ for short) and 2 ($T2$ for short), are facing off in a penalty shootout. Each team shall take n sequential rounds of penalty shots. Each round consists of one team kicking first, and, after observing the outcome of that shot, the second team taking the next shot. If one team scores more goals than the other at the end of n rounds, then it wins the match. We refer to these n rounds as the **regular rounds**. Throughout the paper we will assume that $n = 2$. This is sufficient to characterize sequential fairness and analyze the current scheme as well as other proposed mechanisms, such as the alternating-order mechanism. Thus, with $n = 2$, the analysis is tractable and yet rich enough to capture the multiround feature of penalty shootouts.[14]

If the shootout score is tied at the end of regular rounds, the format reverts to **sudden death**; that is, each team takes on additional round of shots, and then, if one team scores while the other one does not, the former team wins the match; otherwise a further round of sudden-death penalty shots is taken. We refer to the sudden-death rounds as $n+1, n+2, ...$

Since potentially the match can continue forever, we assume that each team consists of an infinite number of kickers and that each kicker takes at most one shot.[15]

A penalty kick consists of a probabilistic event with three outcomes: Either a goal is scored (G), the shot goes wide (O), or the shot is saved by the goalie (S). The latter two outcomes lead to the same score for the team: a goal is not scored.

While each kicker is a strategic player, for tractability the goalie is modeled as a probabilistic machine (alternatively, if one would like to opt for a simple game in which the goalie is also strategic, see the next footnote). The goalie waits in the middle of the goal line prior to the shot. He jumps to one side or the other with probabilities $\frac{1}{2} : \frac{1}{2}$ prior to the penalty shot, as he needs to react early to have any realistic chance to save the kick. So with probability $\frac{1}{2}$ he reaches to the same side of the goal as the kick.[16] Hence, we model

[14] We have $n = 3$ results in an online appendix at http://www2.bc.edu/~unver/research/ASU-socccer-shootouts-appendix-b.pdf, and no extra insight exists in this analysis. Similarly, we skip $n > 3$ as the analysis becomes extremely cumbersome and lengthy without providing any further insight.

[15] In reality, each soccer player can take at most one shot, unless all players in his team have already kicked penalty shots. As each team consists of 11 players, 11 shots need to be taken by each team before any player can kick a second shot. As $n = 5$, this happens very rarely.

[16] Both Palacios-Huerta (2014) and Chiappori et al. (2002) conclude that the penalty kick may be described by a simple 2x2 game with neutral-sided kickers to kick and goalies to dive. Then it would be clear that each kicker would kick the ball right or left with 1/2 probability each and the goalie would dive to each side with 1/2 probability. Thus, the equilibria of these games lead to our reduced-form goalie behavior.

the goal line as a one-dimensional line segment $[0,1]$, where $x=0$ refers to the center of the goal, and $x=1$ refers to the goal pole on the side of the kick.

Each kicker, who is a single-round player in our game, has an action summarized as aiming at coordinate $x \in [0,1]$ of the goal line. When a kicker aims at x, the exact spot the ball reaches on the goal line is determined by a continuous probability density function σ_x in a closed support $[\underline{\epsilon}_x, \overline{\epsilon}_x]$ for some $\overline{\epsilon}_x > x > \underline{\epsilon}_x$. The spot the ball reaches, y, is observable by all other players, but not the intended spot, x. Both x and y are observable by the kicker himself. Moreover, given that the shot is aimed at x, there is a $P_G(x)$ chance that a goal will be scored; and a $P_O(x)$ probability that, the shot will go out (see Palacios-Huerta, 2014, on that). Hence, the shot is saved by the goalie with probability $1 - P_G(x) - P_O(x)$.[17] We assume that P_G, P_O, and σ_x for all $x \in [0,1]$ are all common knowledge.

We assume that P_G is a twice continuously differentiable strictly concave function, which reaches its maximum at some $\overline{x} \in (0,1)$.[18] We assume that $P_G(x) > 1/2$ for all $x \in [0, \overline{x}]$. Function P_O, on the other hand, is an increasing twice continuously differentiable convex function. Increasing P_O is straightforward to motivate: the closer to the middle the ball is aimed, the lower is the chance that the ball will go out. Single-peakedness of P_G is also easy to motivate: Whenever the ball is aimed at low x values, it can be saved with a higher chance by the diving goalie (see the previous footnote). For higher x values, although the goalie's chances of saving the ball decrease as he may no longer be able to reach it, the chances of the ball going out increase. Hence, there is an optimal spot for the highest goal probability $\overline{x}$. Concavity of P_G and convexity of P_O are primarily assumed for the tractability of our analysis, and do not play any other major role for the interpretation of our results.

We assume that each kicker on both teams is identical in ability and has the same goal-scoring and kicking-out probability.

4.1 Shootout Mechanisms and Shootout Game

A **shootout mechanism** is a function, ϕ, that assigns a probability $\phi(h^{k-1}, g_{T1} : g_{T2})$ to $T1$ kicking first in Round k, given the sequence of first-kicking teams in the first $k-1$ rounds is $h^{k-1} = (h_r^{k-1})_{r=1}^{k-1}$ where $h_r^{k-1} \in \{T1, T2\}$ is the team that kicked first in Round r and

[17] Actually P_G and P_O are summary functions obtained from the following process: As mentioned before, the spot the ball reaches, y, is observable by all other players, but not the intended spot, x. If $y > 1$, then the ball goes out. So $P_O(x) = \int_{y=1}^{\overline{\epsilon}_x} \sigma_x(y)dy$. On the other hand, the goalkeeper can save the ball that arrives at spot y with probability $S(y)$, which is a continuous function. Hence, $P_G(x) = \int_{\underline{\epsilon}_x}^{1} [1 - S(y)]\sigma_x(y)dy$. Hence, we assume that the family of densities $\{\sigma_x\}_{x\in[0,1]}$ and save probability function S have all the properties that need the below restrictions to hold for P_G and P_O.

[18] In fact P_G is not concave around 0, and it is decreasing, as the ball can go both sides of the middle, $x = 0$, when it is aimed at $x = 0$. Nevertheless, we assume the goal-maximizing point is farther to the right. Therefore, without loss of generality, we use a strictly concave P_G.

$g_{T1} : g_{T2}$ is the score (i.e., the goals scored by $T1$ and $T2$, respectively) at the beginning of Round k. Thus, the probability of $T2$ kicking first in Round k is $1 - \phi(h^{k-1}; g_{T1} : g_{T2})$.

Each shootout mechanism ϕ induces a hidden action extensive-form game, which we will simply refer to as **the game**, such that the exact spot that each kicker aims the ball on the goal line is unobservable by others. Given the current state $(h^{k-1}; g_{T1} : g_{T2})$, for Rounds $k = 1, 2, ...$, the order of first-kicking teams in the previous $k - 1$ rounds h^{k-1}, and feasible scores $g_{T1} : g_{T2}$, the nature determines with probability $\phi(h^{k-1}; g_{T1} : g_{T2})$ $T1$ kicking next first and probability $1 - \phi(h^{k-1}; g_{T1} : g_{T2})$ $T2$ kicking next first. Then a kicker of the first-kicking team takes the penalty shot, observing the state and the history of the outcomes of all the shots up to that point as goal, out, or save. The kicker aims to some spot $x \in [0, 1]$ to maximize his expected individual payoff (which we explain in the next paragraph). Then nature determines with probability distribution $P_G(x), P_O(x), 1 - P_G(x) - P_O(x)$ whether the penalty kick results in a goal, goes out, or is saved, respectively. After the outcome of this shot is observed, the other team's kicker takes a penalty shot, observing the history of the outcomes of the shots up to that point. We continue until the end of Regular Rounds $k = n$ similarly. If the score is tied, then we continue with the sudden-death rounds until the tie is broken at the end of a sudden-death Round $k > n$.

Each kicker aims to maximize his expected individual payoff in the game. Each kicker's payoff function consists of two additive components. The first is the utility received when his team wins or loses the shootout: V_W is the win payoff and $V_L < V_W$ is the loss payoff. This component of the payoff is common to each kicker. The second component of the individual payoff consists of an individual outcome based valuation: If the kicker scores a goal he gets utility $U_G > 0$, if he kicks the ball out he receives payoff $U_O < 0$, and if the goalie saves the kick he receives payoff $U_S = 0$. This is a normalization that guarantees that scoring a goal is the most desirable outcome, and kicking the ball out is less desirable than kicking the ball inside the goal frame and yet the goalie saves the ball. With this normalization, we can also drop a variable from our notation without affecting our analyses. Thus, overall ex post payoff of a kicker i of Team Tk is then

$$u_{i,Tk} = V_t + U_p \tag{1}$$

where $t \in \{W, L\}$ refers to the overall team outcome, win or loss; and $p \in \{G, O, S\}$ refers to the kicker's penalty outcome, goal, out, or save.

An **information set** is $H \in \mathcal{H}_{i,Tk}$, i.e., the set of information sets that kicker $i \in \{1, 2, ...\}$ of Team $Tk \in \{T1, T2\}$ can move, consists of the exact spot the ball went to for each of the previous kicks, the team of the kick, and whether the kick was scored as a goal, went out, or was saved by the goalie. They are observable by kicker i of Team Tk moving in information set H, but not the intended spots of previous kicks. Each information set also has an associated round (without loss of generality indexed with the kicker, i.e. i'th round), order of kicking in the round as 1^{st} or 2^{nd}, and a current score difference between $T1$ and $T2$. We refer to all of this observable information as the **state** of the information set. Note that from the point of view of the kicker, who is a single-shot player in the game, all payoff-relevant information of an information set is given through its state.

A pure **strategy** $X_{i,Tk} : \mathcal{H}_{i,Tk} \to [0,1]$ is a function from the set of information sets that team Tk's kicker i can move to the spots that each kicker can target while taking the penalty shot.

As alluded to before, this is a sequential hidden action game, as each player observes only where the ball goes and whether the kick was a goal, out, or a save in previous kicks, but not the intended spot towards which the ball was kicked. Hence, as a kicker takes a penalty shot, he has a belief over intended spots of previous kicks. Formally, a **belief** $\mu(H)$ is a function that maps each information set $H \in \mathcal{H}_{i,Tk}$ that Team Tk's i'th kicker moves with positive probability to a probability distribution over histories of actions taken that would lead to the same information set.

4.2 Markov Perfection and State-Symmetric Equilibria

Our solution concept is *state-symmetric perfect Bayesian equilibrium* (SPBE), in which strategies in regular rounds depend only on the state of the game, i.e., on the round number, kicking order, and score difference; strategies in sudden-death rounds depend only on the current kicking order and score difference. The strategies in SPBE are memoryless in that they depend only on the current state.

A **perfect Bayesian equilibrium** in the game of shootout mechanism ϕ is an assessment, i.e., a strategy profile and a belief profile pair $\big[X = (X_{i,Tk})_{i\in\{1,2,...\},Tk\in\{T1,T2\}}, \mu = (\mu(H))_{H\in\mathcal{H}_{i,Tk},i\in\{1,2,...\},Tk\in\{T1,T2\}}\big]$ such that for any $\{Tk, T\ell\} = \{T1, T2\}$, $i \in \{1, 2, ...\}$, and $H \in \mathcal{H}_{i,Tk}$,

- spot $X_{i,Tk}(H) \in [0,1]$ maximizes the expected value over all possible ex post payoffs $u_{i,Tk}$ at information set H, given $(X_{-i,Tk}), (X_{j,T\ell})$ among all spots in $[0,1]$; and
- belief $\mu(H)$ is consistently derived by Bayes' rule from ϕ, X, P_G, P_O, $\mu(H')$ for all $H' \neq H$.

Observe that each kicker is a one-shot player and maximizes his individual expected payoff over his ex post payoffs $u_{i,Tk}$ defined in Equation 1. The exact formulation of this expected payoff will become clear in our analysis.

In this game, once the equilibrium strategies are found, beliefs are straightforward to construct. At any information set H, the kicking player believes with probability one that other kickers before him used equilibrium strategies. This is because the payoffs explicitly depend on the actual outcome of each kick, which is observable as Goal (G) or No Goal (NG), not on the intended spots of kicks (which are not observable). Further, beliefs will not play a role in finding the optimal strategies in equilibria as the kicker decides on his best action by taking into consideration only future players' kicks, not those of the past ones. We will not explicitly calculate the beliefs from this point on, except when we refine the possible multiple equilibria of the fixed-order mechanism.

Since we are making a fairness analysis over different shootout mechanisms, we will focus on a Markovian symmetric equilibrium concept (i.e., unless we refine the possible multiple equilibria of the fixed-order mechanism):

A **state-symmetric assessment** (X, μ) is defined as

- **In regular rounds:** $X_{i,Tk}(H) = X_{i,T\ell}(H')$ and $\mu(H) = \mu(H')$ for teams $Tk, T\ell \in \{T1, T2\}$ where both information sets $H \in \mathcal{H}_{i,Tk}$ and $H' \in \mathcal{H}_{i,T\ell}$ pertain to the same Regular Round $i \leq n$, and the same kicking order, 1^{st} or 2^{nd}, in the round while the score difference between $T1$ and $T2$ in H, s, and in H', s', satisfy $s = -s'$ if $T\ell \neq Tk$ and $s = s'$ if $T\ell = Tk$.

- **In sudden-death rounds:** $X_{i,Tk}(H) = X_{j,T\ell}(H')$ and $\mu(H) = \mu(H')$ for any $Tk, T\ell \in \{T1, T2\}$ where information sets $H \in \mathcal{H}_{i,Tk}$ and $H' \in \mathcal{H}_{j,T\ell}$ involve (possibly different) Sudden-Death Rounds $i, j > n$ but they refer to the same kicking order, 1^{st} or 2^{nd}, while the score difference between $T1$ and $T2$ in H, s, and in H', s', satisfy $s = -s'$ if $Tk \neq T\ell$ and $s = s'$ if $Tk = T\ell$.

A state-symmetric assessment in sudden-death rounds, for instance, dictates that two players on the same team or different teams will exactly aim at the same intended spot and have exactly the same beliefs if they were in each other's shoes. Note that before every sudden-death round the score is identical if the game reaches it, while before each regular round after the first round it could be different. Unlike the sudden-death rounds, the number of regular rounds is finite, and therefore the round number as well as the kicking order and score would matter in regular rounds. Therefore, even if two teams are tied in different regular rounds, the players who kick first need not use the same strategy in those two rounds.

A **state-symmetric equilibrium** of a shootout mechanism ϕ is defined as a state-symmetric Perfect Bayesian equilibrium of the game induced by ϕ. This solution concept is identical to *symmetric Markov-perfect equilibrium* if we ignored the beliefs and focused only on strategies assuming that each state of the game spans a subgame of the game. As noted above, beliefs play no role other than equilibrium selection when there are multiple equilibria; this is without loss of generality.

4.3 Sequential Fairness and Aristotelean Justice Criterion

Using the concept of state-symmetric equilibrium, we now define the key design concept in our analysis as follows: an assessment (X, μ) of the game induced by mechanism ϕ is **sequentially fair** if for all problems with balanced teams (i.e., for any underlying utility values V_W, V_L, U_G, U_O and goal and out probability functions P_G, P_O), at any $(h^{k-1}; g_{T1} : g_{T2})$ with $g_{T1} = g_{T2}$, - i.e., when they are tied at the beginning of Round k for any k -, each team has exactly a 50% chance of winning. We will seek shootout mechanisms whose *all* state-symmetric equilibria are sequentially fair. We will refer to such mechanisms, for short, as **sequentially fair mechanisms**. Note that it is not the shootout mechanism that is inherently fair, but its state-symmetric equilibria that need to be fair.

We will analyze sequential fairness in sudden-death rounds first. It will be useful to formally define this concept. A mechanism is **sequentially fair in sudden-death** rounds if, for all problems with balanced teams, for any sudden-death Round $k > n$, at any $(h^{k-1}; g_{T1} : g_{T2})$ with $g_{T1} = g_{T2}$, - i.e., when they are tied at the beginning of Round k -, each team has exactly a 50% chance of winning.

Our desiderata are determining whether the current mechanism's equilibria are sequentially fair, inspecting other plausible mechanisms, and characterizing the class of sequentially fair mechanisms.

Sequential fairness is the first part of the two-part Aristotelean justice criterion. We say that a mechanism satisfies **Aristotelean justice criterion** if it is sequentially fair and, when there is team with higher-ability kickers than those of the other team, whenever scores are tied at the beginning of a round, the better team wins with a weakly higher probability (and with a strictly higher probability at least at one round) at all state-symmetric equilibria.[19]

5 Analysis: A Kicker's Optimization Problem

We first analyze each kicker's optimization problem for a given mechanism ϕ and other agents' strategies. The i'th kicker of Team Tk, denoted by $\ell \equiv Tk, i$'s best response determination problem boils down to

$$\max_{x_\ell \in [0,1]} U_\ell(x_\ell; W_{G,\ell}, W_{NG,\ell}) \equiv \Big(P_G(x_\ell)W_{G,\ell} + [1 - P_G(x_\ell)]W_{NG,\ell}\Big) + \Big(P_G(x_\ell)U_G + P_O(x_\ell)U_O\Big) \tag{2}$$

where $P_G(x_\ell)W_{G,\ell} + [1 - P_G(x_\ell)]W_{NG,\ell}$ is Kicker ℓ's expected continuation team payoff and $P_G(x_\ell)U_G + P_O(x_\ell)U_O$ is Kicker ℓ's expected individual kick payoff for expected continuation values $W_{G,\ell}$ conditional on he scores and $W_{NG,\ell}$ conditional on he does not score. These

[19]We introduce an instance of unbalanced teams in Section 7.1 such that we can rank the teams as better and worse teams.

values, $W_{G,\ell}$ and $W_{NG,\ell}$, are functions of the shootout mechanism, the score difference, round number (i in this case), kicking order in that round, and the others' strategy profile. We drop them from our notation for simplicity.

Hence, the necessary first-order conditions for an interior maximum turn out to be

$$P_G'(x_\ell^*)(W_{G,\ell} - W_{NG,\ell} + U_G) + P_O'(x_\ell^*)U_O = 0. \tag{3}$$

The second-order conditions lead to the first-order conditions being sufficient, since we have

$$P_G''(x_\ell^*)(W_{G,\ell} - W_{NG,\ell} + U_G) + P_O''(x_\ell^*)U_O < 0, \tag{4}$$

which follows from the facts that $P_G'' < 0$, $W_{G,\ell} - W_{NG,\ell} \geq 0$, $U_G > 0$, $P_O'' \geq 0$ and $U_O < 0$. Hence if an interior maximum exists it is unique given $W_{G,\ell} - W_{NG,\ell}$. We will refer to $W_{G,\ell} - W_{NG,\ell}$ as the **expected marginal contribution** of the kicker to his team. We turn our attention to analyze the properties of the optimum for a kicker.

Proposition 1 *At any interior best response of Kicker ℓ, $x_\ell^* < \overline{x}$ is the optimal goal-scoring spot; and the higher his expected marginal contribution, the higher is his goal-scoring probability.*

Also note that if kicking out and the goal being saved were valued equally, i.e., $U_O = U_S = 0$, then $x_\ell^* = \overline{x}$, i.e., x_ℓ^* would be optimal. But since $x_\ell^* < \overline{x}$ because $U_O < U_S = 0$, a kicker chooses to kick more conservatively. The relative magnitude $P_G'(x_\ell^*)/P_O'(x_\ell^*)$ as well as magnitudes of expected marginal contribution $W_{G,\ell} - W_{NG,\ell}$, U_O, and U_{NG} determine how much he *shaves off* an optimal kick.

Next, we focus on fully solving $W_{G,\ell} - W_{NG,\ell}$ for the current scheme, the fixed-order mechanism.

6 The Current Scheme: The Fixed-Order Mechanism

The current shootout scheme is the fixed-order mechanism, in which the first kicker is determined before Round 1 with an even lottery and then the procedure continues with the same kicking order throughout. Formally, the **fixed-order mechanism** ϕ is defined as follows:

$$\phi(\emptyset; 0:0) = 0.5 \quad \text{and} \quad \phi(h^{k-1}; g_{T1} : g_{T2}) = \begin{cases} 1 & \text{if } h_1^{k-1} = T1 \\ 0 & \text{if } h_1^{k-1} = T2 \end{cases}$$

for all Rounds $k \geq 2$, orders of first-kicking teams in the previous $k-1$ rounds h^{k-1}, and feasible scores $g_{T1} : g_{T2}$ at the beginning of Round k.

We will now characterize the state-symmetric equilibria of the fixed-order mechanism in the sudden-death rounds.

Without loss of generality assume that $T1$ wins the coin toss before Round 1 and kicks first throughout.

At state-symmetric equilibria, if they exist, each $T1$ kicker will use exactly the same action when he kicks in the sudden-death rounds, as $T1$ always goes first and the score is tied at the beginning of each sudden-death round.

Similarly, by symmetry, each $T2$ kicker will use exactly the same action when his team is behind (which can be by one goal at most); and he will use exactly the same action when the score is even (which can happen if the preceding $T1$ kicker kicks out or his kick is saved).

On the other hand, $T1$ and $T2$ kickers may potentially use different actions at state-symmetric equilibria, as they kick in different orders: in each round $T1$ goes first and $T2$ goes second.

Hence, if a state-symmetric equilibrium exists, the probability of Team i winning at the beginning of each sudden-death round is the same for each $i = 1, 2$.

At a state-symmetric equilibrium, let us define V_{T1} to be the *value function of* $T1$, that is the expected utility it contributes by winning or losing to its all kickers, in the first sudden-death round. Denote by x the optimal kicking strategy for $T1$'s kicker. Define V_{T2}^{B} to be the value function of $T2$ in the first sudden-death round when $T2$ is currently behind by one goal, V_{T2}^{E} and to be the value function of $T2$ in the first sudden-death round when the score is currently even. $T2$'s kicker's optimal kicking strategy in each scenario is y_B and y_E respectively.

We can write the following *Bellman* equation for V_{T1}, where recall $P_G(x)$ is the goal probability when the kick is aimed at x, V_W is the team victory payoff for each kicker, and V_L is the team loss payoff for each kicker:

$$V_{T1} = \; P_G(x)W_{G,T1} + [1 - P_G(x)]W_{NG,T1} \tag{5}$$

using the language we developed in the previous section, $W_{G,T1}$ is the future expected future value conditional on the kicker scoring and $W_{NG,T1}$ is the future expected value conditional on the kicker not scoring. We have

$$W_{G,T1} = P_G(y_B)V_{T1}^{*} + [1 - P_G(y_B)]V_W \tag{6}$$

$$W_{NG,T1} = P_G(y_E)V_L + [1 - P_G(y_E)]V_{T1}^{*} \tag{7}$$

such that V_{T1}^{*} is the continuation payoff attributed to T1 in case the game goes to a second sudden-death round.

For $T2$, we have

$$V_{T2}^B = P_G(y_B) \underbrace{V_{T2}^*}_{=W_{G,T2}^B} + [1 - P_G(y_B)] \underbrace{V_L}_{=W_{NG,T2}^B} \tag{8}$$

$$V_{T2}^E = P_G(y_E) \underbrace{V_W}_{=W_{G,T2}^E} + [1 - P_G(y_E)] \underbrace{V_{T2}^*}_{=W_{NG,T2}^E} \tag{9}$$

where

$$V_{T2}^* = V_W + V_L - V_{T1}^* \tag{10}$$

is the continuation payoff attributed to $T2$ in our win-or-lose game.

Next, we solve the decision problem faced by each kicker given other players' actions and beliefs using the first-order necessary and sufficient conditions given in Equation 3. Recall that for a Kicker ℓ

$$P_G'(x_\ell^*)(W_{G,\ell} - W_{NG,\ell} + U_G) + P_O'(x_\ell^*)U_O = 0 \tag{11}$$

where x_ℓ^* is the optimal spot for Kicker ℓ.

At equilibrium, $x = x^*$, $y_B = y_B^*$, and $y_E = y_E^*$, and hence we can solve them by plugging Equations 5 – 10 into Equation 11. To do that we need to resolve the continuation values V_{T1}^* and V_{T2}^* for each team.

Hence, it is useful to note that in any state-symmetric equilibrium $V_{T1} = V_{T1}^*$. Therefore, by Equation 5,

$$V_{T1}^* = \frac{P_G(x)[1 - P_G(y_B)]V_W + [1 - P_G(x)]P_G(y_E)V_L}{P_G(x)[1 - P_G(y_B)] + [1 - P_G(x)]P_G(y_E)} = \alpha V_W + (1 - \alpha)V_L \tag{12}$$

where the winning probability of $T1$, α, is given by

$$\alpha = \frac{P_G(x)[1 - P_G(y_B)]}{P_G(x)[1 - P_G(y_B)] + [1 - P_G(x)]P_G(y_E)}. \tag{13}$$

A value for $\alpha > 0.5$ at a state-symmetric equilibrium will signal that the fixed-order mechanism is biased in favor of the first-kicking team in the sudden-death rounds (and $\alpha < 0.5$ is vice versa for the second-kicking team). On the other hand, the fixed-order mechanism is a sequentially fair mechanism if and only if $\alpha = 0.5$ at every state-symmetric equilibrium. For $T2$, then, we get by Equation 10,

$$V_{T2}^* = (1 - \alpha)V_W + \alpha V_L. \tag{14}$$

Hence, Equations 5 – 10 through Equation 11 become self-contained to solve for x^*, y_B^* and y_E^*. The following theorem characterizes the state-symmetric equilibrium strategy candidates solving these equations:[20]

[20] In our analysis, we did not have to model the beliefs of agents explicitly. We use the summary functions P_G and P_O, and the agents have to best respond to what the other players are doing at equilibrium. The beliefs will be crucial in using the equilibrium selection criterion, though, later in Subsection 6.1.

Theorem 1 (The fixed-order mechanism, sudden-death rounds) *(i) A state-symmetric equilibrium exists if and only if* $P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O \geq 0$.

(ii)When it exists, there may be multiple state-symmetric equilibria with strategy profiles (x^*, y^*_B, y^*_E)*, all of which are to the left of the goal-optimal spot, satisfying*

- $x^* = y^*_E$*, i.e., the* $T1$ *kicker and* $T2$ *kicker, when the score is even, kick at the same spot; and*
- *for every equilibrium with* (y^*_E, y^*_B)*, there exists another equilibrium with* $(\hat{y_E}, \hat{y_B})$ *such that* $\hat{y}_E = y^*_B$ *and* $\hat{y}_B = y^*_E$.

It will be useful to quantify "may be" in the above theorem. The below proposition answers this question:

Proposition 2 *Suppose that in the sudden-death rounds of the fixed-order mechanism, a state-symmetric equilibrium exists. Then, multiple state-symmetric equilibria exist if and only if there are multiple solutions* β *to the equation*

$$\beta = \frac{1 - P_G(y(1-\beta))}{2 - P_G(y(\beta)) - P_G(y(1-\beta))}, \tag{15}$$

where $y(\beta) = f^{-1}\left(\frac{-U_O}{(V_W - V_L)\beta + U_G}\right)$ *for* $f(x) = P'_G(x)/P'_O(x)$ *for all* $x \in [0, 1]$.

Moreover, there is an odd number of solutions with $\beta = \frac{1}{2}$ *always being a solution and others being located symmetrically around it. We also have* $y^*_B = y(\beta)$ *and* $x^* = y^*_E = y(1-\beta)$ *for any solution* β.

Thus, generically, the fixed-order mechanism is *not sequentially fair* as the winning probability of $T1$ $\alpha \neq \frac{1}{2}$ in equilibrium, whenever $y_B \neq y_E$.

Example 1 ***(Sequentially unfair equilibria)*** *Suppose the game has the following structure:*

$$V_W - V_L = 7;\ U_G = \frac{556.08}{879};\ U_O = -15s$$
$$P_G(x) = 0.82 - 1.2(0.5 - x)^2;\ P_O(x) = \frac{270.232704}{879s}x$$

It can be readily verified that for every $s \geq 1$, $(y_E, y_B) = (0.04, 0.03)$ *(and hence* $(y_E, y_B) = (0.03, 0.04)$*) constitutes an equilibrium.*

Theorem 2 *The fixed-order mechanism is not sequentially fair in general.*

Its proof is immediately implied by Theorem 1. We will provide some intuition for this result when we discuss sequentially fair mechanisms in the sudden-death rounds in Section 7.2.

6.1 Equilibrium Refinement

Next, we address the question as to which state-symmetric equilibrium is more likely to be observed when there are multiple state-symmetric equilibria in the fixed-order mechanism. To that end, we use a selection criterion similar to Cho and Kreps's (1987) "Intuitive Criterion." Suppose there are multiple state-symmetric equilibria. Let the state-symmetric equilibrium with (x^*, y_E^*, y_B^*) be the one with highest x, i.e., the intended spot by $T1$'s kickers is the closest to the goal-optimal spot among all state-symmetric equilibria. We will refer to this equilibrium as *the most aggressive equilibrium for* $T1$ for the following reason: As $x^* = y_E^* > y_B^*$, we have the winning probability of $T1$, $\alpha = \frac{1-P_G(y_B^*)}{2-P_G(y_E^*)-P_G(y_B^*)} > \frac{1}{2}$ by Equation 13; and moreover, such a winning probability for $T1$ is the highest among all state-symmetric equilibria.

As a result, $T1$'s kickers can collectively enforce the most aggressive kicking equilibrium for their team and win more often, in which the first kicker can set the tone of aggressiveness for his team. Being the first mover, if $T1$ can credibly "signal" $T2$ that they are indeed playing this most aggressive equilibrium, this would be the most beneficial for $T1$. In this case, we can use such a signaling through beliefs in the state-symmetric equilibrium to obtain a refinement. For example, if σ_{x^*}, the probability density function of the ball reaching a particular spot on the goal line when it is aimed at x^* has the support set $[x^* - \underline{\epsilon_{x^*}}, x^* + \overline{\epsilon_{x^*}}]$. Suppose that this support is disjoint from such support sets of other equilibria. Then, whenever $T2$ kickers observe a kick spot in σ_{x^*}'s support, they can credibly deduce that indeed $T1$ is playing this aggressive equilibrium. Hence, the beliefs of $T2$'s kickers in information sets that are never reached in a state-symmetric equilibrium can be fine-tuned so that less aggressive equilibria can be eliminated.

Definition 1 (Refinement Criterion) *If the most aggressive state-symmetric equilibrium for* $T1$ *involves aiming at* x^* *for each kicker, and the possible spots that the ball can go under* x^* *(as determined by the support of* σ_{x^*}*,* $[x^* - \underline{\epsilon}_{x^*}, x^* + \overline{\epsilon}_{x^*}]$*) are different from any of the spots that the ball can go under all other state-symmetric equilibria, then* $T1$ *can credibly enforce the most aggressive state-symmetric equilibrium.*

Hence, we get the following corollary:

Corollary 1 (Team 1 wins more often) *If the state-symmetric equilibria can be refined, then* $T1$*, the team that kicks first, wins with a higher probability than* $T2$ *in the sudden-death rounds of the fixed-order mechanism.*

Hence, in our analysis with equally-skilled players and goalies, the fixed-order mechanism is biased toward the first-moving team and further multiple equilibria certainly exist. Indeed, empirically as well, these multiple equilibria and the overall first mover advantage are evident. The relative frequency figures regarding the winning probability of the teams that kick first vary significantly across tournaments throughout the world (see Figure 1).

7 Mechanism Design: Sequentially Fair Mechanisms

In the previous section we concluded that the currently used fixed-order mechanism is not sequentially fair. It turns out that even if we introduced a sequentially fair extension to the fixed-order mechanism in sudden-death rounds, it would still be sequentially unfair.

In fact, a large class of intuitive mechanisms turns out to be sequentially unfair. A fitting example of such mechanisms is the alternating-order mechanism, in which the kicking order reverses in every round. Suppose $T1$ starts off the shootout; then $T2$ kicks first in Round 2, $T1$ kicks first in Round 3, and so on. It turns out that even this mechanism is sequentially unfair. In addition, a large class of mechanisms, which we refer to as *exogenous mechanisms*, turns out to be sequentially unfair. A mechanism ϕ is **exogenous** if, for all rounds k, and kicking orders h^{k-1} regarding the beginning of round k, $\phi(h^{k-1} : g_{T1} : g_{T2}) = \rho(k)$ for some function ρ, i.e., who goes first in each round is determined independent of the current score but as a function of the current round. Hence, both fixed-order and alternating-order mechanisms are exogenous, and even the version of the alternating-order mechanism in which the 5th round's kicking order is randomly determined is exogenous.

Another interesting exogenous mechanism is the *random-order* mechanism ϕ, which determines who goes first in every round using an unbiased lottery, that is $\phi(h^{k-1}; g_{T1}, g_{T2}) = \frac{1}{2}$ for all k. However, a biased random-order mechanism where the probability of which team kicks first does not depend on the current score is sequentially unfair.

Despite its impracticality, one may expect this exogenous mechanism to be sequentially fair. Indeed, this turns out to be the case. However, the class of sequentially fair mechanisms is far richer than the random-order mechanism. There are some very practical mechanisms in this class.

We will next characterize all sequentially fair mechanisms in the regular rounds. We will assume that a mechanism that gives sequential fairness in the sudden-death rounds exists (and we then show that there are uncountably many such mechanisms).

We introduce a class of mechanisms that will be crucial in our analysis of sequentially fair mechanisms. A mechanism ϕ is **uneven score symmetric** if for all $(h^{k-1}; g_{T1} : g_{T2})$ and $(h'^{\,k-1}; g_{T2} : g_{T1})$ such that $g_{T1} \neq g_{T2}$ and $k \leq n$, we have $\phi(h^{k-1}; g_{T1} : g_{T2}) = 1 - \phi(h'^{\,k-1}, g_{T2} : g_{T1})$. That is, as long as the score is not tied at the end of a round, the probability of who kicks first in the next round is the same for $T1$ and $T2$ whenever they are in each other's shoes. E.g., when $T1$ is ahead $3-2$ in (the beginning of) Round 4, and when $T2$ is ahead in Round 4 with a score of $2 - 3$, in Round 4 $T1$'s probability of kicking first in the first case is the same as $T2$'s probability of kicking first in the second case.

It turns out that such mechanisms fully characterize the sequentially fair mechanisms in the regular rounds.

Theorem 3 (Sequentially fair mechanisms) *Suppose a mechanism ϕ is sequentially fair in sudden-death rounds. Then ϕ is sequentially fair if and only if it is uneven score symmetric in regular rounds.*

The intuition behind this result can be given as follows: When a round starts even, then the first team's kicker and the second team's kicker both exert the same effort and kick the ball to the same point. This is almost like asserting that when the score is even, kicking order is of minimum importance. The importance of kicking order, on the other hand, stems from the fact that when the score is uneven at the beginning of a round, teams assert different levels of effort in kicking penalties depending on when they kick. Under an uneven score-symmetric mechanism, each team's kickers foresee that their team will be treated symmetrically, as the other team in case either team falls behind or jumps ahead in score. Therefore, this assurance takes the reason behind the importance of kicking order out of the equation.

The theorem makes another interesting point. Interestingly, there is only one sequentially fair exogenous mechanism: The random-order mechanism that determines which team will kick first with a fair coin toss in a round. We formalize it below, and it follows directly from Theorem 3.

Proposition 3 ***Random order*** *is the only* ***exogenous*** *mechanism that is* ***sequentially fair****.*

Note that one does not need to treat both teams symmetrically all the time to obtain sequential fairness. In fact, when the score is tied, it does not matter which team kicks first. However, when the score is not tied, teams need to be treated symmetrically when the score is in their favor. This feature opens the door for some interesting practical mechanisms to be sequentially fair. Two examples of such mechanisms are the **ahead-first** and **behind-first** mechanisms. In ahead-first [behind-first] mechanisms, the team who is ahead [behind] in score after a round kicks first in the next round, and otherwise the order of the teams is determined in some predetermined manner. There are also many other uneven-score symmetric mechanisms in which lotteries play a significant role. For example, a lottery mechanism that forces the behind team to go first in 75% of the time and $T1$ to go first 60% of the time when the score is tied is also sequentially fair.

7.1 Better Teams Under Sequentially Fair Mechanisms

Uneven-score symmetric mechanisms have another nice feature. Theorem 3 states that when two teams have the same kicking ability, they have equal winning probability. What if one team is better than the other? Suppose there is one player who has a better kicking ability than the rest of the players, i.e., the player has a higher $P_G(x)$ and a lower $P_O(x)$ for every $x \in [0, 1]$. We formally define a **better player** as follows: Let $\{P_G, P_O\}$ represent all players' kicking ability except the better player, and $\{\widetilde{P}_G, \widetilde{P}_O\}$ represent the better player's kicking ability. We assume (a) $P_G(x) < \widetilde{P}_G(x)$ and $P_O(x) > \widetilde{P}_O(x)$, and (b) $\frac{P'_G(x)}{\widetilde{P}'_G(x)} = \frac{P'_O(x)}{\widetilde{P}'_O(x)}$ for all $x \in [0, 1]$. We show that the team with this better player – now named the **better team** – has a higher winning probability under uneven-score symmetric mechanisms.

Theorem 4 *Suppose a mechanism that is sequentially fair in sudden-death rounds and uneven-score symmetric in regular rounds is used in the shootout. Then a better team has a higher ex ante chance of winning at the unique state-symmetric equilibrium of the shootout induced by this mechanism, if the better player is used strategically in the best kicking order possible by the better team.*

Therefore, sequentially fair mechanisms satisfy the Aristotelean Justice criterion according to the definition of better/worse teams above.

7.2 Sequential Fairness in Sudden-Death Rounds

The class of sequentially fair mechanisms is larger when sudden-death rounds are also considered.

First we introduce a practical sequentially fair mechanism for the sudden-death rounds.

As we concluded in the previous section, the fixed-order mechanism clearly fails sequential fairness in the sudden-death rounds. So is there a simple and deterministic mechanism that is sequentially fair in the sudden-death rounds? The answer is affirmative, and the alternating-order mechanism *is* sequentially fair in sudden-death rounds, although it is not in regular rounds. The intuition is straightforward: Under the alternating-order mechanism, one can have uneven scores, such as $T1$ being ahead, in an intermediate regular round. Hence, it cannot satisfy uneven-score symmetry as required in a sequentially fair mechanism. On the other hand, in the sudden-death rounds, the score is never uneven at the beginning of a round. Hence, the exogeneity of the alternating order does not prevent sequential fairness.

Theorem 5 *The alternating-order mechanism is sequentially fair in sudden-death rounds.*

The intuition behind this result and its relationship to Theorem 1 about the multiplicity of equilibria in the fixed-order can be given as follows: All mechanisms span an infinite game in the sudden-death rounds. Typically this gives rise to multiplicity of equilibria. However, we are interested in state-symmetric equilibria for sequential fairness. In the fixed-order game, not all histories and information sets are reached in the path of the shootout, since fixed-order always dictates the same team to kick first. That is, two teams are "never in each other's shoes" during sudden-death rounds. Hence, the game has total freedom to choose among many different equilibria, i.e., the rounds that dictate that $T2$ kicks first are never reached and have no restrictions on the equilibrium behavior. On the other hand, the alternating-order mechanism is just the opposite in that sense: both teams are "in each other's shoes" in every other round. This puts more restrictions on the state-symmetric equilibria, and only the 50%-50% winning equilibria survive state-symmetry.

Actually, for such a restriction to hold, we do not even need the teams to be "in each other's shoes" as frequently as in the alternating-order mechanism. In fact, there are uncountably many other mechanisms that are sequentially fair in sudden-death rounds:

Theorem 6 *Take any mechanism ϕ that is uneven score symmetric in regular rounds, and any sequentially fair mechanism φ in sudden-death rounds. Construct a mechanism ψ such that for a given Sudden-death Round k, for all ℓ such that $n < \ell < k$ and feasible scores $g_{T1} : g_{T2}$, and beginning of Round ℓ kicking orders $h^{\ell-1}$, $\psi(h^{\ell-1}; g_{T1} : g_{T2}) = \phi(h^{\ell-1}; g_{T1} : g_{T2})$ and for all $\ell \geq k$ and $\ell \leq n$ and feasible scores $g_{T1} : g_{T2}$ and beginning of Round ℓ kicking orders $h^{\ell-1}$, $\psi(h^{\ell-1}; g_{T1} : g_{T2}) = \varphi(h^{\ell-1}; g_{T1} : g_{T2})$. Then ψ is sequentially fair.*

That is, we can replace the continuation of any uneven-score symmetric mechanism after some sudden-death round with a sequentially fair mechanism in sudden-death rounds (i.e., such as with the alternating-order mechanism), and regardless of initial part of the mechanism, the newly constructed mechanism becomes sequentially fair. The intuition of this result is as follows: Take the last round before sequential fairness kicks in, say Round k. By backward induction, as teams are tied at the beginning of Round k and in Round $k+1$ they have a $50\% - 50\%$ chance of winning, in all situations the two kickers of Round k exert the same effort regardless of kicking order (as we explained in the intuition behind Theorem 3). Therefore, at the beginning of Round k, both teams have an equal chance of winning as well. An example of such a mechanism is a behind-first mechanism such that in the first $n + 10$ rounds T1 kicks first whenever the game is tied, and then we alternate the order. Note that in the first 10 sudden-death rounds $T1$ kicks first, and yet, the mechanism is sequentially fair as it is appended by a sequentially fair mechanism in sudden-death rounds, namely the alternating-order mechanism.

8 Market Design and Practical Criteria

Sequential fairness is capable of ruling out many mechanisms in regular rounds, including the fixed-order mechanism currently used worldwide. Interestingly, it also rules out a seemingly fair exogenous mechanism, namely the alternating-order mechanism. Nevertheless, a case could easily be made for that mechanism over the lone sequentially fair exogenous mechanism, i.e., over the random mechanism, especially in the sudden-death rounds. In terms of endogenous mechanisms, however, sequential fairness does not pose as much of a restriction. In any such mechanism, when the score is even at the end of a round, it does not matter which team kicks first in the next round. In addition, when the score is not even at the end of a round, as long as the same probability is used in determining which team will kick first, whether the winning team or the losing team kicks first does not matter either. Thus, one needs further desirable properties to help refine the set of sequentially fair mechanisms. We will next define additional criteria to provide concrete practical advice in that regard. We start with an efficiency argument.

8.1 Efficiency and Behind-First Mechanisms

It is not difficult to argue that requiring the maximum effort possible in terms of aiming at the optimal spot (by taking the right amount of risk of kicking the ball out) is a desirable property. This is because most soccer fans would want to see kickers aim at the optimal spot as much as possible, leading to higher penalty shootout scores if not to simply higher-quality kicks. Thus, a crucial question is "does one of the sequentially fair mechanisms have an advantage over others in terms of the effort level of kickers and thus goal efficiency?" We use the following property at equilibrium to introduce a powerful efficiency notion.

Dominance in goal production and effort: Mechanism ϕ **dominates** mechanism ϕ' in terms of **goal production and effort** if for any two state-symmetric equilibria of ϕ and ϕ', (X, μ) and (X', μ'), respectively, we have $X_{r,Tk}(H) \geq X'_{r,T\ell}(H')$ (and hence, $P_G(X_{r,Tk}(H)) \geq P_G(X'_{r,T\ell}(H'))$) for any two information sets $H \in \mathcal{H}_{r,Tk}$ and $H' \in \mathcal{H}_{r,T\ell}$ pertaining to the same Regular Round $r \leq n$ and the same kicking order, 1^{st} or 2^{nd}, such that the score difference between $T1$ and $T2$ in H, s, and in H', s', satisfy $s = \begin{cases} s' & \text{if } T\ell = Tk \\ -s' & \text{if } T\ell \neq Tk \end{cases}$, and the inequality is strict for at least one information set.

This means that given a state of the information sets reachable under both ϕ and ϕ', the kicker who kicks at that state will exert (weakly) more effort under ϕ than the kicker who kicks at the same state under ϕ'. Hereafter, we will refer to this property as *goal dominance.*

We now turn our attention to which mechanisms are dominant among sequentially fair mechanisms. We define a subclass of sequentially fair mechanisms to address this issue:

Behind-first mechanisms: *In regular rounds, the team that is behind after Round* $r < n$ *kicks first at Round* $r + 1$*; when the score is even after Round* r*, any (random or fixed, exogenous or endogenous) order is admissible at Round* $r + 1$*. In sudden-death rounds, any mechanism that is sequentially fair in sudden-death rounds can be used.*

Then we have the following result:

Theorem 7 *A mechanism is* ***goal-dominant*** *among sequentially fair mechanisms if and only if it is behind first.*

The intuition behind this result can be summarized as follows. Consider the other sequentially fair extreme of *behind first*, the *ahead-first* mechanisms. By backward induction, first observe that ahead-first and behind-first cannot be compared with each other in Round $n = 2$ whenever the score is not tied: in *ahead first* when the score difference is $s > 0$, $T1$ kicks first, and when the score difference is $s < 0$, $T2$ kicks first, while it is just the opposite for *behind first*. So there are no two comparable information sets, as in the definition of the dominance property, that are reached with positive probability under both mechanisms. On the other hand, when the score is tied, all uneven-score symmetric mechanisms lead to the same goal efforts and are equivalent in Round $n = 2$ in terms of goal dominance.

Therefore, the difference between the two mechanism subclasses boils down to the Round 1 kickers' behavior. To analyze that, first we summarize the incentives facing Round 2 kickers. In Round 2, kicking first is not good at all for goal-production incentives: the first-kicking team's player (if his team is either behind or ahead) will always exert less effort than he would in the case when he kicks second in Round 2. This is true because his marginal contribution will be less in the first case, as the other team's kicker – who will go second – can always miss and overcome the first kicker's failure. So he has higher incentives to shirk when he kicks first.

Now, we turn our attention to Round 1 kickers' marginal contributions under both mechanisms. First, observe that both teams' kickers under any uneven-score symmetric mechanism exert the same effort in Round 1, from our previous analysis. Therefore, understanding the first-kicking team player's incentives is sufficient to draw the difference between the two mechanisms regardless of the kicking order or score during Round 1. A first-round kicker, if he does not exert high effort under *behind first*, may cause his team to fall behind with higher probability. This causes his teammate to shirk more, when he goes first, and the other team's second player to exert higher effort, when he goes second in Round 2. On the other hand, under *ahead first*, the first-round kicker's incentives are exactly the opposite! If he does not exert high effort in Round 1, his team may fall behind with higher probability, but his teammate will exert relatively higher effort under *ahead first* by going second in Round 2 (with respect to *behind first*) and the other team's second kicker will exert less effort in Round 2 (with respect to *behind first*). Hence, the first-round kicker's possible failure can still be salvaged with higher probability under *ahead first*. So he shirks under *ahead first* with respect to *behind first*. Therefore, *behind first* dominates any random (i.e., convex combination of *ahead first* and *behind first*) and *ahead-first* mechanism among all uneven-score symmetric mechanisms.

Behind-first mechanisms have also another natural motivation. One concern we might have is whether a mechanism could increase score rectifiability for the team that is behind in score. Consider the following axiom, which behind-first mechanisms satisfy:

Instant rectifiability: Whenever any Tk is behind in score after Round r, $T\ell$ should obtain with probability one the chance to make the score discrepancy smaller before Tk (where $k \neq \ell$) can obtain a chance to make the score discrepancy larger in Round $r+1$.

Thus, instant rectifiability means that the team that is behind gets a chance to catch up with the team that is ahead as soon as possible, before a larger score deficit may arise. Without instant rectifiability, a larger score deficit may put the losing team in a more non-rectifiable position, especially as the end of the shootout nears. We have the following observation:

Observation 1 *Behind-first mechanisms are the only sequentially fair mechanisms that satisfy* ***instant rectifiability****.*

Observe that instant rectifiability also has implications for the probability that all regular-round penalty shots are taken by kickers. To see that, suppose that $n = 2$ and the score is $1 - 0$ after the first round. Consider the fixed-order mechanism. Suppose that $T1$'s last kicker scores in Round 2; then there is no need for $T2$'s last kicker to kick in Round 2. Observe that this is an event with a probability of more than $\frac{1}{2}$ by assumption. Thus, with a high probability, $T2$'s last kicker will make no contribution. Only in the case where $T1$'s last kicker misses, i.e., with a probability less than $\frac{1}{2}$ by assumption, will $T2$'s last kicker's kick be needed by his team. Now, again with the $1 - 0$ score after the first round, consider instead any mechanism that satisfies instant rectifiability. Suppose $T2$'s last kicker scores (which, again, is a more-than-$\frac{1}{2}$-probability event by assumption); then $T1$'s last kicker will get to kick. Only in the case that $T2$'s last kicker misses, i.e., with a probability of less than $\frac{1}{2}$, will $T1$'s last kicker's kick no longer be needed. Thus, instant rectifiability will also increase the overall probability that all regular penalty shots are taken.

Thus, we have the following remark:

Remark 1 *Suppose $n = 2$. Conditional on reaching a state of $1 - 0$ score led by $T1$, the probability of all kickers using their kicks is higher in behind-first mechanisms than that in non-behind-first mechanisms.*

8.2 Alternating-Order Mechanisms and Sequential Fairness

Although behind-first mechanisms have nice features when the score is uneven, as mentioned before they are silent on how to define the kicking order when the score is tied. Sequential fairness in regular rounds, by our characterization in Theorem 3, is also mute on this issue, but reversing the kicking order is a sure way of establishing sequential fairness in sudden-death rounds (Theorem 5).

The alternating-order mechanism, which is not sequentially fair in regular rounds since it does not satisfy uneven score symmetry, does possess a nice property, at least when the score is tied in most crucial rounds, i.e., in sudden-death rounds: namely, it gives both teams an equality of opportunity in sudden-death rounds. Clearly, such an equality-of-opportunity property is nowhere more important than in sudden-death rounds in which the score must be tied before every round.

The alternating-order mechanism is also used in tiebreak-serve patterns in tennis and has been popular since its inception. We would like to preserve the equality-of-opportunity feature of this mechanism, especially in sudden-death rounds. The behind-first mechanism defined below has this feature.

The alternating-order behind-first mechanism: *The team that is behind in score after any Round r kicks first at Round $r + 1$. If the score is tied after Round r, then the team that kicked second at Round r kicks first in Round $r + 1$.*[21]

Besides its simplicity, this mechanism possesses several nice features. We will start with *sudden-death equality of opportunity*. This property would emerge naturally since a simple but strong case could be easily made against the same team kicking multiple times in a row in those rounds in a lop-sided fashion:

Sudden-death equality of opportunity: Whenever the shootout ends after the Sudden-death Round $n + r$ with r even, each team will have kicked first exactly $r/2$ times in the sudden-death rounds.

Then we have the following corollary:

[21] We are agnostic about how the first round order is determined in the definition of the mechanism. It can be determined in any manner. However, in practice we suggest it be determined by a fair coin toss as in the current fixed-order mechanism.

Corollary 2 *The alternating-order behind-first mechanism satisfies* ***sudden-death equality of opportunity****.*

Another justification of alternating-order behind-first mechanisms is as follows: Eclectic mechanisms could be confusing for players, coaches, referees, and fans. One can combine a sequentially fair mechanism in regular rounds with another sequentially fair mechanism in sudden-death rounds in an eclectic fashion to come up with an overall sequentially fair mechanism. For example, consider the following mechanism in regular rounds coupled with the alternating-order mechanism in sudden-death rounds: $T1$ kicks first in Round r as long as the score is tied or $T1$ is behind in Round $r-1$; once $T2$ falls behind after some Round $r' > r$, $T2$ kicks first until $T1$ falls behind in score after some Round $r'' > r'$, after which $T1$ kicks first. One can improve on such a patchy mechanism by requiring that such an eclecticism should be eliminated. We will introduce two axioms such that the latter uses the former in its definition to formalize this intuition of *simplicity*. Before introducing the first axiom, we formally introduce how an order pattern can be recognized in a mechanism:

A **finite machine representation** of a mechanism is a triple (Q, A, t) such that

- Q is a finite set of **(machine) states** such that state $q = (Tk)_w \in Q$ denotes that team Tk taking the first penalty shot in the round associated with this state and w is just an index number. Thus, Q can be partitioned into two as $Q_{T1} = \{(T1)_1, ..., (T1)_{w_1}\}$ and $Q_{T2} = \{(T2_1, ..., (T2)_{w_2}\}$ for some w_1 and w_2 as the sets of states in which team $T1$ and $T2$ kick first, respectively.

- $A = \{(g_1 : g_2)\}$ is the set of **possible scores**.

- $t : Q \cup \{\emptyset\} \times A \times Q \to [0, 1]$ is a **state transition probability function** such that $\sum_{q' \in Q} t(q, (g_1 : g_2), q') = 1$ for all $q \in Q \cup \{\emptyset\}$ and $(g_1 : g_2) \in A$. Here, $t(q, (g_1 : g_2), q')$ is the probability of moving from state q to state q' when after round associated with q is played and the score is $g_1 : g_2$ just before q' and after q.

We refer to null state $\emptyset$, as the **start of the shootout**. In this representation, we envision that each machine state is associated with a round of penalty kicks taken by each team consecutively. However, as round numbers proceed, the game will have to come back to some previous machine state, as the set of states is finite whereas a game can last arbitrarily long in theory.

A mechanism ϕ is said to **have finite machine representation** (Q, A, t), if (1) $t(\emptyset, (0 : 0), (T1)_1) = \phi(\emptyset, 0 : 0)$ and $t(\emptyset, (0 : 0), (T2)_1) = 1 - \phi(\emptyset, 0 : 0)$; and (2) recursively, for any kicking-order history h^{r-1} at the beginning of Round r, and feasible score $g_{T1} : g_{T2}$ at the beginning of Round r, if the associated machine state with round $r-1$ was $q \in Q$, then we have $t(q,(g_{T1}:g_{T2}),(T1)_w)=\phi(h^{r-1},g_{T1}:g_{T2})$ for some state $(T1)_w\in Q_{T1}$ and $t(q,(g_{T1}:g_{T2}),(T2)_w)=1-\phi(h^{r-1},g_{T1}:g_{T2})$ for some state $(T2)_w\in Q_{T2}$; and once a transition occurs to a state q' from q, ex post we refer to q' as the machine state associated with round r.

Note that a machine representation does not specify when the shootout game ends, as no round information is kept in the machine representation. It only keeps track of how transitions are made between different kicking orders in a well-defined pattern. We are ready to introduce our next axiom:

Stationarity: A mechanism is **stationary** if it has a finite machine representation (Q,A,t) such that for all states $q_i\in Q\cup\{\emptyset\}$ and $q_j\in Q$, $t(q_i,(g_{T1}:g_{T2}),q_j)=t(q_i,(g'_{T1}:g'_{T2}),q_j)$ for all scores such that $g_{T1}-g_{T2}=g'_{T1}-g'_{T2}$.

Thus, stationarity implies that state transitions are made in the same manner whenever score differences are the same.

For example, the alternating-order behind-first mechanism has this type of a representation as shown in Figure 2.

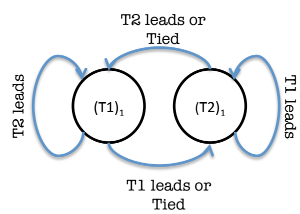

Figure 2: The state transition representation for the alternating-order behind-first mechanism. Transitions from the start of the shootout are omitted for simplicity. In general one of the two states in the figure will be chosen randomly with an unbiased lottery.

We state the following proposition whose proof is given in the figure:

Proposition 4 *The alternating-order behind-first mechanism is* ***stationary****.*

Machine representations can be used to measure the complexity of an algorithm.[22] However, very complicated mechanisms can also be stationary. We consider a modified Prouhet-Thue-Morse behind-first mechanism. First we define the fractal *Prouhet-Thue-Morse* mechanism (cf. Palacios-Huerta, 2014): The kicking order proceeds in an exogenous manner as follows: $T1-T2-T2-T1-T2-T1-T1-T2-...$ i.e. the order sequence since the beginning of the shootout reverses after 2^k rounds for each $k=1,2,....$ We define the following modified

[22]For example, in game theory, they are used to represent the recall requirement needed for implementing a repeated game strategy (cf. Rubinstein, 1998, for an excellent survey).

Prouhet-Thue-Morse behind-first mechanism: If one team is behind, it kicks first; otherwise, at even scores the first-kicking team follows the sequence $T1{-}T2{-}T2{-}T1{-}T2{-}T1{-}T1{-}T2$; then this sequence reverses starting with T2 and keeps reversing until the shootout ends. Any behind-first mechanism compatible with a Prouhet-Thue-Morse order is stationary, and the simplest stationary machine representation of such a mechanism cannot have fewer than $|Q| = 16$ states.[23] On the other hand, if we would like to have a chance of both teams kicking first in at least one round, we need at least two states, one $T1$-kicking-first state and one $T2$-kicking-first state. Thus, $|Q| = 2$ is the minimum we can hope for in a reasonable mechanism. Indeed, the current (fixed-order) mechanism has $|Q| = 2$, as according to the initial coin toss, either team can go first. However, it is not sequentially fair. The random-order mechanism has also $|Q| = 2$ and is sequentially fair (but not goal-dominant). Our behind-first alternating-order mechanism also has this property (cf. Figure 2). We formalize this axiom as follows:

Simplicity: A mechanism is **simple** if it has a stationary machine representation with only two states such that in one state $T1$ kicks first and in the other $T2$ kicks first.

Another example of a stationary behind-first mechanism that does not satisfy simplicity is as follows: Consider the following twist in our alternating-order behind-first mechanism. Instead of teams switching order whenever the score is tied, let's reinforce an explicit sequence of kicking first as $T1 - T2 - T1 - T2$, etc.; that is, the team that kicked first last time when the score is even now kicks last. This mechanism is stationary; however, it is not simple. Its representation in Figure 3 has four states, and we cannot find another stationary representation with fewer states.

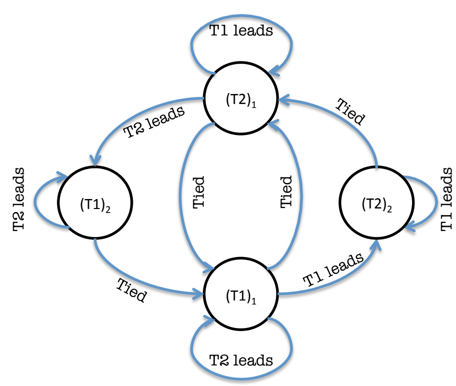

Figure 3: The state transition representation for the modified alternating-order behind-first mechanism. Transitions from the start of the shootout are omitted for simplicity. In general $(T1)_1$ or $(T2)_1$ will be chosen randomly as the initial state with an unbiased lottery.

[23]On the other hand, the truly fractal Prouhet-Thue-Morse sequence is not stationary.

An important motivation for simplicity stems from the FIFA soccer rules. These rules state that a rule violation by the referees during a game necessitates replay of the game. Shootout mechanisms that satisfy the simplicity axiom will make the process easier to administer for the referees and will make the process less prone to rule violations. We see simplicity as a vital requirement of a real-life shootout mechanism. The current mechanism satisfies simplicity but none of the other properties we have introduced in this paper. Our mechanism also satisfies simplicity, along with all the other important properties. We formalize the simplicity of the alternating-order behind-first mechanism with the following proposition. We gave its proof earlier:

Proposition 5 *The alternating-order behind-first mechanism is **simple**.*

We state the main result of this section as follows:

Theorem 8 ***Alternating order behind first** is the unique **goal-dominant sequentially fair** mechanism that satisfies **simplicity** and **sudden-death equality of opportunity**.*

Moreover, this theorem and our earlier Observation 1 about behind-first mechanisms lead to the following corollary:

Corollary 3 ***Alternating order behind first** is the unique **sequentially fair** mechanism that satisfies **instant rectifiability**, **simplicity**, and **sudden-death equality of opportunity**.*

We next demonstrate the independence of properties in Theorem 8: A sequentially fair mechanism that satisfies all axioms but violates the goal-dominance property is the alternating-order ahead-first mechanism. A sequentially fair mechanism that satisfies all properties but the sudden-death equality of opportunity is a behind-first mechanism, which randomly determines with an even lottery who goes first when the score is tied. A sequentially fair mechanism that satisfies all properties but is not simple is a Prouhet-Thue-Morse behind-first mechanism.

Finally, given that soccer is part of the entertainment sector, we will elaborate on the relevance of the criteria considered in this section in that respect. Dominance's implication regarding goal efficiency is already embraced by fans' desire to see more goals or at least higher-quality penalty kicks in a match, including the shootout. Instant rectifiability will help make the penalty score closer, which should be preferred to the current non-rectifiability of a score gap in a shootout. Simplicity will make the process easier to follow for the fans and players and easier to administer for the referees. Sequential fairness and sudden-death equality of opportunity will make the process fairer such that it would be harder to dispute the legitimacy of the winner of the shootout.

9 Discussion and Concluding Remarks

Like the current fixed-order shootout mechanism in soccer, some sequential tournaments may be conducive to a first-mover advantage, which may impede the efficiency and/or fairness of these tournaments.[24] Further analysis of related specific exogenous and endogenous tiebreak mechanisms may be modified to design new tournament structures with more desirable efficiency or fairness characteristics in these other real-life tournaments as well.

Also note that our behind-first mechanisms and the additional criteria or properties we have considered here can be of help in sports competitions other than soccer. For example, ice hockey and field hockey, as well as water polo, handball, cricket, and rugby, also have tiebreak or penalty shootout mechanisms the same as or similar to that of soccer, and thus can benefit from our properties.

A relevant question would be whether any of these properties are already being used in real life and how successful they are in their domains. In that regard, we will first give examples from player draft mechanisms in North American major sports and then from the age-old game Petanque (a.k.a. Boules or Bocce). First, note that all of the sequential player draft mechanisms in major professional leagues in the US such as the National Football League, the National Basketball Association, Major League Baseball, and the National Hockey League can be considered special cases of behind-first mechanisms in that domain, where the more disadvantaged teams (in terms of their league records from the previous season) go *ahead first* and pick better players before less disadvantaged teams. This feature is credited widely with the competitive balance of North American major sports.

Perhaps the behind-first property is nowhere more blatant and effectively at work than in the rules of "Petanque," which was invented in ancient times by the Greeks, later modified by Romans, and is now popular in various parts of the world including France and Italy - and currently expanding. In this game, the goal is to throw metal or wooden balls (boules) as close as possible to a small special wooden target, while standing inside a small starting circle. The rules are as follows: A player from the team that threw (and established) the target also throws the first ball. Then a player from the other team throws the second ball. The team with the ball that is closest to the target is said to "have the point" or "be winning" and other team is "losing." Then the "losing" team gets to throw the next ball.[25] Thus, in essence, just like our behind-first mechanisms, petanque too intends to give the "losing" team a chance to recover. Further, if the two balls closest to the target are from opposing teams and equidistant, teams play alternately until one team becomes the

[24] See Fudenberg et al (1983) and Harris and Vickers (1985) for sequential patent race models.

[25] See Article 15 of the world governing body of Petanque's, FIPJP's, official rulebook at http://www.fipjp.com/en/rules-texts-downloads.

"winning" team and the other one the "losing" team.[26]

A Appendix: Proofs of the Results in the Main Text

Proof of Proposition 1. First observe that $\overline{x}$ solves Equation 4 when $U_O = 0$. As the partial derivative w.r.t. U_O on the (left-hand side of) first-order condition is $P'_O(x^*_\ell) > 0$, the implicit function theorem implies that $x^*_\ell < \overline{x}$. Moreover, as the partial derivative w.r.t. $W_{G,\ell} - W_{NG,\ell}$ on the first-order condition is $P'_G(x^*_\ell) > 0$ (as $x^*_\ell < \overline{x}$), the implicit function theorem again implies that the higher the expected marginal contribution, $W_{G,\ell} - W_{NG,\ell}$, the higher is x^*_ℓ; and the higher is $P_G(x^*_\ell)$. ■

Proof of Theorem 1. We drop "*" superscripts for convenience. We write the three first-order conditions using Equation 11 (or 3) as:

$$P'_G(x)[P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - (1 - P_G(y_E))V_{T1} + U_G] + P'_O(x)U_O = 0$$

$$P'_G(y_B)[V_{T2} - V_L + U_G] + P'_O(y_B)U_O = 0$$

$$P'_G(y_E)[V_W - V_{T2} + U_G] + P'_O(y_E)U_O = 0$$

We first prove that $x = y_E$.

<u>*Claim 1.*</u> $x = y_E$.

<u>*Proof of Claim 1*</u>. Define

$$\Delta = P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - [1 - P_G(y_E)]V_{T1} - V_W + V_{T2}.$$

From the first-order conditions of x and y_E, $x \geq y_E$ if and only if $\Delta \geq 0$. Recall that the winning probability of $T1$ in equilibrium, α is given in Equation 13. Hence,

$$\begin{aligned}
\Delta &= P_G(y_B)(V_{T1} - V_W) + P_G(y_E)(V_{T1} - V_L) + V_{T2} - V_{T1} \\
&= P_G(y_B)(1 - \alpha)(V_L - V_W) + P_G(y_E)\alpha(V_W - V_L) + (1 - 2\alpha)(V_W - V_L) \\
&= [-P_G(y_B)(1 - \alpha) + P_G(y_E)\alpha + 1 - 2\alpha](V_W - V_L) \\
&= [1 - P_G(y_B) + (P_G(y_E) + P_G(y_B) - 2)\alpha](V_W - V_L)
\end{aligned}$$

[26] See Article 28 of the FIPJP's official rulebook.

We substitute α from Equation 13 as follows:

$$\begin{aligned}
\Delta &= [1 - P_G(y_B) + (P_G(y_E) + P_G(y_B) - 2)\frac{P_G(x)(1 - P_G(y_B))}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}](V_W - V_L) \\
&= (1 - P_G(y_B))[1 + \frac{(P_G(y_E) + P_G(y_B) - 2)P_G(x)}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}](V_W - V_L) \\
&= \left[\frac{(1 - P_G(y_B))(V_W - V_L)}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}\right] \\
&\quad \times [P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E) + (P_G(y_E) + P_G(y_B) - 2)P_G(x)] \\
&= \frac{(1 - P_G(y_B))(V_W - V_L)}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)}[P_G(y_E) - P_G(x)]
\end{aligned}$$

Suppose $x > y_E$, then as both $x, y_E < \overline{x}$ and P_G is increasing on the left of $\overline{x}$, we have $P_G(x) > P_G(y_E)$. But then $\Delta < 0$, contradicting that $x > y_E$. Supposition $x < y_E$ leads to a similar contradiction. Therefore, we must have $x = y_E$. $\diamondsuit$

Given $x = y_E$, α can be simplified as

$$\alpha = \frac{P_G(x)(1 - P_G(y_B))}{P_G(x)(1 - P_G(y_B)) + (1 - P_G(x))P_G(y_E)} = \frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)},$$

and $\alpha = \frac{1}{2}$ iff $x = y_B$. Then the first-order condition w.r.t. y_B can be simplified as:

$$\begin{aligned}
P'_G(y_B)[V_{T2} - V_L + U_G] + P'_O(y_B)U_O = 0 \\
\Longrightarrow P'_G(y_B)[(1 - \alpha)(V_W - V_L) + U_G] + P'_O(y_B)U_O = 0 \\
\Longrightarrow P'_G(y_B)[(V_W - V_L)\frac{1 - P_G(y_E)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P'_O(y_B)U_O = 0 \qquad (16)
\end{aligned}$$

Similarly, the first-order condition w.r.t. y_E can be simplified as:

$$\begin{aligned}
P'_G(y_E)[V_W - V_{T2} + U_G] + P'_O(y_E)U_O = 0 \\
\Longrightarrow P'_G(y_E)[\alpha(V_W - V_L) + U_G] + P'_O(y_E)U_O = 0 \\
\Longrightarrow P'_G(y_E)[(V_W - V_L)\frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P'_O(y_E)U_O = 0 \qquad (17)
\end{aligned}$$

Now we are ready to prove part (i). First we show that $P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O \geq 0$ implies the existence of equilibrium. Define $H(z) \equiv P'_G(z)[\frac{V_W - V_L}{2} + U_G] + P'_O(z)U_O$. $H(z)$ is continuous with $H'(z) < 0$ as $P''_G(z) < 0$ and $P''_O(z) \geq 0$. Then $H(0) = P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O \geq 0$ and $H(\overline{x}) = P'_O(\overline{x})U_O < 0$ implies that there exists some $a \in [0, \overline{x})$ such that $H(a) = 0$. It can readily be seen that $(x, y_B, y_E) = (a, a, a)$ solves the two first-order conditions and hence constitutes an equilibrium.

On the other hand, assume now $P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O = H(0) < 0$. As $H'(z) < 0$, $H(z) < 0$ for every $z \in [0, 1]$. Suppose to the contrary that there exists an equilibrium

(x, y_B, y_E). Clearly $y_B \neq y_E$, for otherwise $\frac{1-P_G(y_E)}{2-P_G(y_B)-P_G(y_E)} = \frac{1}{2}$ and the first-order condition of y_B becomes $H(y_B) < 0$. Suppose $y_B > y_E$. Then the first-order condition w.r.t. y_E in Equation 17 becomes:

$$P'_G(y_E)[(V_W - V_L)\frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P'_O(y_E)U_O$$
$$< P'_G(y_E)[\frac{V_W - V_L}{2} + U_G] + P'_O(y_E)U_O = H(y_E) < 0,$$

a contradiction! Then $y_B < y_E$; and similarly, the first-order condition for y_B is negative, leading to a contradiction. Therefore, an equilibrium exists if and only if $P'_G[\frac{V_W-V_L}{2} + U_G] + P'_O(0)U_O = H(0) \geq 0$.

There may be multiple solutions (y_E, y_B), and whenever one exists, then $(\hat{y_E}, \hat{y_B})$ satisfying $\hat{y_E} = y_B$ and $\hat{y_B} = y_E$ also lead to a state-symmetric equilibrium. ■

Proof of Proposition 2. The first-order conditions are given by Equations 16 and 17 for y_B and y_E in the proof of Theorem 1, respectively (dropping the superscript "$*$"). We get $y_B = y(\beta)$ and $y_E = y(1-\beta)$, since $f = P'_G/P'_O$ is an invertible differentiable decreasing function in the region $[0, \bar{x}]$ by assumption that P_O is convex and increasing and P_G is strictly concave and increasing in the interval $[0, \bar{x}]$. Thus, circularly, plugging in y_B and y_E, we get Equation 15. Optimal spots y_B and $x = y_E$ are multiple valued if and only if β is multiple valued. $\beta = \frac{1}{2}$ always solves Equation 15, and if $\beta = \alpha$ is a solution then $\beta = 1 - \alpha$ is also a solution. Thus, there is an odd number of solutions. ■

Proof of Theorem 3. We solve it by backward induction. As both teams have an equal chance of winning in sudden-death rounds, the value function is $\frac{V_W+V_L}{2}$ for each team at the end of the regular rounds.

Second Round, Second Kick

Whether the last-kicking team is currently even or behind, it can readily be verified that the optimal kicking strategy is always $\xi < \overline{x}$, where ξ is determined by the following first-order condition:

$$P'_G(\xi)[\frac{V_W - V_L}{2} + U_G] + P'_O(\xi)U_O = 0$$

Second Round, First Kick

Next we look at the optimal kicking strategy for the first-kicking team in Round 2. Consider two cases:

<u>When $T2$ kicks first in Round 2</u>

There are three possible states: when the score is currently even, when $T2$ is currently behind (by one goal), and when $T2$ is currently ahead (by one goal).

When the score is currently even Let y_{2E} denote the optimal kicking strategy for $T2's$ kicker in Round 2 when the score is even. The value function for $T2$ is

$$V_{T2,P2,E} = (P_G(y_{2E})P_G(\xi)+(1-P_G(y_{2E}))(1-P_G(\xi)))\frac{V_W + V_L}{2}+P_G(y_{2E})(1-P_G(\xi))V_W+(1-P_G(y_{2E}))P_G(\xi)V$$

By Equation 11, y_{2E} solves the following first-order condition:

$$P'_G(y_{2E})[(P_G(\xi) - (1 - P_G(\xi)))\frac{V_W + V_L}{2} + (1 - P_G(\xi))V_W - P_G(\xi)V_L + U_G] + P'_O(y_{2E})U_O = 0$$
$$\implies P'_G(y_{2E})[\frac{V_W - V_L}{2} + U_G] + P'_O(y_{2E})U_O = 0$$

Therefore $y_{2E} = \xi$ and $V_{T2,P2,E} = \frac{V_W+V_L}{2}$.

When $T2$ is currently behind Let y_{2B} denote the optimal kicking strategy for $T2's$ kicker in Round 2 when the score is currently behind. The value function for $T2$ is

$$V_{T2,P2,B} = P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(y_{2B}))V_L$$

y_{2B} satisfies the following first-order condition:

$$P'_G(y_{2B})[P_G(\xi)V_L + (1 - P_G(\xi))\frac{V_W + V_L}{2} - V_L + U_G] + P'_O(y_{2B})U_O = 0$$
$$\implies P'_G(y_{2B})[(1 - P_G(\xi))\frac{V_W - V_L}{2} + U_G] + P'_O(y_{2B})U_O = 0$$

When $T2$ is currently ahead Let y_{2A} denote the optimal kicking strategy for $T2's$ kicker in Round 2 when the score is currently ahead. The value function for $T2$ is

$$V_{T2,P2,A} = P_G(y_{2A})V_W + (1 - P_G(y_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}]$$

The optimal kicking strategy, y_{2A}, satisfies the following first-order condition:

$$P'_G(y_{2A})[V_W - (1 - P_G(\xi))V_W - P_G(\xi)\frac{V_W + V_L}{2} + U_G] + P'_O(y_{2A})U_O = 0$$
$$\implies P'_G(y_{2A})[P_G(\xi)\frac{V_W - V_L}{2} + U_G] + P'_O(y_{2A})U_O = 0$$

As $P_G(\xi) > \frac{1}{2}$, $y_{2A} > y_{2B}$.

When $T1$ kicks first in Round 2

Let x_{2E}, x_{2B}, and x_{2A} denote the optimal kicking strategy for $T1's$ kicker in Round 2 when the score is even, when $T1$ is behind, and when $T1$ is ahead respectively. By symmetry, we have the following results:

When the score is currently even

The optimal kicking strategy is $x_{2E} = y_{2E} = \xi$, and the value function for $T1$ is $V_{T1,P2,E} = \frac{V_W+V_L}{2}$.

When $T1$ is currently behind

The optimal kicking strategy is $x_{2B} = y_{2B} < \xi$, and the value function for $T1$ is

$$V_{T1,P2,B} = P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(x_{2B}))V_L$$

When $T1$ is currently ahead

The optimal kicking strategy is $x_{2A} = y_{2A} < \xi$, and the value function for $T1$ is

$$V_{T1,P2,A} = P_G(x_{2A})V_W + (1 - P_G(x_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}]$$

First Round, Second Kick

Next we study the second team's optimal kicking strategy in Round 1. There are two possible states:

When $T1$ does not score in Round 1

The value function for $T2$ in this case is

$$V_{T2,P1,E} = P_G(y_{1E})[\phi(T1;0:1)(V_W+V_L-V_{T1,P2,B})+(1-\phi(T1;0:1))V_{T2,P2,A}]+(1-P_G(y_{1E}))\frac{V_W + V_L}{2},$$

where

$$\begin{aligned} V_{T1,P2,B} &= P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(x_{2B}))V_L \\ &= \frac{V_W + V_L}{2} - [1 - P_G(x_{2B})(1 - P_G(\xi))]\frac{V_W - V_L}{2} \\ V_{T2,P2,A} &= P_G(y_{2A})V_W + (1 - P_G(y_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}] \\ &= \frac{V_W + V_L}{2} + [1 - (1 - P_G(y_{2A}))P_G(\xi)]\frac{V_W - V_L}{2} \end{aligned}$$

We substitute the equations of $V_{T1,P2,B}$ and $V_{T2,P2,A}$ into $V_{T2,P1,E}$ as follows:

$$V_{T2,P1,E} = \frac{V_W + V_L}{2} + P_G(y_{1E})\{\phi(T1;0:1)[1 - P_G(x_{2B})(1 - P_G(\xi))]$$
$$+(1 - \phi(T1;0:1))[1 - (1 - P_G(y_{2A}))P_G(\xi)]\}\frac{V_W - V_L}{2}$$

The optimal kicking strategy, y_{1E}, satisfies the following first-order condition:

$$P'_G(y_{1E})\{\alpha_1\frac{V_W - V_L}{2} + U_G\} + P'_O(y_{1E})U_O = 0, \text{ where}$$
$$\alpha_1 = \phi(T1;0:1)[1 - P_G(x_{2B})(1 - P_G(\xi))] + (1 - \phi(T1;0:1))[1 - (1 - P_G(y_{2A}))P_G(\xi)].$$

When $T1$ scores in Round 1, the value function for $T2$ is

$$V_{T2,P1,B} = P_G(y_{1B})\frac{V_W + V_L}{2}+(1-P_G(y_{1B}))[(1-\phi(T1;1:0))V_{T2,P2,B}+\phi(T1;1:0)(V_W+V_L-V_{T1,P2,A})],$$

where

$$\begin{aligned} V_{T2,P2,B} &= P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(y_{2B}))V_L \\ &= \frac{V_W + V_L}{2} - [1 - P_G(y_{2B})(1 - P_G(\xi))]\frac{V_W - V_L}{2} \\ V_{T1,P2,A} &= P_G(x_{2A})V_W + (1 - P_G(x_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}] \\ &= \frac{V_W + V_L}{2} + [1 - (1 - P_G(x_{2A}))P_G(\xi)]\frac{V_W - V_L}{2} \end{aligned}$$

We substitute the equations of $V_{T2,P2,B}$ and $V_{T1,P2,A}$ into $V_{T2,P1,B}$ as follows:

$$V_{T2,P1,B} = \frac{V_W + V_L}{2} - (1 - P_G(y_{1B}))[(1 - \phi(T1;1:0))[1 - P_G(y_{2B})(1 - P_G(\xi))]$$
$$+\phi(T1;1:0)[1 - (1 - P_G(x_{2A}))P_G(\xi)]]\frac{V_W - V_L}{2}$$

The optimal kicking strategy, y_{1B}, satisfies the following first-order condition:

$$P'_G(y_{1B})\{\alpha_2 \frac{V_W - V_L}{2} + U_G\} + P'_O(y_{1B})U_O = 0, \text{ where}$$
$$\alpha_2 = (1 - \phi(T1;1:0))[1 - P_G(y_{2B})(1 - P_G(\xi))] + \phi(T1;1:0)[1 - (1 - P_G(x_{2A}))P_G(\xi)]$$

Then $y_{1B} = y_{1E}$ iff $\alpha_1 = \alpha_2$ iff

$$\begin{aligned}
&\phi(T1;0:1)[1 - P_G(x_{2B})(1 - P_G(\xi))] + (1 - \phi(T1;0:1))[1 - (1 - P_G(y_{2A}))P_G(\xi)]]\\
&= (1 - \phi(T1;1:0))[1 - P_G(y_{2B})(1 - P_G(\xi))] + \phi(T1;1:0)[1 - (1 - P_G(x_{2A}))P_G(\xi)]\\
&\iff (1 - \phi(T1;0:1) - \phi(T1;1:0))[1 - (1 - P_G(y_{2A}))P_G(\xi)]\\
&= (1 - \phi(T1;0:1) - \phi(T1;1:0))[1 - P_G(x_{2B})(1 - P_G(\xi))]\\
&\iff (1 - \phi(T1;0:1) - \phi(T1;1:0))[(1 - P_G(y_{2A}))P_G(\xi) - P_G(x_{2B})(1 - P_G(\xi))] = 0
\end{aligned}$$

However, $(1 - P_G(y_{2A}))P_G(\xi) - P_G(x_{2B})(1 - P_G(\xi)) > 0$ as $\overline{x} > \xi > x_{2B}$ and $y_{2A} < \xi$. Accordingly, $y_{1B} = y_{1E}$ if and only if $\phi(T1;0:1) + \phi(T1;1:0) = 1$.

First Round, First Kick

Finally, we solve for $T1$'s optimal kicking strategy in Round 1. The value function for $T1$ is

$$\begin{aligned}
V_{T1} &= P_G(x_1)[V_W + V_L - V_{T2,P1,B}] + (1 - P_G(x_1))[V_W + V_L - V_{T2,P1,E}]\\
&= V_W + V_L - P_G(x_1)V_{T2,P1,B} - (1 - P_G(x_1))V_{T2,P1,E}
\end{aligned}$$

We substitute the equations of $V_{T2,P1,B}$ and $V_{T2,P1,E}$ into V_{T1} as follows:

$$V_{T1} = \frac{V_W + V_L}{2} + [P_G(x_1)(1 - P_G(y_{1B}))\alpha_2 - (1 - P_G(x_1))P_G(y_{1E})\alpha_1]\frac{V_W - V_L}{2}$$

The optimal kicking strategy, x_1, satisfies the following first-order condition:

$$P'_G(x_1)\{[(1 - P_G(y_{1B}))\alpha_2 + P_G(y_{1E})\alpha_1]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_1)U_O = 0$$

Therefore

$$x_1 \gtreqless y_{1E} \iff (1 - P_G(y_{1B}))\alpha_2 \gtreqless (1 - P_G(y_{1E}))\alpha_1$$

On the other hand, we have

$$V_{T1} = \frac{V_W + V_L}{2} \iff P_G(x_1)(1 - P_G(y_{1B}))\alpha_2 = (1 - P_G(x_1))P_G(y_{1E})\alpha_1$$

Given that both teams have an equal chance of winning in sudden-death rounds and $V_{T2,P2,E} = V_{T1,P2,E} = \frac{V_W+V_L}{2}$, ϕ is sequentially fair if and only if $V_{T1} = \frac{V_W+V_L}{2}$. We first make the following claim:

Claim 1 $P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$ if and only if $(1-P_G(y_{1B}))\alpha_2 = (1-P_G(y_{1E}))\alpha_1$.

Proof of Claim 1

(i) $P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$ implies $(1-P_G(y_{1B}))\alpha_2 = (1-P_G(y_{1E}))\alpha_1$.

Suppose to the contrary that $(1-P_G(y_{1B}))\alpha_2 \neq (1-P_G(y_{1E}))\alpha_1$ but $P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$. If $(1-P_G(y_{1B}))\alpha_2 > (1-P_G(y_{1E}))\alpha_1$, then from the first-order condition of x_1 we have $\overline{x} > x_1 > y_{1E}$. Then $P_G(x_1)(1-P_G(y_{1B}))\alpha_2 > P_G(y_{1E})(1-P_G(y_{1B}))\alpha_2 > P_G(y_{1E})(1-P_G(y_{1E}))\alpha_1 > (1-P_G(x_1))P_G(y_{1E})\alpha_1$, a contradiction. The other case can be analyzed in a similar fashion.

(ii) $(1-P_G(y_{1B}))\alpha_2 = (1-P_G(y_{1E}))\alpha_1$ implies $P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$.

If $(1-P_G(y_{1B}))\alpha_2 = (1-P_G(y_{1E}))\alpha_1$, then from the first-order condition of x_1 we have $x_1 = y_{1E}$, which in turn implies

$$P_G(x_1)(1-P_G(y_{1B}))\alpha_2 = P_G(y_{1E})(1-P_G(y_{1B}))\alpha_2 = P_G(y_{1E})(1-P_G(y_{1E}))\alpha_1 = (1-P_G(x_1))P_G(y_{1E})\alpha_1$$

Hence the Claim is established. $\diamondsuit$

Accordingly, ϕ is sequentially fair if and only if $(1-P_G(y_{1B}))\alpha_2 = (1-P_G(y_{1E}))\alpha_1$. This equality holds for an arbitrary pair of (feasible) probabilities, $\{P_G, P_O\}$, if and only if $\alpha_1 = \alpha_2$, which holds if and only if $\phi(T1; 0:1) + \phi(T1; 1:0) = 1$, i.e., ϕ is uneven score symmetric. $\blacksquare$

Proof of Theorem 4. We show that by putting the better player in the first round, the better team has a higher chance of winning under an uneven-score symmetric mechanism. Consider two subcases:

(i) When the better player is in T2.

Since the better player is placed in the first round, the second-round maximization problems remain unchanged. Following the proof of Theorem 3, we have $x_{2A} = y_{2A} > x_{2B} = y_{2B}$, and the last kicker's optimal kicking strategy is ξ.

Next we study the second team's optimal kicking strategy in Round 1. When $T1$ does not score in Round 1, the value function for $T2$ is[27]

$$V_{T2,P1,E} = \widetilde{P}_G(y_{1E})[\phi(T1;0:1)(V_W+V_L-V_{T1,P2,B})+(1-\phi(T1;0:1))V_{T2,P2,A}]+(1-\widetilde{P}_G(y_{1E}))\frac{V_W+V_L}{2},$$

[27]Recall that kicking order $T1$ in expression $\phi(T1; g_{T1} : g_{T2})$ refers to the beginning of the second round when T1 kicked first in the first round.

where

$$V_{T1,P2,B} = P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(x_{2B}))V_L$$
$$= \frac{V_W + V_L}{2} - [1 - P_G(x_{2B})(1 - P_G(\xi))]\frac{V_W - V_L}{2}$$

$$V_{T2,P2,A} = P_G(y_{2A})V_W + (1 - P_G(y_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}]$$
$$= \frac{V_W + V_L}{2} + [1 - (1 - P_G(y_{2A}))P_G(\xi)]\frac{V_W - V_L}{2}$$

Therefore

$$V_{T2,P1,E} = \frac{V_W + V_L}{2} + \widetilde{P}_G(y_{1E})\{\phi(T1; 0:1)[1 - P_G(x_{2B})(1 - P_G(\xi))]$$
$$+(1 - \phi(T1; 0:1))[1 - (1 - P_G(y_{2A}))P_G(\xi)]\}\frac{V_W - V_L}{2}$$

The optimal kicking strategy, y_{1E}, satisfies the following first-order condition:

$$\widetilde{P}'_G(y_{1E})\{\alpha_1 \frac{V_W - V_L}{2} + U_G\} + \widetilde{P}'_O(y_{1E})U_O = 0, \text{ where}$$

$$\alpha_1 = \phi(T1; 0:1)[1 - P_G(x_{2B})(1 - P_G(\xi))] + (1 - \phi(T1; 0:1))[1 - (1 - P_G(y_{2A}))P_G(\xi)].$$

When $T1$ scores in Round 1, the value function for $T2$ is

$$V_{T2,P1,B} = \widetilde{P}_G(y_{1B})\frac{V_W + V_L}{2} + (1 - \widetilde{P}_G(y_{1B}))[(1 - \phi(T1; 1:0))V_{T2,P2,B} + \phi(T1; 1:0)(V_W + V_L - V_{T1,P2,A})],$$

where

$$V_{T2,P2,B} = P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(y_{2B}))V_L$$
$$= \frac{V_W + V_L}{2} - [1 - P_G(y_{2B})(1 - P_G(\xi))]\frac{V_W - V_L}{2}$$

$$V_{T1,P2,A} = P_G(x_{2A})V_W + (1 - P_G(x_{2A}))[(1 - P_G(\xi))V_W + P_G(\xi)\frac{V_W + V_L}{2}]$$
$$= \frac{V_W + V_L}{2} + [1 - (1 - P_G(x_{2A}))P_G(\xi)]\frac{V_W - V_L}{2}$$

We substitute the equations of $V_{T2,P2,B}$ and $V_{T1,P2,A}$ into $V_{T2,P1,B}$ as follows:

$$V_{T2,P1,B} = \frac{V_W + V_L}{2} - (1 - \widetilde{P}_G(y_{1B}))[(1 - \phi(T1; 1:0))[1 - P_G(y_{2B})(1 - P_G(\xi))]$$
$$+\phi(T1; 1:0)[1 - (1 - P_G(x_{2A}))P_G(\xi)]]\frac{V_W - V_L}{2}$$

The optimal kicking strategy, y_{1B}, satisfies the following first-order condition:

$$\widetilde{P}'_G(y_{1B})\{[(1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))] \\ +\phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)]]\frac{V_W-V_L}{2}+U_G\}+\widetilde{P}'_O(y_{1B})U_O=0$$

Given that $y_{2B}=x_{2B}$ and $x_{2A}=y_{2A}$, the first-order condition can be rewritten as

$$P'_G(y_{1B})\{\alpha_2\frac{V_W-V_L}{2}+U_G\}+P'_O(y_{1B})U_O=0,\ \text{where}$$

$$\alpha_2=(1-\phi(T1;1:0))[1-P_G(x_{2B})(1-P_G(\xi))]+\phi(T1;1:0)[1-(1-P_G(y_{2A}))P_G(\xi)].$$

Under a sequentially fair mechanism, $\phi(T1;0:1)+\phi(T1;1:0)=1$, and we have $\alpha_1=\alpha_2$. Accordingly, $y_{1E}=y_{1B}$. Finally, we solve for $T1$'s optimal kicking strategy in Round 1. The value function for $T1$ is

$$\begin{aligned} V_{T1} &= P_G(x_1)[V_W+V_L-V_{T2,P1,B}]+(1-P_G(x_1))[V_W+V_L-V_{T2,P1,E}] \\ &= V_W+V_L-P_G(x_1)V_{T2,P1,B}-(1-P_G(x_1))V_{T2,P1,E} \\ &= \frac{V_W+V_L}{2}+[P_G(x_1)(1-\widetilde{P}_G(y_{1B}))\alpha_2-(1-P_G(x_1))\widetilde{P}_G(y_{1E})\alpha_1]\frac{V_W-V_L}{2} \end{aligned}$$

The optimal kicking strategy, x_1, satisfies the following first-order condition:

$$\begin{aligned} &P'_G(x_1)\{[(1-\widetilde{P}_G(y_{1B}))\alpha_2+\widetilde{P}_G(y_{1E})\alpha_1]\frac{V_W-V_L}{2}+U_G\}+P'_O(x_1)U_O=0 \\ &\Longrightarrow P'_G(x_1)\{\alpha_1\frac{V_W-V_L}{2}+U_G\}+P'_O(x_1)U_O=0 \end{aligned}$$

Therefore $x_1=y_{1E}=y_{1B}$, and

$$V_{T1}=\frac{V_W+V_L}{2}+[P_G(x_1)(1-\widetilde{P}_G(x_1))-(1-P_G(x_1))\widetilde{P}_G(x_1)]\alpha_1\frac{V_W-V_L}{2}<\frac{V_W+V_L}{2}.$$

Hence T2 has a higher chance of winning.

(ii) When the better player is in T1. Following the same procedure in (i), we conclude $x_1=y_{1E}=y_{1B}$. But now V_{T1} becomes

$$\begin{aligned} V_{T1} &= \widetilde{P}_G(x_1)[V_W+V_L-V_{T2,P1,B}]+(1-\widetilde{P}_G(x_1))[V_W+V_L-V_{T2,P1,E}] \\ &= V_W+V_L-\widetilde{P}_G(x_1)V_{T2,P1,B}-(1-\widetilde{P}_G(x_1))V_{T2,P1,E} \\ &= \frac{V_W+V_L}{2}+[\widetilde{P}_G(x_1)(1-P_G(x_1))-(1-\widetilde{P}_G(x_1))P_G(x_1)]\alpha_1\frac{V_W-V_L}{2}>\frac{V_W+V_L}{2}. \end{aligned}$$

Again, the team with a better player has a higher chance of winning. ■

Proof of Theorem 5. Without loss of generality, assume T1 kicks first in the first sudden-death round (i.e., in Round $n+1$). In a state-symmetric equilibrium, denote by x_I the optimal kicking strategy for the first kicker in each sudden-death round, and x_B (x_E) the optimal kicking strategy for the second kicker in each sudden-death round when the score is behind (tied). Let V_{T1} (V_{T2}) denote T1's (T2's) value function at the beginning of the first sudden-death round (Round $n+1$). Then

$$\begin{aligned} V_{T1} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]V_{T2} \\ &\quad + P_G(x_I)(1-P_G(x_B))V_W + (1-P_G(x_I))P_G(x_E)V_L \\ V_{T2} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]V_{T1} \\ &\quad + P_G(x_I)(1-P_G(x_B))V_L + (1-P_G(x_I))P_G(x_E)V_W \end{aligned}$$

We substitute V_{T2} into the equation of V_{T1} as follows:

$$\begin{aligned} V_{T1} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]^2 V_{T1} \\ &\quad + \{[P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]P_G(x_I)(1-P_G(x_B)) + (1-P_G(x_I))P_G(x_E)\}V_L \\ &\quad + \{[P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))](1-P_G(x_I))P_G(x_E) + P_G(x_I)(1-P_G(x_B))\}V_W \end{aligned}$$

Then V_{T1} can be solved as:

$$V_{T1} = \gamma V_W + (1-\gamma)V_L, \text{ where}$$

$$\gamma = \frac{1-(1-P_G(x_I))P_G(x_E)}{2-(1-P_G(x_I))P_G(x_E) - P_G(x_I)(1-P_G(x_B))}.$$

As this is a zero-sum game, we have $V_{T2} = (1-\gamma)V_W + \gamma V_L$.

The optimal kicking strategy, x_I, satisfies the following first-order condition:

$$P'_G(x_I)\{[P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L + U_G\} + P'_O(x_I)U_O = 0.$$

Similarly, the optimal kicking strategies x_B and x_E are determined by the following conditions:

$$\begin{aligned} P'_G(x_B)\{V_{T1} - V_L + U_G\} + P'_O(x_B)U_O = 0 \\ P'_G(x_E)\{V_W - V_{T1} + U_G\} + P'_O(x_E)U_O = 0 \end{aligned}$$

We are going to claim that all three kicking strategies are equivalent, i.e., $x_I = x_B = x_E$, which in turn implies that $V_{T1} = V_{T2} = \frac{V_W+V_L}{2}$ as $\gamma = \frac{1}{2}$, and sequential fairness is established.

First we compare x_I and x_E. Define

$$\Delta_{IE} = [P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L - (V_W - V_{T1})$$

By comparing the first-order conditions of x_I and x_E, we observe that

$$\Delta_{IE} \gtreqless 0 \text{ if and only if } x_I \gtreqless x_E.$$

Substituting the equations of V_{T2} and V_{T1} into Δ_{IE} gives us

$$\begin{aligned}\Delta_{IE} &= V_{T1} - V_{T2} - P_G(x_B)(V_W - V_{T2}) + P_G(x_E)(V_{T2} - V_L) \\ &= [2\gamma - 1 - P_G(x_B)\gamma + P_G(x_E)(1-\gamma)](V_W - V_L) \\ &= [(2 - P_G(x_B) - P_G(x_E))\gamma - 1 + P_G(x_E)](V_W - V_L).\end{aligned}$$

Plugging in the expression of γ and doing some simplifications, we have

$$\Delta_{IE} = \frac{P_G(x_I)(1-P_G(x_B)) - P_G(x_B)(1-P_G(x_E))}{2-(1-P_G(x_I))P_G(x_E) - P_G(x_I)(1-P_G(x_B))}(V_W - V_L)$$

We can then conclude that $x_I \gtreqless x_E$ if and only if $x_I \gtreqless x_B$. Next we compare x_I and x_B. Define

$$\Delta_{IB} = [P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L - (V_{T1} - V_L)$$

By comparing the first-order conditions of x_I and x_B, we observe that

$$\Delta_{IB} \gtreqless 0 \text{ if and only if } x_I \gtreqless x_B.$$

By the same token, we can simplify Δ_{IB} as

$$\Delta_{IB} = \frac{P_G(x_E)(1-P_G(x_I)) - P_G(x_B)(1-P_G(x_E))}{2-(1-P_G(x_I))P_G(x_E) - P_G(x_I)(1-P_G(x_B))}(V_W - V_L)$$

Therefore

$$x_I \gtreqless x_B \text{ if and only if } x_E \gtreqless x_B.$$

Finally we compare x_E and x_B. Define

$$\Delta_{EB} = V_W - V_{T1} - (V_{T1} - V_L)$$

Δ_{EB} can be simplified as

$$\Delta_{EB} = \frac{P_G(x_E)(1-P_G(x_I)) - P_G(x_I)(1-P_G(x_B))}{2-(1-P_G(x_I))P_G(x_E) - P_G(x_I)(1-P_G(x_B))}(V_W - V_L)$$

Accordingly,

$$x_E \gtreqless x_B \text{ if and only if } x_E \gtreqless x_I.$$

Combining all three observations (inequalities) above, we conclude that in a state-symmetric equilibrium we must have $x_I = x_E = x_B$. ■

Proof of Theorem 6. Take any mechanism ϕ and any sequentially fair mechanism φ. Construct a mechanism ψ such that for a given Sudden-death Round k, for all $n < \ell < k$, kicking-order histories $h^{\ell-1}$, and feasible scores $g_{T1} : g_{T2}$, $\psi(h^{\ell-1}; g_{T1} : g_{T2}) = \phi(h^{\ell-1}; g_{T1} :$

g_{T2}) and for all $\ell \geq k$ and $\ell \leq n$, kicking-order histories $h^{\ell-1}$, and feasible scores $g_{T1} : g_{T2}$, $\psi(h^{\ell-1}; g1 : g_{T2}) = \varphi(h^{\ell-1}; g_{T1} : g_{T2})$.

Now in the Sudden-death Round k and after, whenever the game reaches this round, the probability of winning is given as $\frac{1}{2}$ for each team. By backward induction, consider Round $k-1$. Consider the team that kicks second. Without loss of generality suppose it is $T2$, and $T1$ goes first in Round $k-1$. We can reuse the same first-order conditions for both teams that we used in the proof of Theorem 1, setting

$$V_{T1} = V_{T2} = \frac{V_W + V_L}{2}$$

as the continuation value under the sequentially fair mechanism in Round k. Suppose x is $T1$'s kicker's optimal spot, y_E is $T2$'s kicker's optimal spot when they are still tied, and y_B is $T1$'s kicker's optimal spot when $T1$ is ahead (by one goal). Recall the first-order conditions through Equation 11 (or 3):

$$P'_G(x)[P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - (1 - P_G(y_E))V_{T1} + U_G] + P'_O(x)U_O = 0$$

$$P'_G(y_B)[V_{T2} - V_L + U_G] + P'_O(y_B)U_O = 0$$

$$P'_G(y_E)[V_W - V_{T2} + U_G] + P'_O(y_E)U_O = 0$$

We rewrite T2's kicker's first-order conditions plugging in $V_{T1} = V_{T2}$:

$$P'_G(y_B)[\frac{V_W - V_L}{2} + U_G] + P'_O(y_B)U_O = 0$$

$$P'_G(y_E)[\frac{V_W - V_L}{2} + U_G] + P'_O(y_E)U_O = 0$$

The last two equations yield $y_B = y_E$ (each has a unique solution by assumptions). Given that $T1$'s equation yields:

$$P'_G(x)[\frac{V_W - V_L}{2} + U_G] + P'_O(x)U_O = 0$$

As $T1$ has the same first-order conditions as $T2$, we get $x = y_B = y_E$. So each team's winning probability is the same, $\frac{1}{2}$ in Round k, as well. The mechanism ψ is sequentially fair starting from Round k. We repeat this argument for each Sudden-death Round $\ell = k-2, k-3, ..., n+1$ and obtain the desired result. ■

Proof of Theorem 7. First observe that all kickers exert the same effort in sudden-death rounds across all sequentially fair mechanisms. Now consider a state in Round 2. It can readily be seen from the proof of Theorem 3 that the optimal kicking strategy at that state is solely determined by "the state" (the score difference and the kicking order in Round 2), and hence is independent of which sequentially fair mechanism leads to that state.

Therefore, the difference across sequentially fair mechanisms boils down to Round 1 kickers' behavior. From the proof of Theorem 3, we observe that any sequentially fair mechanism ϕ must satisfy the condition $\phi(T1; 0:1) + \phi(T1; 1:0) = 1$.[28] Moreover, under this condition, the three optimal kicking strategies in the first round are the same: $x_1 = y_{1E} = y_{1B}$, and they are determined by the following first-order condition:

$$P'_G(x_1)\{\alpha_1 \frac{V_W - V_L}{2} + U_G\} + P'_O(x_1)U_O = 0, \text{ where}$$

$$\alpha_1 = \phi(T1; 1:0)[1 - (1 - P_G(y_{2A}))P_G(\xi)] + (1 - \phi(T1; 1:0))[1 - P_G(x_{2B})(1 - P_G(\xi))].$$

Hence the higher the value of α_1, the higher the goal efficiency. As $x_{2B} < \xi$ and $y_{2A} < \xi$, $1 - P_G(x_{2B})(1 - P_G(\xi)) > 1 - (1 - P_G(y_{2A}))P_G(\xi)$. Therefore maximum effort is achieved when $\phi(T1; 1:0) = 0$, i.e., when ϕ is a behind-first mechanism. ■

Proof of Theorem 8. Observe that the mechanisms that satisfy the axioms should be behind-first, since behind-first mechanisms are the only ones that satisfy sequential fairness and goal dominance (by Theorem 7). The mechanisms that satisfy the sudden-death equality of opportunity (SDEO from now on) have to have each team kicking first in every two sudden-death rounds exactly once. Hence, the only kicking order that is simple and SDEO in the sudden-death rounds is alternating-order. Stationarity (as implied by simplicity) implies that the order of kicking switches when the score stays even between two rounds - i.e., if the state was reached after a tie in score, the order switches after this state if the tied score continues. But this does not imply how the kicking order changes if we transition to a tied score from an uneven score. Simplicity implies that we have two states as $Q = \{(T1)_1, (T2)_1\}$. Thus, we need to use the same states of sudden-death rounds also in the regular rounds. Hence, as kicking order switches when the score is tied, i.e. we transition from $(T1)_1$ to $(T2)_1$ or the other way around in the sudden-death rounds, we should do the same in the regular rounds as well. Thus, whenever a round ends with a tied score, we should reverse the kicking order. We end up with the unique machine representation in Figure 2, i.e. with the alternating-order behind-first. ■

References

[1] Abdulkadiroğlu, A. and T. Sönmez. 2013. "Matching Markets: Theory and Practice." In D. Acemoglu, M. Arello, and E. Dekel, Editors, *Advances in Economics and Econometrics*, Vol. 1, Cambridge, 3-47.

[28] Recall that kicking order $T1$ in expression $\phi(T1; g_{T2} : g_{T2})$ refers to the beginning of second round when T1 kicked first in the first round.

[2] Apesteguia, J., and I. Palacios-Huerta. 2010. "Psychological Pressure in Competitive Environments: Evidence from a Randomized Natural Experiment." *American Economic Review*, 100, 2548-2564.

[3] Aristotle. 1999. *Nicomachean Ethics* (translated by W.D. Ross), Batoche Books.

[4] Baggio, R. 2001. *Una Porta Nel Cielo: Un'autobiografia* (*A Goal in The Sky: An Autobiography* in English), Limina; see the following website for the translated quote used in this paper: http://www.theguardian.com/sport/2002/may/19/worldcupfootball2002.football.

[5] Bar-Eli, M., O.H. Azar, I. Ritov, Y. Keidar-Levin, and G. Schein. 2007. "Action Bias among Elite Soccer Goalkeepers: The Case of Penalty Kicks." *Journal of Economic Psychology*, 28, 606–621

[6] Carrillo, J.D. 2007. "Penalty Shoot-Outs: Before or After Extra Time?" *Journal of Sports Economics*, 8, 505-518.

[7] Che, Y-K. 2010. "Discussion of 'Matching Markets: Theory and Practice' by Abdulkadiroglu and Sönmez." Prepared for the 2010 World Congress of Econometric Society.

[8] Che, Y-K. and T. Hendershott. 2008. "How to Divide the Possession of a Football?" *Economics Letters*, 99, 561-565.

[9] Che, Y-K. and T. Hendershott. 2009. "The NFL Should Auction Possession in Overtime Games." *Economists' Voice* 6.9, Article 5.

[10] Chiappori, P.-A., S. Levitt, and T. Groseclose. 2002. "Testing Mixed-Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer." *American Economic Review*, 92, 1138-1151.

[11] Durham, W. 1979. *Scarcity and Survival in Central American-Ecological. Origins of the Soccer War*, Stanford University Press.

[12] Economic Sciences Nobel Prize Committee, Royal Swedish Academy. 2012. "Stable Allocations and the Practice of Market Design." *Scientific Background on the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012* (http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2012/advanced.html)

[13] Edmans, A., D. Garcia, and O. Norli. 2007. "Sports Sentiment and Stock Returns." *Journal of Finance*, 62, 1967-1998.

[14] Fudenberg, D., G. Richard, J. Stiglitz, and J. Tirole. 1983. "Preemption, Leapfrogging, and Competition in Patent Races." *European Economic Review*, 22, 3-31.

[15] Harford, T. 2006. "World Cup Game Theory: What Economics Tells Us about Penalty Kicks." The Undercover Economist (http://www.slate.com/articles/arts/the_undercover_economist/2006/06/world_cup_game_theory.html).

[16] Harris, C. and J. Vickers. 1985. "Perfect Equilibrium in a Model of a Race." *Review of Economic Studies*, 52, 193-209.

[17] Kocher, M., M. Lenz, and M. Sutter. 2012. "Psychological Pressure in Competitive Environments: New Evidence from Randomized Natural Experiments." *Management Science*,

[18] Lenten, L., J. Libich, and P. Stehlík. 2013. "Policy Timing and Footballers' Incentives: Penalties Before or After Extra Time?" *Journal of Sports Economics*, 14, 629-655.

[19] Ozanian, M. 2013. "Soccer's Most Valuable Teams: At $3.3 Billion, Real Madrid Knocks Manchester United From Top Spot." *Forbes* (http://www.forbes.com/sites/mikeozanian/2013/04/17/soccers-most-valuable-teams-real-madrid-dethrones-manchester-united-from-top-spot-at-3-3-billion/).

[20] Palacios-Huerta, I. 2003. "Professionals Play Minimax." *Review of Economic Studies*, 70, 395-415.

[21] Palacios-Huerta, I. 2014. *Beautiful Game Theory: How Soccer Can Help Economics.* Princeton.

[22] Sönmez, T. and M. U. Ünver. 2011. "Matching, Allocation, and Exchange of Discrete Resources." In J. Benhabib, A. Bisin, and M. Jackson, Editors, *Handbook of Social Economics*, 781-852, Vol. 1A, North-Holland.

[23] Sönmez, T. and M. U. Ünver. 2013. "Market Design for Kidney Exchange." In Z. Neeman, A. Roth, N. Vulkan, Editors, *Oxford Handbook of Market Design*, Oxford.

[24] Szymanski, S. 2003. "The Economic Design of Sporting Contests." *Journal of Economic Literature*, 41, 1137-1187.

[25] Zygband, P. and H. Collignon. 2011. "The Sports Market." *AT Kearney* (http://www.atkearney.com/paper/-/asset_publisher/dVxv4Hz2h8bS/content/the-sports-market/10192).

B Online Appendix: Three Regular Round sequentially fair mechanisms

Let us define $V_{i,j,s}$ to be the value function for the kicker who is the j^{th} kicker to kick in Round k when the state is $s = (s_1, s_2)$, where s_i is the score for the team who kicks i^{th} in Round k. Denote by $x_{i,j,s}$ the optimal kicking strategy for this kicker.

Third Round, Second Kick

Whether the team is currently even or behind, the optimal kicking strategy is always x^*, where x^* is determined by the following first-order condition:

$$P_G'(x^*)[\frac{V_W - V_L}{2} + U_G] + P_O'(x^*)U_O = 0$$

Third Round, First Kick

When the score is currently even ($s = (2, 2)$, $(1, 1)$ or $s = (0, 0)$), the value function for the team is $\frac{V_W+V_L}{2}$.

When $s = (0, 1)$, the value function for the kicker is

$$\begin{aligned}V_{3,1,(0,1)} &= P_G(x_{3,1,(0,1)})P_G(x^*)V_L + P_G(x_{3,1,(0,1)})(1 - P_G(x^*))\frac{V_W + V_L}{2} + (1 - P_G(x_{3,1,(0,1)}))V_L \\ &= \frac{V_W + V_L}{2} - [1 - P_G(x_{3,1,(0,1)})(1 - P_G(x^*))]\frac{V_W - V_L}{2} = \frac{V_W + V_L}{2} - \alpha_{3,1,(0,1)}\frac{V_W - V_L}{2}\end{aligned}$$

The optimal kicking strategy, $x_{3,1,(0,1)}$, satisfies the following first-order condition:

$$P_G'(x_{3,1,(0,1)})[(1 - P_G(x^*))\frac{V_W - V_L}{2} + U_G] + P_O'(x_{3,1,(0,1)})U_O = 0$$

Similarly, we have $V_{3,1,(0,1)} = V_{3,1,(1,2)}$ and $x_{3,1,(0,1)} = x_{3,1,(1,2)}$.

When $s = (1, 0)$, the value function for the kicker is

$$\begin{aligned}V_{3,1,(1,0)} &= P_G(x_{3,1,(1,0)})V_W + (1 - P_G(x_{3,1,(1,0)}))[(1 - P_G(x^*))V_W + P_G(x^*)\frac{V_W + V_L}{2}] \\ &= \frac{V_W + V_L}{2} + [1 - (1 - P_G(x_{3,1,(1,0)}))P_G(x^*)]\frac{V_W - V_L}{2} = \frac{V_W + V_L}{2} + \alpha_{3,1,(1,0)}\frac{V_W - V_L}{2}\end{aligned}$$

The optimal kicking strategy, $x_{3,1,(1,0)}$, satisfies the following first-order condition:

$$P_G'(x_{3,1,(1,0)})[P_G(x^*)\frac{V_W - V_L}{2} + U_G] + P_O'(x_{3,1,(1,0)})U_O = 0$$

Similarly, we have $V_{3,1,(1,0)} = V_{3,1,(2,1)}$ and $x_{3,1,(1,0)} = x_{3,1,(2,1)}$.

Second Round, Second Kick

Denote by $\phi_3(s)$ the prob. that the first-kicking team in Round 2 kicks first in Round 3 when the state at the end of Round 2 is s.

When $s = (0,0)$, the value function for the kicker is

$$V_{2,2,(0,0)} = P_G(x_{2,2,(0,0)})[\phi_3(0,1)(V_W + V_L - V_{3,1,(0,1)}) + (1-\phi_3(0,1))V_{3,1,(1,0)}] + (1 - P_G(x_{2,2,(0,0)}))\frac{V_W + V_L}{2}$$
$$= \frac{V_W + V_L}{2} + P_G(x_{2,2,(0,0)})\alpha_{2,2,(0,0)}\frac{V_W - V_L}{2},$$

where

$$\alpha_{2,2,(0,0)} = \phi_3(0,1)\alpha_{3,1,(0,1)} + (1-\phi_3(0,1))\alpha_{3,1,(1,0)}$$

The optimal kicking strategy, $x_{2,2,(0,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(0,0)})\{\alpha_{2,2,(0,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,2,(0,0)})U_O = 0$$

When $s = (1,0)$, the value function for the kicker is

$$V_{2,2,(1,0)} = P_G(x_{2,2,(1,0)})\frac{V_W + V_L}{2} + (1 - P_G(x_{2,2,(1,0)}))[\phi_3(1,0)(V_W + V_L - V_{3,1,(1,0)}) + (1-\phi_3(1,0))V_{3,1,(0,1)}]$$
$$= \frac{V_W + V_L}{2} - (1 - P_G(x_{2,2,(1,0)}))\alpha_{2,2,(1,0)}\frac{V_W - V_L}{2},$$

where

$$\alpha_{2,2,(1,0)} = \phi_3(1,0)\alpha_{3,1,(1,0)} + (1-\phi_3(1,0))\alpha_{3,1,(0,1)}.$$

The optimal kicking strategy, $x_{2,2,(1,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(1,0)})\{\alpha_{2,2,(1,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,2,(1,0)})U_O = 0$$

When $s = (1,1)$, the value function for the kicker is

$$V_{2,2,(1,1)} = P_G(x_{2,2,(1,1)})[\phi_3(1,2)(V_W + V_L - V_{3,1,(1,2)}) + (1-\phi_3(1,2))V_{3,1,(2,1)}] + (1 - P_G(x_{2,2,(1,1)}))\frac{V_W + V_L}{2}$$
$$= \frac{V_W + V_L}{2} + P_G(x_{2,2,(1,1)})\alpha_{2,2,(1,1)}\frac{V_W - V_L}{2},$$

where

$$\alpha_{2,2,(1,1)} = \phi_3(1,2)\alpha_{3,1,(1,2)} + (1-\phi_3(1,2))\alpha_{3,1,(2,1)}$$

The optimal kicking strategy, $x_{2,2,(1,1)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(1,1)})\{\alpha_{2,2,(1,1)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,2,(1,1)})U_O = 0$$

When $s = (2,1)$, the value function for the kicker is

$$V_{2,2,(2,1)} = P_G(x_{2,2,(2,1)})\frac{V_W + V_L}{2} + (1 - P_G(x_{2,2,(2,1)}))[\phi_3(2,1)(V_W + V_L - V_{3,1,(2,1)}) + (1-\phi_3(2,1))V_{3,1,(1,2)}]$$
$$= \frac{V_W + V_L}{2} - (1 - P_G(x_{2,2,(2,1)}))\alpha_{2,2,(2,1)}\frac{V_W - V_L}{2},$$

where

$$\alpha_{2,2,(1,0)} = \phi_3(2,1)\alpha_{3,1,(2,1)} + (1 - \phi_3(2,1))\alpha_{3,1,(1,2)}.$$

The optimal kicking strategy, $x_{2,2,(2,1)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(2,1)})\{\alpha_{2,2,(1,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,2,(2,1)})U_O = 0$$

When $s = (0,1)$, the value function for the kicker is

$$\begin{aligned}V_{2,2,(0,1)} &= P_G(x_{2,2,(0,1)})V_W + (1 - P_G(x_{2,2,(0,1)}))[\phi_3(0,1)(V_W + V_L - V_{3,1,(0,1)}) + (1 - \phi_3(0,1))V_{3,1,(1,0)}] \\ &= \frac{V_W + V_L}{2} + \alpha_{2,2,(0,1)}\frac{V_W - V_L}{2},\end{aligned}$$

where

$$\alpha_{2,2,(0,1)} = P_G(x_{2,2,(0,1)}) + (1 - P_G(x_{2,2,(0,1)}))[\phi_3(0,1)\alpha_{3,1,(0,1)} + (1 - \phi_3(0,1))\alpha_{3,1,(1,0)}].$$

The optimal kicking strategy, $x_{2,2,(0,1)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(0,1)})\{[1 - [\phi_3(0,1)\alpha_{3,1,(0,1)} + (1 - \phi_3(0,1))\alpha_{3,1,(1,0)}]]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,2,(0,1)})U_O = 0$$

When $s = (2,0)$, the value function for the kicker is

$$\begin{aligned}V_{2,2,(2,0)} &= P_G(x_{2,2,(2,0)})[\phi_3(2,1)(V_W + V_L - V_{3,1,(2,1)}) + (1 - \phi_3(2,1))V_{3,1,(1,2)}] + (1 - P_G(x_{2,2,(2,0)}))V_L \\ &= \frac{V_W + V_L}{2} - \alpha_{2,2,(2,0)}\frac{V_W - V_L}{2},\end{aligned}$$

where

$$\alpha_{2,2,(2,0)} = P_G(x_{2,2,(2,0)})[\phi_3(2,1)\alpha_{3,1,(2,1)} + (1 - \phi_3(2,1))\alpha_{3,1,(1,2)}] + 1 - P_G(x_{2,2,(2,0)}).$$

The optimal kicking strategy, $x_{2,2,(2,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(2,0)})\{[1 - [\phi_3(2,1)\alpha_{3,1,(2,1)} + (1 - \phi_3(2,1))\alpha_{3,1,(1,2)}]]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,2,(2,0)})U_O = 0$$

Second Round, First Kick

When $s = (0,0)$ or $s = (1,1)$, the value function for the team is $\frac{V_W + V_L}{2}$.

When $s = (0,1)$, the value function for the kicker is

$$\begin{aligned}V_{2,1,(0,1)} &= P_G(x_{2,1,(0,1)})(V_W + V_L - V_{2,2,(1,1)}) + (1 - P_G(x_{2,1,(0,1)}))(V_W + V_L - V_{2,2,(0,1)}) \\ &= \frac{V_W + V_L}{2} - \alpha_{2,1,(0,1)}\frac{V_W - V_L}{2},\end{aligned}$$

where

$$\alpha_{2,1,(0,1)} = P_G(x_{2,1,(0,1)})P_G(x_{2,2,(1,1)})\alpha_{2,2,(1,1)} + (1 - P_G(x_{2,1,(0,1)}))\alpha_{2,2,(0,1)}.$$

The optimal kicking strategy, $x_{2,1,(0,1)}$, satisfies the following first-order condition:

$$P'_G(x_{2,1,(0,1)})\{[\alpha_{2,2,(0,1)} - P_G(x_{2,2,(1,1)})\alpha_{2,2,(1,1)}]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,1,(0,1)})U_O = 0$$

When $s = (1, 0)$, the value function for the kicker is

$$\begin{aligned} V_{2,1,(1,0)} &= P_G(x_{2,1,(1,0)})(V_W + V_L - V_{2,2,(2,0)}) + (1 - P_G(x_{2,1,(1,0)}))(V_W + V_L - V_{2,2,(1,0)}) \\ &= \frac{V_W + V_L}{2} + \alpha_{2,1,(1,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{2,1,(1,0)} = P_G(x_{2,1,(1,0)})\alpha_{2,2,(2,0)} + (1 - P_G(x_{2,1,(1,0)}))(1 - P_G(x_{2,2,(1,0)}))\alpha_{2,2,(1,0)}.$$

The optimal kicking strategy, $x_{2,1,(1,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,1,(1,0)})\{[\alpha_{2,2,(2,0)} - (1 - P_G(x_{2,2,(1,0)}))\alpha_{2,2,(1,0)}]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,1,(1,0)})U_O = 0$$

First Round, Second Kick

When $s = (0, 0)$, the value function for the kicker is

$$\begin{aligned} V_{1,2,(0,0)} &= P_G(x_{1,2,(0,0)})[\phi_2(0,1)(V_W + V_L - V_{2,1,(0,1)}) + (1 - \phi_2(0,1))V_{2,1,(1,0)}] + (1 - P_G(x_{1,2,(0,0)}))\frac{V_W + V_L}{2} \\ &= \frac{V_W + V_L}{2} + P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{1,2,(0,0)} = \phi_2(0,1)\alpha_{2,1,(0,1)} + (1 - \phi_2(0,1))\alpha_{2,1,(1,0)}.$$

The optimal kicking strategy, $x_{1,2,(0,0)}$, satisfies the following first-order condition:

$$P'_G(x_{1,2,(0,0)})\{\alpha_{1,2,(0,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{1,2,(0,0)})U_O = 0.$$

When $s = (1, 0)$, the value function for the kicker is

$$\begin{aligned} V_{1,2,(1,0)} &= P_G(x_{1,2,(1,0)})\frac{V_W + V_L}{2} + (1 - P_G(x_{1,2,(1,0)}))[\phi_2(1,0)(V_W + V_L - V_{2,1,(1,0)}) + (1 - \phi_2(1,0))V_{2,1,(0,1)}] \\ &= \frac{V_W + V_L}{2} - (1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{1,2,(1,0)} = \phi_2(1,0)\alpha_{2,1,(1,0)} + (1 - \phi_2(1,0))\alpha_{2,1,(0,1)}.$$

The optimal kicking strategy, $x_{1,2,(1,0)}$, satisfies the following first-order condition:

$$P'_G(x_{1,2,(1,0)})\{\alpha_{1,2,(1,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{1,2,(1,0)})U_O = 0.$$

First Round, First Kick

The value function for the kicker is

$$\begin{aligned} V_{1,1,(0,0)} &= P_G(x_{1,1,(0,0)})[V_W + V_L - V_{1,2,(1,0)}] + (1 - P_G(x_{1,1,(0,0)}))[V_W + V_L - V_{1,2,(0,0)}] \\ &= \frac{V_W + V_L}{2} + [P_G(x_{1,1,(0,0)})(1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} \\ &- (1 - P_G(x_{1,1,(0,0)}))P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}]\frac{V_W - V_L}{2} \end{aligned}$$

The optimal kicking strategy, $x_{1,1,(0,0)}$, satisfies the following first-order condition:

$$P'_G(x_{1,1,(0,0)})\{[(1-P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)}+P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}]\frac{V_W - V_L}{2}+U_G\}+P'_O(x_{1,1,(0,0)})U_O = 0$$

Therefore

$$x_{1,1,(0,0)} \gtreqless x_{1,2,(0,0)} \iff (1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} \gtreqless P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}$$

On the other hand, we have

$$\begin{aligned} V_{1,1,(0,0)} = \frac{V_W + V_L}{2} &\iff P_G(x_{1,1,(0,0)})(1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} = (1 - P_G(x_{1,1,(0,0)}))P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)} \\ &\iff (1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} = (1 - P_G(x_{1,1,(0,0)}))\alpha_{1,2,(0,0)} \end{aligned}$$

The condition holds if $\phi_2(1,0) + \phi_2(0,1) = 1$.