Consumer Referrals

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Abstract

In many industries, firms reward their customers for making referrals. We analyze a monopoly's optimal policy mix of price, advertising intensity, and referral fee when buyers choose to what extent to refer other consumers to the firm. When the referral fee can be optimally set by the firm, it will charge the standard monopoly price. The firm always advertises less when it uses referrals. We extend the analysis to the case where consumers can target their referrals. In particular, we show that referral targeting could be detrimental for consumers in a low-valuation group.

Keywords: consumer referral policy, word of mouth, referral reward program, targeted advertising, targeted referrals.

JEL numbers: C7, D4, D8, L1.

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1 Introduction

Firms often pay existing customers for referring potential customers to the firms' products or services. For example, DIRECTV's Referral Offer has promised a $100 credit to any customer for referring a friend who signs up for the company's service. Referral policies are adopted in a variety of industries, including banking, health care, web design services, home remodeling, housing, vacation packages, home alarm systems, and high-speed Internet connection. They are used in the recruitment of nurses and technicians, as well as in selling cars, houses, and tickets to sporting events. Private schools, doctors, and daycare centers give out referral bonuses as well.[1] Such referral programs are often seen as "Win/Win/Win" because existing customers, potential customers, and firms all benefit. This is not surprising, given that consumer referrals raise consumer awareness about the product.[2]

How can firms efficiently manage consumer referrals? We ask whether a firm would set a higher or lower price in the presence of consumer referrals. On the one hand, the referral fee adds to the marginal cost of selling the product, which prompts the firm to raise its price. On the other hand, a higher price reduces purchase probability, diminishing referral incentives. It is therefore not clear in which direction the optimal price would move. We also answer the following questions: when would a firm use consumer referrals, would it engage in more or less advertising under referrals, and what are the overall welfare effects of referral

[1] Casual observation of referral policies suggests that referral rewards are usually paid out to existing customers for referring new customers who buy the product. Referral payments are typically made in the form of cash, deposit, gift certificate, bonus points, free product or service, or entry into a lottery.

[2] Recommendations from other people are considered more trustworthy than direct advertising. According to Nielsen's 2012 Global Trust in Advertising Survey of 28,000 Internet respondents from 56 countries, consumers around the world continue to find recommendations from personal acquaintances by far the most credible: 92 percent of respondents trust ("completely" or "somewhat") recommendations from people they know, and 90 percent find these recommendations ("highly" or "somewhat") relevant. In comparison, ads are found trustworthy or relevant by only 30-50 percent of respondents, depending on the media.

policies?

Designing an optimal referral policy is complicated by the interactions among the firm's pricing, advertising, and referral policies. A critical part of our analysis is that we endogenize consumers' decisions about how engaged they wish to be in making referrals. To our knowledge, no other study has taken such a comprehensive approach to developing an analytical model of consumer referrals.

We introduce consumer referrals into a new product market served by a monopoly. The firm has two alternative means of raising consumer awareness about its product. It can inform consumers directly about the existence of the product and its price through advertising, or it can do so indirectly through consumer referrals. Consumers who receive the firm's ads and decide to purchase the product choose the extent to which they refer other consumers. The firm's referral policy provides a monetary reward (referral fee) for each successful referral. Consumers can make multiple referrals at a constant marginal cost. Since referrals are sent independently and at random, in equilibrium there is congestion in referral messages. The firm can manage referral incentives in our model by changing its policy mix (price, advertising intensity, and referral fee).

We first characterize the consumer referral equilibrium for any finite number of referring consumers and any policy mix chosen by the firm (Proposition 1). Considering a large population of consumers, we then analyze the firm's optimal policy and, in particular, its pricing strategy. We find that the profit-maximizing price is the monopoly price (Proposition 2). The firm uses its referral fee to manage the referral activity and its price to maximize the profitability of sales to consumers aware of its product. However, when the referral fee is capped below the optimal level or the marginal referral cost is increasing, the firm sets its price below the monopoly level. We also find that the firm always advertises less when it uses consumer referrals. In Proposition 3, we provide comparative static results for the optimal policy mix. In Proposition 4, we show that the firm chooses to use referrals as long as the referral cost is not too high. Welfare results are stated in Proposition 5. We show that allowing the firm to use consumer referrals results in a higher product awareness and a Pareto-superior allocation.

Oftentimes, consumers have superior information about other consumers' preferences. For example, when consumers belong to a social group or network, the members of that group tend to have similar tastes or simply know more about each other. This informational advantage allows consumers to target their referrals. To study targeted referrals, we assume that there are two groups of consumers: high-type and low-type. High-type consumers tend to have higher valuations than low-type consumers. Although the firm does not know to which group consumers belong, fellow consumers do. If willingness-to-pay distributions are significantly different across groups, then only high-type consumers receive referrals, the price is higher, the ratio of referral fee to profit margin is lower, and the advertising level is lower under targeted referrals than if consumers do not know to which group others belong (Proposition 6). Quite intuitively, if consumers have better information, the monopoly relies less on advertising and more on targeted referrals. Interestingly, low-type consumers can suffer from superior consumer information that enables referral targeting. Proofs are delegated to Appendix A.

2 Brief Literature Review

The related literature on consumer referrals is fairly scant: Jun and Kim (2008), Byalogorsky et al. (2005), and Galeotti and Goyal (2009). Jun and Kim (2008) assume a finite chain of consumers with i.i.d. random valuations. Consumers are rational and forward looking: they consider the expected benefit from giving a referral when making their purchase decisions. The authors show that even though the firm sets a common price and referral fee, it effectively price-discriminates between the consumers located early in the chain (who are more valuable to the monopoly) and those later in the chain.[3]

Byalogorsky et al. (2005) use the same setup as Jun and Kim (2008), but adopt a behavioral assumption that consumers make referrals whenever their overall expected utility from buying a product and making a referral exceeds a critical level of "consumer delight." When consumers are easy to please, a referral program would not be used because referrals would be made without it. But when consumers are not so easy to please, the firm would use both a positive referral fee and a lower price. These papers do not consider the optimal use of advertising as an alternative communication channel, rule out referral congestion, and consider a simple chain network.[4]

In contrast, Galeotti and Goyal (2009) consider a more complex network model in which consumers make multiple referrals at no cost. They analyze the optimal advertising policy

[3] Arbatskaya and Konishi (forthcoming) justify the tie-breaking rule used in Jun and Kim (2008), showing that effective price discrimination is indeed a common feature of the model in the second-best environment (with a common referral fee and same price for all consumers) and the first-best environment (with a sufficient number of policy tools). Jun et al. (2006) extend the framework of Jun and Kim (2008), allowing for more complex networks and transforming them into tree networks by imposing a tie-breaking rule.

[4] An alternative model was offered by Mayzlin (2006), who looks at the case where advertising and word of mouth are both used to influence consumer choices between vertically differentiated products. While in her model, consumers cannot distinguish between promotional chat and consumer recommendations, in our model, advertising and referrals are two distinct information channels.

and show that using consumer referrals would unambiguously increase profits. At the same time, an increase in the level of social interaction can increase or decrease the level of advertising and profits. While they concentrate on the relationship between network structure and optimal advertising strategy, we assume a simple complete network and analyze the optimal policy mix for the firm when consumer referral decisions are determined endogenously.

A related stream of literature focuses on advertising and congestion. In his pioneering paper, Butters (1977) formulated a *competitive* model of advertising in which firms send a number of ads to consumers randomly, informing them about the existence of the product and its price. Butters shows that price dispersion occurs in equilibrium. In his model, some portion of ads is wasted due to congestion, but the level of congestion (the number of ads) is socially optimal. Van Zandt (2004), Anderson and de Palma (2009), and Johnson (2013) present alternative information congestion models in which consumers ignore some of the advertisements they receive.[5] They all show that, as the number of ads decreases, both firms and consumers are better off because of a reduction in congestion, though the reasons for this result differ. In contrast, in our model, referrals are subject to congestion because we endogenize the referral intensity. Despite the presence of referral congestion, referrals are underprovided in our monopoly model.

There is also literature on targeted advertising. Van Zandt (2004) and Johnson (2013) assume that firms sell heterogeneous products and have some information about consumer preferences. They analyze targeted advertising policies in oligopolistic markets (Van Zandt, 2004) and competitive markets (Johnson, 2013). Although the mechanisms are different,

[5] Van Zandt (2004) assumes that all consumers can process up to a certain number of ads, while Anderson and de Palma (2009) assume that a consumer's cost of processing ads depends on the number of ads she receives. Johnson (2013) allows consumers to decide what fraction of ads to block.

both papers show that improved targeting increases firms' profits and makes consumers better off. Esteban et al. (2001) consider a monopoly choosing between mass and targeted advertising. With targeted advertising, the number of wasted ads is reduced, but the monopoly power increases. The authors show that the latter welfare loss tends to exceed the former benefit. Galeotti and Moraga-Gonzalez (2008) analyze a simple oligopolistic model of targeted advertising and show that market segmentation generates higher profits in equilibrium. These papers assume that firms possess information on consumer types and therefore can conduct targeted advertising.[6] In contrast, we assume that consumers have superior information, and the firm uses them to raise awareness about its product.

3 The Model of Consumer Referrals

In this section, we describe the equilibrium consumer referral behavior in a market served by a monopoly that sets a price, advertises its product, and pays referral rewards to existing customers for referring new customers.

Each of N consumers purchases at most one unit of a product, where N is a large finite number. Consumers' values for the product v follow a known distribution function $G(\cdot)$, with a log-concave survival function $1 - G(\cdot)$ and a continuously differentiable density g, defined on $[\underline{v}, \overline{v}]$ with $\underline{v} \geq 0$. The firm chooses its (nondiscriminatory) price $p \geq 0$, its level of advertisement $a \in [0, 1]$ (a fraction of consumers reached by advertisements), and a referral policy characterized by a referral fee $r \geq 0$ that it pays to its consumers for successfully referring others to its product. The marginal cost of production is $c \geq 0$. Advertisements

[6] Kim (2010) considers a monopoly that can target a single consumer in a social network with information about its product. The informed consumer then starts the diffusion of product information across the consumer network. Kim shows that the optimal targeting strategy for the firm is based on a concept of δ-closeness centrality, which is a modification of closeness centrality to an environment where discounting matters.

and referrals inform consumers about the existence of the firm's product and its price. Only consumers who are informed of the product through an advertisement ("the informed") or a referral can purchase the product.

Consumers can attempt to collect referral fees by referring other people. In the baseline model, we assume a complete consumer network. That is, each consumer can refer any other consumer.[7] We also assume that the firm informs consumers about its referral program only after they purchase the product. Although this assumption may reflect reality,[8] it could be relaxed since, as we will show, in equilibrium consumers derive zero net benefit from making referrals. It follows that the informed consumers purchase the product whenever their values v exceed price p, which happens with probability $1 - G(p)$. The number of referrers is denoted by n. We treat n as fixed in this section and endogenize it in the next section. Each referral attempt costs $\rho > 0$, which captures the cost of informing a contact about the product. On the benefit side, referral attempts can be successful or unsuccessful. If a referrer's contact has a low willingness-to-pay and/or is already informed, the referral attempt will not be successful. Furthermore, potential referrals may have been contacted by others and may assign credit for the referral to another person.

We assume that $n \geq 2$ referrers simultaneously and independently choose a fraction $\theta \in [0, 1]$ of consumers to send their referrals at random.[9] Our focus is on the symmetric consumer referral equilibrium, in which each of $n < N$ referrers suggests the product to a

[7] This assumption can be relaxed. For example, if we assume that only a fraction of consumers can make referrals, all the results will be the same. In Section 6, we look at targeted referrals and assume that there are two disjoint groups of people (Group H and Group L), with consumers making referrals to others in their group.

[8] For example, a daycare might send the following message to its current families: "You may or may not be aware that you can earn a free week of childcare by referring a family to our program."

[9] Ignoring the integer problem, we treat θ as a continuous variable. We could equivalently consider each referrer's strategy to be a probability of making a referral to each consumer in i.i.d. manner.

fraction θ^E of the total consumer population at random. As more referrals are made, an increasingly smaller fraction of referrals are successful. The *referral reach* R is the share of consumers (within any given population of consumers) that are reached by the referral messages. The referral reach is described by $R = 1 - (1-\theta)^n$ since with probability $(1-\theta)^n$ a consumer will not receive any referrals.

The per-consumer number of referrals sent by n referrers is called the *referral intensity* $S = n\theta$. One prominent feature of referrals is congestion: for any fraction $\theta > 0$, we define *referral congestion* as the expected ratio of the number of referral messages sent by all referrers to the expected number of referrals registered by them:

$$\phi(\theta; n) \equiv \frac{S(\theta; n)}{R(\theta; n)} = \frac{n\theta}{1-(1-\theta)^n}. \tag{1}$$

There is always congestion in referral messages and $\phi(\theta; n) > 1$.

To find the (symmetric) equilibrium strategy θ^E, we need to look at the incentives of consumers to refer. The probability that a consumer who receives a referral buys the good is $(1-a)(1-G(p))$, which is the fraction of consumers who have not received an ad and are willing to buy the product.

Proposition 1. *Suppose the firm chooses a policy mix of price p, advertising intensity a, and referral fee r. Then, for any number of referrers n, the equilibrium consumer referral strategy $\theta^E > 0$ exists for all r above the critical level $r_0 \equiv \rho/[(1-a)(1-G(p))]$. No referrals are sustained for $r \leq r_0$. If the equilibrium strategy is interior, $\theta^E \in (0,1)$, it is uniquely determined by*

$$r(1-a)(1-G(p)) = \rho\phi(\theta^E; n) \tag{2}$$

and the equilibrium referral probability θ^E and referral congestion ϕ increase when the referral *fee r increases and when price p, advertising intensity a, and referral cost ρ decrease.*

In Proposition 1, we describe the unique symmetric referral equilibrium.[10] In the referral equilibrium, each referring consumer is indifferent between sending and not sending an additional referral. Moreover, due to constant referral cost ρ, the expected net benefit from making each referral is zero for any referrer and at the aggregate level. Interestingly, even if there were another period for referrals to be made after the initial referral market clears, no consumer would make an additional referral.

The comparative statics results of Proposition 1 are intuitive. The factors that increase the benefit of making referrals (a higher referral fee r, lower price p, or lower advertising intensity a) or reduce referral cost ρ must increase referral probability and congestion in order for consumers to remain indifferent between referring and not referring. In the proof of Proposition 1, we show that, quite intuitively, congestion increases in θ and n; therefore, the equilibrium referral intensity θ^E is also negatively affected by the number of referring consumers n. At the same time, the equilibrium referral congestion is independent of the number of referrers n.

4 Monopoly Choice of Price, Advertising, and Referral Policy

In this section, we characterize the optimal (profit-maximizing) monopoly policy mix (p^*, a^*, r^*) and derive the conditions under which a firm would choose to support consumer referrals.[11]

[10] Although there also exists a continuum of asymmetric equilibria, the equilibrium referral congestion is identical in all equilibria.

[11] We describe the firm's decision in terms of a choice of a triple (p, a, r), but the order in which the monopoly chooses the components of its policy mix does not matter. Even in the case where the firm announces the product's price p but does not mention its referral policy in the ads it sends to consumers, our results are the same. This is because competition between consumers at the referral stage drives the

The cost of advertising per consumer is described by a twice continuously differentiable function $C(a)$, which increases at an increasing rate in the fraction a of consumers reached: $C'(a) > 0$ and $C''(a) > 0$. We also assume that $C(0) = 0$, $C'(0) = 0$, and $C(\bar{a}) > \bar{v} - c$ holds for some $\bar{a} \in (0, 1)$, where $\bar{a}$ is a prohibitive level of advertising.

From now on, we assume that N is large enough that we can treat the firm's per consumer profit function as a deterministic function and calculate the number of consumers who purchase the product and can make referrals as $n = a\left(1 - G\left(p\right)\right)N$. With a large N, we can describe referral reach as a function of referral intensity by using an approximation[12]

$$R = 1 - (1 - \theta)^n \simeq 1 - e^{-n\theta} = 1 - e^{-S}$$

since $\ln(1 - \theta)^n = n \ln(1 - \theta) \simeq -n\theta$ for large N and $S = n\theta$. Inverting this relationship, we can then write referral intensity and congestion as functions of only referral reach R:

$$S = S\left(R\right) = -\ln\left(1 - R\right)$$

and the referral congestion $\phi(\theta, n)$ is approximated by

$$\varphi = \varphi(R) = \frac{S}{R} = \frac{-\ln\left(1 - R\right)}{R},$$

where congestion as a function of referral reach $R \in (0, 1)$ satisfies $\varphi(R) \geq 1$, $\varphi'(R) > 0$, $\varphi''(R) > 0$, $\lim_{R \to 0} \varphi(R) = 1$, and $\lim_{R \to 1} \varphi(R) = \infty$. In particular, for any firm's policy mix $(p,\, a,\, r)$, the equilibrium referral congestion $\phi(\theta^E; n)$ can be written as a function of only the equilibrium referral reach R^E: $\phi(\theta^E; n) = \varphi(R^E)$.

expected net benefit from making referrals to zero for any policy mix of the firm, as long as the firm supports active referrals.

[12] This approximation argument is used, for example, in the presentation of Butters (1977) advertisement model in Tirole (1988).

We can then use Proposition 1 to describe the equilibrium referral reach R^E as a function of policy variables p, a, and r: $R^E = R^E(p, a, r)$. The equilibrium referral condition (2) of Proposition 1 implies that the aggregate benefit of referring equals the aggregate cost of making referrals:

$$r(1-a)(1-G(p))R^E = \rho S\left(R^E\right). \tag{3}$$

The firm can achieve a higher equilibrium referral reach when it sets a lower price, advertises less, and offers a higher referral fee. That is, $\frac{\partial R^E}{\partial p} < 0$, $\frac{\partial R^E}{\partial a} < 0$, and $\frac{\partial R^E}{\partial r} > 0$. The referral reach is also higher when the referral cost is lower: $\frac{\partial R^E}{\partial \rho} < 0$.

The firm's per-consumer profit is:

$$\Pi(p, a, r) = a(p-c)(1-G(p)) + R^E(p-c-r)(1-a)(1-G(p)) - C(a), \tag{4}$$

where $R^E = R^E(p, a, r)$. The first term captures profits from consumers who purchase after receiving an ad, and the second one from consumers who purchase the product by referrals.

This permits us to rewrite the firm's profit in (4) as:

$$\Pi(p, a, r) = \left[a + (1-a)R^E\right]\pi(p) - \rho S\left(R^E\right) - C(a), \tag{5}$$

where $\pi(p) = (p-c)(1-G(p))$ is the expected profit from an informed consumer.

We next describe the optimal policy mix of the firm that uses referrals – its choice of price, advertising, and referral fee. It is not clear if the optimal price p^* in the presence of consumer referrals is higher or lower than the standard monopoly price $p^m \equiv \arg\max_p (\pi(p))$. On the one hand, referral fees raise the cost of selling the product for the firm, and we would expect the firm to have a higher price under referrals. On the other hand, from Proposition 1 a higher price means lower referral incentives, and therefore the firm may want to set a lower price to promote referrals. Perhaps surprisingly, we can give a definite answer to this question. The two effects cancel each other out when the firm is optimally choosing its price and referral fee.

Proposition 2 states that the firm continues to set the standard monopoly price $p^m \equiv \arg\max_p (\pi(p))$ when it supports consumer referrals with an optimally chosen referral fee.

Proposition 2. *There exists an optimal policy mix* (p^*, a^*, r^*). *The firm sets its price* p^* *equal to the standard monopoly price* p^m *regardless of whether it supports consumer referrals. The firm advertises less when it uses consumer referrals.*

The existence of the optimal policy mix (p^*, a^*, r^*) follows from the continuity of the profit function with respect to policy variables and the compactness of the strategy space. The key observation of Proposition 2 is that the optimal price is the monopoly price even when the firm supports consumer referrals. As the proof shows, this holds true irrespective of the level of advertising. It is significant that we can separate the pricing decision from consumer referral considerations when the referral fee is optimally set. In such a case, the firm can ignore the effect of its price on referral reach and simply use the price to maximize the profitability of sales to consumers who are informed by ads and/or referrals.

To see why this is the case, note that the firm's profit in (5) is affected by referral fee r only through referral reach $R^E = R^E(p, a, r)$. The first-order condition with respect to r is, therefore,

$$\frac{d\Pi}{dr} = \frac{\partial \Pi}{\partial R}\frac{\partial R^E}{\partial r} = 0. \tag{6}$$

Since $\frac{\partial R^E}{\partial r} > 0$ for $r > r_0$, the referral reach R is optimized under the optimal r^*:

$$\frac{\partial \Pi}{\partial R} = (1-a)\,\pi(p) - \rho S'(R) = 0, \tag{7}$$

where $S'(R) = 1/(1-R)$. Then, the indirect effects of p and a on profits through their effects on R^E are zero. This dramatically simplifies our analysis: as long as the referral fee is optimally set, the first-order conditions for p and a set partial derivatives with respect to p and a to zero: $\frac{d\Pi}{dp} = \frac{\partial \Pi}{\partial p} = 0$ and $\frac{d\Pi}{da} = \frac{\partial \Pi}{\partial a} = 0$.

From the firm's profit equation (5), it follows that for an optimally-set referral fee, the profit-maximizing price is the standard monopoly price $p^* = p^m \equiv \arg\max_p (\pi(p))$ and the firm advertises less under referrals; i.e., $a^* < a^m$, where $a^m \equiv \arg\max_a (a\pi(p^m) - C(a))$. Intuitively, the marginal benefit of advertising is lower when the firm supports active consumer referrals because ads are wasted on consumers who become aware of the product through referrals. This result does not depend on a price level.[13]

What if for some reason r cannot be optimally set? Suppose that there is a cap on r, but the cap is sufficiently high so that referrals still occur. Then, the resulting referral reach is lower than the optimal level, i.e. $\frac{\partial \Pi}{\partial R} > 0$. Since $\frac{\partial R^E}{\partial p} < 0$ holds, the optimal price p is set below the monopoly price in this case. Similarly, the optimal advertising level a is lower than when there is no cap on r. In Section 7.1, we analyze the case where the marginal cost of making referrals increases as a consumer makes more referrals. In this case, the optimal price would also be lower than the monopoly price, since a lower price implies more referrers

[13] We can show that the optimal policy mix is unique, assuming that $r^* = 0$ when the firm does not support referrals and that $C(a)$ is sufficiently convex; i.e., $C''(a) > \rho/(1-a)^2$ for all $a \in (0, \bar{a})$. The convexity condition guarantees that the optimal advertising level is unique. Clearly, $p^* = p^m$ always holds. The interior profit-maximizing referral reach is uniquely determined by equation (7). Finally, referral fee r^* is uniquely determined by equation (3). We can also show that the profit function has a local maximum at the critical point (characterized by the system of first-order conditions) when the cost of advertising function is sufficiently convex at the equilibrium level of advertising; i.e., $C''(a) > \rho/(1-a)^2$ holds at the critical point. To demonstrate this fact, we write down the Hessian for the profit function and apply the second partial derivative test for the local maximum.

and lower total referral costs.

Proposition 3 describes the comparative static responses of the optimal monopoly policy mix (p^*, a^*, r^*) to changes in the cost of making referrals ρ.

Proposition 3. *Under the optimal monopoly policy (p^*, a^*, r^*) that supports consumer referrals, a higher referral cost results in a higher referral fee, more advertising, and a lower referral reach.*

Since referrals and ads are alternative information channels that the firm can use, it is natural that as referrals become more costly, the firm supports fewer referrals and increases its reliance on advertising. It is also intuitive that the firm offers a higher referral fee to compensate consumers for a higher referral cost. On the other hand, when it is less costly for consumers to refer their contacts, more of them attempt to make referrals, which results in a higher level of referral congestion. The firm responds to this by lowering incentives for referrals: it reduces the referral fee and increases its advertising level, leaving fewer consumers uninformed.

Proposition 4 provides a necessary and sufficient condition for the firm to use consumer referrals.

Proposition 4. *For any given p and $a > 0$, the firm supports consumer referrals if and only if $\rho < \overline{\rho}$, where $\overline{\rho} = (1 - a)\,\pi(p)$.*

Note that the threshold for referral cost $\overline{\rho} = \overline{\rho}\,(p, a)$ is high when the profits from the referral consumer, $(1 - a)\,\pi(p)$, are high. That is, firms are more likely to adopt a referral policy when the profitability of the product $\pi(p)$ is high and advertising a is low.

Corollary 1. *The firm supports consumer referrals if and only if the referral cost is sufficiently small,* $\rho < \rho_0$*, where* $\rho_0 = (1 - a^m)\,\pi(p^m)$ *with* $p^m \equiv \arg\max_p (\pi(p))$ *and* $a^m \equiv \arg\max_a (a\pi(p^m) - C(a))$.

The result is intuitive. A firm with no referral policy can improve its profits by introducing a referral policy with a referral fee $r \in (r_0, p^m - c)$, while keeping its price p^m and advertising a^m at the same level. Such a referral fee exists when ρ is sufficiently low: $\rho < \rho_0$. The firm can then earn additional profits from referred consumers.

5 Welfare Considerations

In this section, we discuss the welfare effects of referrals. We compare the profit-maximizing allocation in a model with consumer referrals to one in which the firm cannot use referrals. We show that for a sufficiently small referral cost, an introduction of consumer referrals results in higher profits and that this cannot make consumers ex ante worse off. We also look at whether the profit-maximizing policy mix is socially optimal.

To study the welfare effects of referrals, we need to define consumer product awareness A as a measure of consumers informed about the product through either advertising or consumer referrals: $A = a + (1 - a)\,R$. In what follows, we assume that the cost of making referrals is sufficiently low so that the profit-maximizing monopoly policy (p^*, a^*, r^*) has a positive referral reach. We then compare the level of product awareness for such a policy to the level of product awareness in the case of no referrals (which is a^m since consumers become informed of the product only through advertisements). A higher product awareness under consumer referrals implies a higher (ex ante) benefit to a consumer willing to buy the product.

To obtain further welfare results, we need to define social welfare. In the referral equilibrium, the expected net benefit to consumers from making referrals is zero. Then, the expected (aggregate) consumer benefits are $B(p, a, r) = A(p, a, r) * CS(p)$, where $CS(p) = \int_p^{\overline{v}} (v - p)\, g(v) dv$ is the consumer surplus from buying the product. We define the social welfare as the sum of monopoly profits and consumer benefits. The social welfare is then $W = \Pi + B$, where monopoly profits are as seen in (5).

Proposition 5. *Allowing a firm to use consumer referrals when the referral cost is sufficiently small results in a higher product awareness and a Pareto-superior allocation. The profit-maximizing policy mix provides a lower referral reach and lower product awareness than is socially optimal, while at the same time supporting the socially optimal level of advertising.*

The Pareto dominance of referrals is intuitive. From Corollary 1, we know that the firm finds it beneficial to support referrals when the referral cost is sufficiently small. Since the price is unchanged (Proposition 2) and more consumers are aware of the product under referrals, we find that regardless of the valuation for the product, a consumer cannot be worse off in the presence of consumer referrals. This means that if the firm supports consumer referrals, it is socially optimal to do so.

The underprovision of product awareness by a monopoly is quite standard and is due to the nonappropriability of consumer surplus. In the proof of Proposition 5, we show that for any level of advertising intensity a and price p, the firm chooses a lower referral reach and awareness than is socially optimal. Intuitively, since the gains to society from an elevated product awareness are higher than the benefit to the firm, the firm underprovides product awareness.

The socially optimal level of advertising is achieved if the firm is free to set a profit-maximizing referral fee. Intuitively, the choice of advertising is motivated by arguments of efficiency. The firm minimizes the cost of making consumers aware of the product using two information channels – advertising and consumer referrals. Therefore, the trade-off that the firm faces is the same as that experienced under social-welfare maximization.

Proposition 5 implies that if we start at a profit-maximizing policy mix (p^*, a^*, r^*) and consider marginal changes in the firm's policy mix, a higher referral fee would be socially beneficial because referrals are underprovided by the firm. At the same time, if the firm continues to choose its profit-maximizing referral fee, the advertising intensity a^* is socially optimal. As usual, the firm's price is higher than the socially optimal price.

6 The Two-Group Model: Targeted Referrals

In this section, we assume that consumers have an informational advantage over the firm: they know to which group other consumers belong, while the firm cannot distinguish between consumer groups.[14]

There are two groups of consumers: H and L with fractions λ^H and λ^L, respectively $(\lambda^H + \lambda^L = 1)$. Group H consumers tend to have a higher willingness-to-pay in the sense of the *hazard-rate dominance* than group L, i.e., for all p, $\frac{g^H(p)}{1-G^H(p)} < \frac{g^L(p)}{1-G^L(p)}$ holds, where $G^H(v)$ and $G^L(v)$ are the cumulative distribution functions of values for groups H and L. The supports of the distribution functions overlap, so that some consumers who belong to group L have a higher willingness-to-pay than some consumers in group H. The general

[14] The model can also be interpreted as a social circles model. Consumers in a group know each other and referrals are made only within that group. However, this assumption is not critical. The same results are obtained under an alternative assumption of a complete network, where anyone can refer anyone else.

distribution $G(v)$ is a weighted average of $G^H(v)$ and $G^L(v)$: $G(v) \equiv \lambda^H G^H(v) + \lambda^L G^L(v)$ for all v. Let $\pi^\tau(p) \equiv (p-c)(1-G^\tau(p))$ for $\tau \in \{H, L\}$ and $\pi(p) \equiv (p-c)(1-G(p))$. We assume the log concavity of $1-G^\tau(\cdot)$ for $\tau \in \{H, L\}$. This assures the uniqueness of profit-maximizing prices: $p^\tau \equiv \arg\max_p \pi^\tau(p)$ and $p^m \equiv \arg\max_p \pi(p)$. The hazard-rate dominance condition implies $p^H > p^m > p^L$.

Consumer i who receives the firm's advertisement can choose q_i^H and q_i^L as referral intensities for two different groups because she can distinguish which of her friends belong to H and L groups. We have the same referral equilibrium as before, but the condition applies for each group $\tau = H, L$. The equilibrium referral reach for group τ consumers $R^\tau = R^\tau(p, a, r)$ is defined implicitly by

$$(1-a)(1-G^\tau(p))rR^\tau = \rho S(R^\tau) \tag{8}$$

for all $r > r_0^\tau \equiv \frac{\rho}{(1-a)(1-G^\tau(p))}$, and the equilibrium referral intensity is higher when referral fee r is higher and price p, advertising intensity a, and referral cost ρ are lower. Note that λ^τ has no effect in determining the referral reach in each group.

The firm's per-consumer profit in this environment is:

$$\Pi(p, a, r) = \sum_{\tau \in \{H, L\}} \lambda^\tau (a + (1-a)R^\tau)\, \pi^\tau(p) - \rho \sum_{\tau \in \{H, L\}} \lambda^\tau S(R^\tau) - C(a), \tag{9}$$

where $R^\tau = R^\tau(p, a, r) > 0$ for $r > r_0^\tau$ and $R^\tau = 0$ otherwise.

Equation (9) is clearly a natural extension of (5), but there is an important difference. The firm can no longer control the referral reach in both groups independently by using a single referral fee r. We cannot use the technique we used in the base model to simplify $\frac{d\Pi}{dp}$ and $\frac{d\Pi}{da}$ because $(1-a)\pi^\tau(p) - \rho S'(R^\tau) = 0$ is not assured for either τ.[15] For this reason,

[15] Of course, if the firm could use differentiated referral fees (r^H and r^L), the optimal referral reach can be

calculating the optimal monopoly price under active referrals for both groups is no longer simple. There is no dichotomy in the firm's decision problem, where price is used to maximize profit per consumer and referral fee is used to control referral reach. However, we can show that the firm chooses to increase its price after the introduction of consumer referrals when only group H receives consumer referrals (i.e., when $r_0^H < r \leq r_0^L$). In this case, referral fee needs to control only the referral reach in group H, and we can apply the same technique as before.

Assuming the following sufficient condition for no referrals to be extended to group-L consumers under targeted referrals: $\pi^L(p^m) \leq \rho$, in Proposition 6 we compare the optimal firm's policy (p^*, r^*, a^*) in the case when consumer referrals are random to the optimal firm's policy (p^{T*}, a^{T*}, r^{T*}) when consumer referrals are targeted (because consumers know to which group other consumers belong). We also compare the corresponding referral reach R^* and $R^{T*} = R^{H*}$ and ratios of referral fee to profit margin, $r^*/(p^m - c)$ and $r^{T*}/\left(p^{T*} - c\right)$.

Proposition 6. *Suppose that $\pi^L(p^m) \leq \rho$ holds. Under targeted referrals, the firm's optimal policy (p^{T*}, a^{T*}, r^{T*}) is such that group-L consumers receive no referrals, and the firm advertises less under targeted referrals than under random referrals, $a^{T*} < a^*$. Moreover,* the optimal price p^{T*} is *higher than the standard monopoly price p^m and $p^H > p^{T*} > p^m > p^L$ holds. The equilibrium referral reach among group-H consumers is higher under targeted referrals than under random referrals $R^{T*} > R^*$, while the ratio of referral fee to profit margin is lower $r^{T*}/\left(p^{T*} - c\right) < r^*/(p^m - c)$.*

Proposition 6 shows that if consumers possess superior information about which con-

set for each group separately. However, it is unreasonable to assume that the firm can set type-dependent referral fees because the whole point of this extension is to examine how the firm may use consumer referrals to utilize superior consumer information.

sumers are likely to purchase the product, then the firm would reduce its reliance on mass advertising and would support a higher referral reach for targeted referrals. Interestingly, consumers can be made better or worse off by the firm's use of referrals when consumers have an information advantage. Under no referrals, every consumer has an equal probability of receiving information about the product. However, with targeted referrals, group-L consumers are less likely to receive the information, although some of them may have high valuations of the product. Thus, the impact of targeted referrals on consumers may depend on consumer type.

In particular, if the valuations of group-H consumers and group-L consumers are sufficiently dissimilar, then only group-H consumers will be targeted with referrals. In this case, all consumers face a higher price and lower advertising intensity, which means that group-L consumers would prefer that the firm not be able to use consumer referrals at all.

7 Discussion and Extensions

7.1 Increasing Marginal Referral Cost

The constant marginal referral cost assumption is important for establishing the monopoly pricing result of Proposition 2.[16] If the marginal referral cost $\rho(k)$ is increasing in the number of referrals k each consumer makes ($\rho'(k) > 0$), the referral equilibrium formula becomes[17]

$$r\,(1-a)\,(1-G\,(p)) = \rho(k^E)\phi(\frac{k^E}{N};n). \tag{10}$$

The proof is in Appendix B. The number of referrals each referring consumer makes k^E is affected by the number of referring consumers n even when N is large. An increase in p and

[16] We owe this insight to Ben Hermalin.

[17] For the analysis of this subsection, we assume that the referrers choose the number of referrals k instead of referral intensity θ, treating k as a real number.

a reduces n, and this results in an increase in the marginal referral cost. Thus, the optimal price is lower in this case. Similarly, the optimal advertisement level under increasing $\rho(k)$ is higher than the one with constant ρ. This is an intuitive result: if $\rho(k)$ is increasing in k, the firm has an incentive to reduce the equilibrium k. Price cuts and higher advertising intensity increase the number of referrers. Therefore, to achieve the same level of referral reach R, fewer referrals per referrer are made and referral costs are lower.

7.2 Private Referral Benefits

We can allow for consumer referrals to be motivated by reasons other than monetary payoffs. For example, suppose that whenever a successful referral is made, the referrer receives not only a referral fee r but also a nonmonetary private benefit $b > 0$. Then, the consumer referral equilibrium is $(r + b)(1 - G(p))(p - c) = \rho\varphi(R^E(p, a, r))$. The private referral benefit b effectively reduces the firm's marginal cost of selling by referrals. Not surprisingly, we find that the firm then supports more referrals and advertises less. Note that if the private benefit b is so large that the firm chooses not to further support referrals with monetary rewards, the firm's price is set above the standard monopoly price.

7.3 Consumers Know Valuations of Others

Suppose consumers know other consumers' valuations of a product (or know that the valuations are high enough for consumers to buy the product). Then, consumers target referrals to individuals whose valuations are sufficiently high. In this case, the firm chooses a higher price than in the base model. A price increase has the additional benefit of more precisely targeted referrals. Although referral messages are not wasted on unlikely prospects, the savings are not fully captured by either consumers or the firm because of a higher congestion level. To reduce congestion, the monopoly sets a lower referral fee, relative to its profit margin, when referrals are targeted than when they are random. Less advertising is sustained in this case than in the base model because referrals are more targeted and are therefore cheaper to use. An additional reason for less advertising is that because of price distortion, the profitability of each sale is lower.

7.4 Cap on the Number of Referrals

We explore the implications of a cap on the number of referrals each referrer can make. Let us assume that consumers' (marginal) referral cost is constant at ρ up to K referrals, but that they cannot make more referrals than K. This modification requires the model to have multiple periods in which consumer referrals are made. For simplicity, we assume that advertising is done only initially (at period 0); in subsequent periods, consumer referrals spread the information about the product. Therefore, from period 1 on, referrals are the only medium used to transmit product information to other consumers. This model connects our model with the models of referral chains in industrial organization and marketing, in which consumers are located on a line and each consumer can make at most one referral without congestion (Jun and Kim, 2008; Byalogorsky et al., 2008; Arbatskaya and Konishi, forthcoming). Notice that with the cap on the number of referrals, the initial probability of a successful referral is high and the net benefit of the referral is positive because many consumers are unaware of the product. As time goes by, more and more consumers become aware of the product, and the net referral benefit goes down. Depending on the size of K, two things can happen. If K is small, then the referral chain may fall short of achieving the referral reach for which the net referral benefit is zero. If K is large enough, then the referral chain terminates in a finite number of periods, thus achieving the level of awareness for which additional referrals are no longer beneficial. In either case, consumer referrals and advertising are no longer substitutes. A larger number of referring consumers speeds up the process of consumer referrals. The firm may have an incentive to increase advertising intensity (and lower its price), especially if the firm is not very patient. This extension seems worthwhile to pursue.

8 Conclusion

Several information channels are available to sellers who market their products to consumers. These include traditional mass advertising on TV and in newspapers and consumer referral policies. We look at a monopoly's optimal advertising, referral, and price policies. We find that the profit-maximizing price is the standard monopoly price, provided that the referral fee is optimally chosen. Intuitively, a monopoly does not use its price to manage consumer referrals, but instead directly uses a referral fee. A consumer referral program improves the firm's profit when referral cost is sufficiently low. The firm relies more heavily on referrals when consumers have superior information about other consumers' preferences and when they derive nonmonetary private benefits from making helpful recommendations.

For any given level of advertising and price, referrals are underprovided because of the nonappropriability of consumer surplus. The welfare effects of referrals tend to be positive because referrals increase consumer awareness about the product. Indeed, in all the cases where referral cost is sufficiently small for the firm to adopt a referral program and where the price does not increase, referrals result in a Pareto improvement. No consumer is worse off and some are better off because they are better informed. It follows that in such cases, if the firm supports consumer referrals, it is socially optimal to do so. This includes cases of increasing marginal referral costs and caps on referral rewards, in which the firm is incentivized to reduce its price in an attempt to stimulate referrals. On the contrary, the ability of consumers to target their referrals to more likely buyers may reduce benefits that accrue to consumers in a low-valuation group because they may be less informed.

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Appendix A: Proofs

Proof of Proposition 1. There are n consumers who have purchased the product and are potential referrers. Focusing on a symmetric equilibrium, suppose that $n-1$ referrers choose the fraction θ of consumers to make referrals to, while the remaining referrer i chooses θ_i. Referral attempts are made randomly. With probability $1-a$, referral attempts reach uninformed consumers. We assume that if a consumer receives $h>0$ referral attempts, then she chooses one with equal probability $1/h$. Then, the probability that a given consumer registers a referral from i is:

$$\psi_i(\theta_i,\theta)=\sum_{h=0}^{n-1}\frac{1}{h+1}\theta_i(1-\theta)^{n-1-h}\theta^h\times C(n-1,h), \tag{11}$$

where $C(n-1,h)=(n-1)!/(n-1-h)!h!$. Note that the term $(1-\theta)^{n-1-h}\theta^h\times C(n-1,h)$ denotes the probability that the given consumer receives h referral attempts from other $n-1$ referrers. By rearranging the formula, we obtain:

$$\begin{aligned}
\psi_i(\theta_i,\theta)&=\sum_{h=1}^{n}\frac{1}{h}\theta_i(1-\theta)^{n-h}\theta^{h-1}\times C(n-1,h-1) \qquad (12)\\
&=\sum_{h=1}^{n}\frac{1}{h}\theta_i(1-\theta)^{n-h}\theta^{h-1}\times\frac{(n-1)!}{(h-1)!(n-h)!}\\
&=\frac{1}{n}\sum_{h=1}^{n}\theta_i(1-\theta)^{n-h}\theta^{h-1}\times\frac{n!}{h!(n-h)!}\\
&=\frac{1}{n}\times\frac{\theta_i}{\theta}\sum_{h=1}^{n}(1-\theta)^{n-h}\theta^h\times C(n,h)\\
&=\frac{\theta_i}{n\theta}\left[1-(1-\theta)^n\right]=\theta_i\frac{1}{\phi(\theta;n)},
\end{aligned}$$

for $\theta>0$, where $\phi(\theta;n)=\frac{n\theta}{1-(1-\theta)^n}$; $\psi_i(\theta_i,0)=\theta_i$ for $\theta=0$. Assuming $\theta>0$, the expected referral reward to referrer i from each referral she makes is $\frac{(1-a)(1-G(p))r}{\phi(\theta;n)}$. The cost of making a referral is ρ. Therefore, referrer i's objective function is linear in θ_i. Note that $\phi(\theta;n) > 1$ for $\theta \in (0,1]$ and $n \geq 2$.[18]

In a symmetric interior equilibrium, consumers are indifferent between which θ_i to choose. The symmetric equilibrium θ^E is implicitly calculated as a solution to

$$(1-a)\,(1-G\,(p))\,r = \rho\phi(\theta^E;n). \tag{13}$$

Since $\phi(\theta;n) > 1$ for $\theta > 0$, this equation has an interior solution only if $r > r_0 \equiv \frac{\rho}{(1-a)(1-G(p))}$.[19] Then, given that others are choosing θ^E, consumer i obtains a zero payoff for any strategy, and she might as well choose θ^E. Thus, θ^E is the symmetric referral equilibrium when $r > r_0$. The symmetric equilibrium referral strategy θ^E is unique when it exists because $\phi(\theta;n)$ is strictly increasing in θ. To see this, we use $(1+\theta)^n > 1+n\theta$ in the following:

$$\begin{aligned}\frac{\partial\phi(\theta;n)}{\partial\theta} &= n\frac{(1-(1-\theta)^n)-nq(1-\theta)^{n-1}}{(1-(1-\theta)^n)^2} \\ &= n\frac{1-(1-\theta)^{n-1}(1+(n-1)\theta)}{(1-(1-\theta)^n)^2} \\ &> n\frac{1-(1-\theta)^{n-1}(1+\theta)^{n-1}}{(1-(1-\theta)^n)^2} = n\frac{1-(1-\theta^2)^{n-1}}{(1-(1-\theta)^n)^2} > 0.\end{aligned} \tag{14}$$

The equilibrium referral strategy θ^E exists because $\phi(\theta;n)$ is strictly increasing in θ for $\theta \in (0,1)$, $\lim_{\theta\to 0}\phi(\theta;n) = 1$, and $\lim_{\theta\to 1}\phi(\theta;n) = n$. If $r \leq r_0$, then $\theta^E = 0$; if $r \in (r_0, nr_0)$, then $\theta^E \in (0,1)$; and if $r \geq nr_0$, then $\theta^E = 1$.

[18] We can show by induction that $(1-\theta)^n > 1-n\theta$ for $\theta \in (0,1]$ and $n \geq 2$. When $n=2$, $(1-\theta)^2 = 1-2\theta+\theta^2 > 1-2\theta$, and the result is true for $n=2$. Suppose it is true for some n: $(1-\theta)^n > 1-n\theta$. We need to show that it is then true for $n+1$: $(1-\theta)^{n+1} > 1-(n+1)\,\theta$. Note that $(1-\theta)^{n+1} = (1-\theta)(1-\theta)^n > (1-\theta)(1-n\theta)$ by the inductive hypothesis, and therefore $(1-\theta)^{n+1} > 1-(n+1)\,\theta+n\theta^2 > 1-(n+1)\,\theta$.

[19] When $\theta = 0$, the net benefit of making a referral is $(1-a)\,(1-G\,(p))\,r-\rho$. Hence, for $r \leq r_0$, the only symmetric equilibrium is $\theta^E = 0$.

For the interior solution $\theta^E \in (0,1)$, the equilibrium referral congestion $\phi^E = \phi(\theta^E; n)$ equals $\frac{r(1-a)(1-G(p))}{\rho}$. It is increasing in r and decreasing in ρ, p, and a. From $\frac{\partial \phi(\theta;n)}{\partial n} = \frac{\theta(1-\theta)^n}{((1-\theta)^n-1)^2} \left[\ln(1-\theta)^n - 1 + (1-\theta)^{-n}\right]$ and $\ln x > 1 - \frac{1}{x}$ for $x \neq 1$, it follows that $\phi(\theta; n)$ is strictly increasing in n for $\theta > 0$. Hence, the equilibrium referral strategy $\theta^E = \theta^E(p, a, r; \rho, n)$ increases in r and decreases in p, a, ρ, and n for $r > r_0 \equiv \frac{\rho}{(1-a)(1-G(p))}$.■

Proof of Proposition 2. First, we will show that there exists a profit-maximizing policy mix (p^*, a^*, r^*). Notice that profit-maximizing p, a, and r must belong to intervals $[c, \bar{v}]$, $[0, \bar{a}]$, and $[0, \bar{v} - c]$, respectively, and that for any finite r, $R^E < 1$ since it follows from equation (3) that $\varphi(R) = r/r_0$ and $\lim_{R \to 1} \varphi(R) = \infty$. Then, for any $(p, a, r) \in [c, \bar{v}] \times [0, \bar{a}] \times [0, \bar{v} - c]$, the firm's profit function is continuous in p, a, and R^E, and $R^E = R^E(p, a, r)$ is continuous in (p, a, r). To see that profits are continuous at $R^E = 0$, consider any p and a. From Proposition 1, if $r > r_0$, then there exists a unique $R^E > 0$ and it is continuous in (p, a, r) because of the monotonicity of $\varphi(R)$. If $r \leq r_0$, then $R^E = 0$. Finally, R^E goes to 0 as r approaches r_0 from above. Hence, R^E is continuous in (p, a, r) and we conclude that the profit function $\Pi(p, a, r)$ is continuous in (p, a, r).

Thus, by the Weierstrass extreme value theorem, there is a profit-maximizing policy mix $(p^*, a^*, r^*) \in [c, \bar{v}] \times [0, \bar{a}] \times [0, \bar{v} - c]$. Moreover, $p^* \in (c, \bar{v})$ and $a^* \in (0, \bar{a})$ must hold because $C'(0) = 0$ and $C(\bar{a}) > \bar{v} - c > \pi(p)$ holds for any p. So, p^* and a^* are interior solutions that must be characterized by the first-order conditions. Regarding r^*, there are two possibilities: in Case I, (p^*, a^*, r^*) is such that $r^* \leq r_0(p^*, a^*)$ and there are no referrals $R^E(p^*, a^*, r^*) = 0$; and in Case II, (p^*, a^*, r^*) is such that $r^* > r_0(p^*, a^*)$ and there are referrals $R^E(p^*, a^*, r^*) > 0$.

In Case I, $R^E = 0$. Since $p = p^*$ and $a = a^*$ are interior solutions, they have to satisfy the first-order conditions:

$$\frac{d\Pi}{dp} = a\pi'(p) = 0,$$
$$\frac{d\Pi}{da} = \pi(p) - C'(a) = 0.$$

Thus, $p^* = p^m \equiv \arg\max_p (\pi(p))$ holds trivially in this case and $a^* = a^m$ is the unique solution to the first-order condition $\pi(p^m) - C'(a) = 0$.

In Case II, since $R^E > 0$, the profit-maximizing monopoly policy mix (p^*, a^*, r^*) is such that $r^* > r_0(p^*, a^*)$ and the interior solution has to satisfy the following three first-order conditions:

$$\frac{d\Pi}{dp} = \left(a^* + (1-a^*)R^E\right)\pi'(p^*) + \left[(1-a^*)\pi(p^*) - \rho S'(R^E)\right]\frac{\partial R^E}{\partial p} = 0, \tag{15}$$
$$\frac{d\Pi}{da} = \left(1 - R^E\right)\pi(p^*) + \left[(1-a^*)\pi(p^*) - \rho S'(R^E)\right]\frac{\partial R^E}{\partial a} - C'(a^*) = 0, \tag{16}$$
$$\frac{d\Pi}{dr} = \left[(1-a^*)\pi(p^*) - \rho S'(R^E)\right]\frac{\partial R^E}{\partial r} = 0. \tag{17}$$

From the last condition, we find that as long as the referral reach is responsive to the referral fee (i.e., $\frac{\partial R^E}{\partial r} > 0$, which occurs when $r > r_0(p^*, a^*)$), the referral fee must be set by the firm to equalize the marginal benefit of expanding the referral reach and the marginal cost of such an expansion:

$$(1-a^*)\pi(p^*) = \rho S'(R^E). \tag{18}$$

This means that for an optimal referral fee, the marginal net benefit of expanding the referral reach is zero.

We can use the observation that referral expansion has zero first-order effects on profits to further characterize the optimal choice of price and advertising. Using equation (18), the first-order conditions for p and a can be written as:

$$\frac{d\Pi}{dp}|_{r^*} = \left(a^* + (1-a^*)\,R^E\right)\pi'\left(p^*\right) = 0, \tag{19}$$

$$\frac{d\Pi}{da}|_{r^*} = \left(1 - R^E\right)\pi\left(p^*\right) - C'(a^*) = 0, \tag{20}$$

where $R^E = R^E\left(p^*, a^*, r^*\right)$. Since $a^* \in (0, \bar{a})$ and $R^E > 0$, the profit-maximizing price satisfies $\pi'\left(p^*\right) = 0$, and the firm's pricing policy under active consumer referrals remains unchanged from the standard monopoly pricing, $p^* = p^m$.

Next, we turn to the firm's advertising strategy. It is easy to see that when $R^E > 0$, $a^* < a^m$. The reason is that the marginal benefit of advertising in Case II, $\left(1 - R^E\right)\pi\left(p^m\right)$, is lower when the firm supports active consumer referrals. Hence, the firm advertises less when it supports referrals with an optimally-set referral fee than when it does not support referrals. Note that this result would still hold if the price were set at a suboptimal level.■

Proof of Proposition 3. Suppose the firm supports consumer referrals and its optimal policy as a function of ρ is $(p,\, a,\, r) = (p^*, a^*, r^*) = (p^*(\rho), a^*(\rho), r^*(\rho))$. Using shorthand notations, $\pi = \pi(p)$, $C = C(a)$, $R = R^E(p,\, a,\, r)$, and $S = S(R)$, we totally differentiate the system of first-order conditions for r, p, and a :

$$\left\{\begin{array}{c} (1-a)\pi = \rho S' \\ \frac{d\Pi}{dp} = \frac{\partial \Pi}{\partial p} = \left[a + (1-a)\,R\right]\pi' = 0 \\ \frac{d\Pi}{da} = \frac{\partial \Pi}{\partial a} = (1-R)\pi - C' = 0 \end{array}\right. , \tag{21}$$

to find that

$$\begin{pmatrix} -\rho S'' & (1-a)\pi' & -\pi \\ (1-a)\pi' & \left[a + (1-a)\,R\right]\pi'' & (1-R)\pi' \\ -\pi & (1-R)\pi' & -C'' \end{pmatrix} \begin{pmatrix} dR \\ dp \\ da \end{pmatrix} = \begin{pmatrix} S' \\ 0 \\ 0 \end{pmatrix} d\rho \tag{22}$$

at $(p,\, a,\, r) = (p^*,\, a^*,\, r^*)$. Since the firm maximizes its profit, the matrix in the LHS is negative semidefinite. Note that $\pi'(p) = 0$ at the optimum solution. Thus, we have the determinant D and the principal minors of the matrix in the LHS satisfy:

$$\begin{aligned} D &= \begin{vmatrix} -\rho S'' & 0 & -\pi \\ 0 & [a + (1-a)\, R]\, \pi'' & 0 \\ -\pi & 0 & -C'' \end{vmatrix} \\ &= \left(\rho S'' C'' - \pi^2\right) [a + (1-a)\, R]\, \pi'' < 0, \end{aligned} \tag{23}$$

$\rho S'' C'' - \pi^2 > 0$, $-C'' [a + (1-a)\, R]\, \pi'' > 0$, $-\rho S'' < 0$, $[a + (1-a)\, R]\, \pi'' < 0$, and $-C'' < 0$ at $(p,\, a,\, r) = (p^*,\, a^*,\, r^*)$.

The impacts of an increase in ρ on the optimal policies are:

$$\begin{aligned} \frac{dR}{d\rho} &= \frac{S' [a + (1-a)\, R]\, \pi''}{D} [-C''], \\ \frac{dp}{d\rho} &= 0, \\ \frac{da}{d\rho} &= \frac{S' [a + (1-a)\, R]\, \pi''}{D} \pi. \end{aligned} \tag{24}$$

Thus, we conclude that $\frac{da^*}{d\rho} > 0$, $\frac{dp^*}{d\rho} = 0$, and $\frac{dR^*}{d\rho} < 0$.

It is left to show that $\frac{dr^*}{d\rho} > 0$. From the first-order condition for referral fee r^* in equation (21), the optimal referral reach $R^* = R^E(p^*, a^*, r^*)$ is such that

$$(1 - a^*)\, (1 - R^*)\, \pi(p^*) = \rho. \tag{25}$$

Since $S = -\ln(1 - R)$, it follows that $S\,(R^*) = \ln\left(\frac{(1-a^*)}{\rho} \pi(p^*)\right)$. From Proposition 1, in the referral equilibrium,

$$r = \frac{S\left(R^E\right)}{R^E} \frac{\rho}{(1-a)(1 - G\,(p))}. \tag{26}$$

Rewriting this expression using R^* and $S\,(R^*)$, we obtain

$$r^* = p^* \frac{\ln\left(\frac{(1-a^*)}{\rho} \pi(p^*)\right)}{\frac{(1-a^*)}{\rho} \pi(p^*) - 1}. \tag{27}$$

Since $p^* \frac{\ln x}{x-1}$ is decreasing in $x = \frac{(1-a^*)\pi(p^*)}{\rho}$, p^* is independent of ρ, and $\frac{da^*}{d\rho} > 0$, we find that the optimal referral fee increases in the referral cost, $\frac{dr^*}{d\rho} > 0$.■

Proof of Proposition 4. For any fixed p and $a > 0$, a referral policy adoption is profitable if and only if the firm can introduce a referral policy with a referral fee $r \in (r_0, p - c)$, where $r_0 \equiv \frac{\rho}{(1-a)(1-G(p))}$. Such r exists if and only if $\rho < \overline{\rho}$ because $r_0 < p - c$ is equivalent to $\rho < (1 - a)\,\pi(p)$. From Proposition 1, the firm that chooses $r > r_0$ supports referrals. Since $r < p - c$, the firm would receive positive additional profits from consumers buying via referrals without altering its profits from the informed consumers. Hence, for any fixed a and p, the firm can increase its profits by introducing a referral program if and only if $\rho < \overline{\rho}$.■

Proof of Corollary 1. When the firm does not use referrals, it chooses price p^m and advertising level a^m such that $\pi(p^m) - C'(a^m) = 0$. By Proposition 4, the firm benefits from introducing referrals while keeping p^m and a^m if $\rho < \rho_0 \equiv \overline{\rho}(p^m, a^m) = (1 - a^m)\,\pi(p^m)$. At the optimal policy mix (p^*, a^*, r^*), the profits can only be higher. Thus, the firm supports referrals if $\rho < \rho_0$.

Consider the optimal monopoly policy as a function of ρ: $(p^*(\rho), a^*(\rho), r^*(\rho))$, with the associated equilibrium referral reach $R^* = R^*(\rho)$. We next show that there exists a threshold for referral cost $\widehat{\rho}$ such that the firm supports consumer referrals if and only if $\rho < \widehat{\rho}$. According to Proposition 2, $p^*(\rho) = p^* = p^m$, and from Proposition 3, $\frac{dR^*}{d\rho} < 0$ and $\frac{da^*}{d\rho} > 0$ under active referrals. From Proposition 4, it follows that the firm supports consumer referrals if and only if $\rho - (1 - a^*(\rho))\,\pi(p^m) < 0$, where the expression on the LHS is strictly increasing in ρ. Thus, there is a unique $\widehat{\rho}$, such that $R^*(\rho) = 0$ for all $\rho \geq \widehat{\rho}$ and $R^*(\rho) > 0$

for all $\rho < \widehat{\rho}$.

Finally, we show that $\widehat{\rho} = \rho_0$. There are no referrals if $\rho > \pi(p^m)$, but we have proved that referrals exist for $\rho < \rho_0$. In the limit, as ρ approaches ρ_0 from below, $\lim\limits_{\rho \to \rho_0 -} R^E(p^*, a^*, r^*) = 0$; $\lim\limits_{\rho \to \rho_0 -} a^* = a^m$; and $\lim\limits_{\rho \to \rho_0 -} r^* = r_0$. Since the referral reach cannot be negative and advertising can never rise above a^m, the firm must offer no referrals when $\rho \geq \rho_0$, and therefore $R^E = 0$, $a^* = a^m$, and $r^* = 0$ in this case since we assume that referral fee is zero when the firm does not support referrals. The optimal $R^*(\rho)$ and $a^*(\rho)$ are continuous for all $\rho > 0$.$\blacksquare$

Proof of Proposition 5. Consider the optimal monopoly policy as a function of ρ, $(p^*(\rho), a^*(\rho), r^*(\rho))$, with the associated equilibrium referral reach $R^* = R^*(\rho)$ and product awareness $A^* = A^*(\rho)$.

From Corollary 1, it follows that the firm supports consumer referrals if and only if $\rho < \rho_0$. We next show that $\frac{dA^*}{d\rho} < 0$ for all $\rho < \rho_0$. Assume that $\rho < \rho_0$, so that the firm chooses to support referrals. We can show that product awareness A^* decreases in the cost of making referrals ρ. Totally differentiating $A^* = a^* + (1 - a^*) R^*$, we find that

$$\frac{dA^*}{d\rho} = (1 - a^*) \frac{dR^*}{d\rho} + (1 - R^*) \frac{da^*}{d\rho}. \tag{28}$$

From equation (24) in Proposition 3's proof, we find that

$$sign\left(\frac{dA^*}{d\rho}\right) = sign\left[(1 - a^*)(-C'') + (1 - R^*)\pi\right], \tag{29}$$

because $\left[\frac{S'}{D} A^* \pi''\right] > 0$. By assumption, $C''(a) > \rho / (1 - a)^2$. From equation (7), the optimal R^* satisfies

$$(1 - a^*)(1 - R^*)\pi = \rho. \tag{30}$$

Hence, $\frac{dA^*}{d\rho} < 0$.

Finally, suppose $\rho = \rho_0$, with $R^*(\rho_0) = 0$. The optimal advertising policy is $a^*(\rho_0) = a^m$, the level of product awareness is $A^*(\rho_0) = a^m$, and $A^*(\rho)$ is continuous at ρ_0. We know that consumer awareness is decreasing in ρ, $\frac{dA^*}{d\rho} < 0$. Then, for any $\rho < \rho_0$, consumer awareness is higher under the optimal monopoly policy with active referrals than when a firm cannot use referrals.

We turn to the proof of the second statement in Proposition 5. From (7), the firm's choice of referral reach equalizes the marginal benefit of extending the referral reach and its marginal cost, $MB_R = (1-a)\pi(p) = MC_R = \rho/(1-R)$. Although the firm cannot directly control referral reach, it can set referral fee r in such a way as to achieve a referral reach through endogenous consumer referral decisions, which are based on the cost and benefit to a consumer of making a referral in the referral equilibrium (3). In contrast, the socially optimal referral reach equalizes the marginal social benefit of extending referral reach and its marginal cost, $MSB_R = (1-a)(\pi(p) + CS(p)) = MC_R = \rho/(1-R)$. We conclude that the firm underprovides referrals. Similarly, the firm underprovides product awareness by setting the marginal benefit of raising product awareness equal to the marginal cost of reaching a consumer who is unaware of the product through a referral, $\pi(p) = \frac{\rho}{1-A}$; whereas the socially optimal level of awareness is guided by the marginal social cost, $(\pi(p) + CS(p)) = \frac{\rho}{1-A}$.

We totally differentiate social welfare, evaluate it at $a = a^*$ and $r = r^*$, and use equation (30) and the first-order condition for a in equation (21) to find that $\frac{dW}{da}|_{\substack{a=a^* \\ r=r^*}} = 0$. Hence, at the optimal monopoly policy (p^*, a^*, r^*), marginal changes in advertising intensity are not welfare-improving.■

Proof of Proposition 6. By the assumption $\pi^L(p^m) \leq \rho$, the marginal cost of extending referrals to low-type consumers exceeds the marginal benefit if $p \geq p^m$.

Consider $p \geq p^m$. In the referral equilibrium under targeted consumer referrals,

$$(1-a)(1-G^H(p))rR^{T*} = \rho S^*(R^{T*}), \tag{31}$$

if $r_0^H < r \leq r_0^L$. The firm's profit function is:

$$\Pi(p,a,r) = a\pi(p) + \lambda^H(1-a)\pi^H(p)R^{T*} - \lambda^H \rho S^*(R^{T*}) - C(a). \tag{32}$$

The first-order condition for the referral fee r is

$$\frac{d\Pi}{dr} = \lambda^H \left[(1-a)\pi^H(p) - \frac{\rho}{1-R^{T*}}\right] \frac{\partial R^{T*}}{\partial r} = 0, \tag{33}$$

and, therefore, the expression in the square brackets is zero at the optimal r^{T*} (that is, the marginal net benefit of extending referral reach among group-H consumers is zero). It follows that

$$(1-a)\,\pi^H(p)(1-R^{T*}) = \rho. \tag{34}$$

First, consider pricing. Conditional on the optimal choice of r^{T*}, the first-order condition for price p is

$$\left.\frac{d\Pi}{dp}\right|_{r=r^{T*}} = \frac{\partial \pi(p)}{\partial p}a + \lambda^H\,(1-a)\,\frac{\partial \pi^H(p)}{\partial p}R^{T*} = 0. \tag{35}$$

The standard monopoly price p^m satisfies $\frac{\partial \pi(p^m)}{\partial p} = 1 - G(p^m) - (p^m - c)\,g(p^m) = 0$. Since $\frac{g^H(p)}{1-G^H(p)} < \frac{g(p)}{1-G(p)}$ holds for all p, we have $\frac{p^m g^H(p^m)}{1-G^H(p^m)} < \frac{p^m g(p^m)}{1-G(p^m)} = 1$. Thus, $\frac{\partial \pi^H(p^m)}{\partial p} = 1 - G^H(p^m) - p^m g^H(p^m) > 0$ holds. By log concavity of $1-G$ and $1-G^H$, $\frac{\partial \pi^H(p)}{\partial p} > 0$ for all $p < p^H$ and $\frac{\partial \pi(p)}{\partial p} < 0$ for all $p > p^m$. This argument proves that $p^{T*} > p^m$. Similarly, we can show that $p^{T*} < p^H$.

Second, we consider advertising. Under the optimally chosen r^{T*}, the derivative of the profit with respect to a can be written as:

$$\begin{aligned}
\left.\frac{d\Pi}{da}\right|_{r=r^{T*}} &= \pi(p^{T*}) - \lambda^H \pi^H(p^{T*}) R^{T*} - C'(a^{T*}) \\
&= \lambda^L \pi^L(p^{T*}) + \lambda^H \pi^H(p^{T*})(1 - R^{T*}) - C'(a^{T*}) \\
&= \lambda^L \pi^L(p^{T*}) + \lambda^H \frac{\rho}{1 - a^{T*}} - C'(a^{T*}) \\
&< \lambda^L \pi(p^m) + \lambda^H \frac{\rho}{1 - a^{T*}} - C'(a^{T*}) \\
&\leq \lambda^L \rho + \lambda^H \frac{\rho}{1 - a^{T*}} - C'(a^{T*}) \\
&< \frac{\rho}{1 - a^{T*}} - C'(a^{T*}).
\end{aligned} \tag{36}$$

In the benchmark model of random referrals, the profit-maximizing level of advertising a^* (conditional on the optimal choice of r^*) is described by

$$\pi(p^m)\,(1 - R^*) - C'(a^*) = \frac{\rho}{1 - a^*} - C'(a^*) = 0. \tag{37}$$

Thus, we have

$$\frac{\rho}{1 - a^{T*}} - C'(a^{T*}) > \frac{\rho}{1 - a^*} - C'(a^*) = 0.$$

By the second-order condition, $\frac{d}{da}\left(\frac{\rho}{1-a} - C'(a)\right) < 0$ must hold. This implies $a^* > a^{T*}$.

Third, we compare the optimal levels of referral reach under targeted and random consumer referrals (R^{T*} and R^*, respectively):

$$(1 - a^*)\,(p^m - c)\,(1 - G(p^m))(1 - R^*) = \rho \tag{38}$$

and

$$(1 - a^{T*})\left(p^{T*} - c\right)(1 - G^H(p^{T*}))(1 - R^{T*}) = \rho. \tag{39}$$

These equations imply $R^{T*} > R^*$ because we know that $p^H > p^{T*} > p^m$, $a^{T*} < a^*$, and $\left(p^{T*} - c\right)(1 - G^H(p^{T*})) > (p^m - c)\,(1 - G^H(p^m)) > (p^m - c)\,(1 - G(p^m))$. To see why, note that $\frac{\partial\pi^H}{\partial p} > 0$ holds for $p \in (p^m, p^H)$ by the log concavity of $1 - G^H(p)$ and $1 - G^H(p) > 1 - G(p)$.

Finally, we compare the ratios of referral fees to profit margins in the case of random and targeted referrals:

$$(1 - a^*)(1 - G(p^m))r^* = \rho\varphi\left(R^*\right) \tag{40}$$

and

$$(1 - a^{T*})(1 - G^H(p^{T*}))r^{T*} = \rho\varphi\left(R^{T*}\right). \tag{41}$$

Using equations (38), (39), (40), and (41), we obtain:

$$\frac{r^*}{p^m - c} = \varphi\left(R^*\right)\left(1 - R^*\right) \tag{42}$$

$$\frac{r^{T*}}{p^{T*} - c} = \varphi\left(R^{T*}\right)\left(1 - R^{T*}\right). \tag{43}$$

Since $R^{T*} > R^*$, to show that $\frac{r^*}{p^m - c} > \frac{r^{T*}}{p^{T*} - c}$, we need only prove that $\zeta(R) \equiv \varphi\left(R\right)\left(1 - R\right)$ is a decreasing function, where $\varphi\left(R\right) = -\left(\ln(1 - R)\right)/R$. To see this, differentiate $\zeta\left(R\right)$ to obtain $\zeta'\left(R\right) = \frac{1}{R^2}\ln(1 - R) + \frac{1}{R}$. Note that $\ln x$ is a strictly concave function with $\ln(1) = 0$ and $(\ln x)' = 1$ at $x = 1$. Thus, $\ln(x) < x - 1$ for all $x \neq 1$. This implies $\ln(1 - R) < -R$. Therefore, for all $R \in (0, 1)$ we have $\zeta'\left(R\right) < -\frac{1}{R} + \frac{1}{R} = 0.$■

Appendix B: Convex Cost of Referral

Suppose that the cost of making referrals is not linear. If that is the case, the net benefit of making referrals is positive, but consumers do not anticipate these net benefits when deciding whether to buy the product upon receiving an advertisement. Let $\rho(k)$ be the marginal cost of making the kth referral. We treat k as a continuous variable.

Proposition 1'. *The equilibrium number of referrals k^E made by each of n referrers is defined implicitly by:*

$$r\left(1-a\right)\left(1-G\left(p\right)\right)=\rho\left(k^E\right)\phi(\frac{k^E}{N};n) \tag{44}$$

for $r>r_0\equiv\frac{\rho(0)}{(1-a)(1-G(p))}$, and no referrals are sustained for lower levels of the referral fee.

Proof of Proposition 1'. Suppose each of n referrers, except for referrer i, makes k referrals, while referrer i makes k_i referrals. Referrers send referrals independently and at random to N people without contacting the same person more than once. As in the proof of Proposition 1, a proportion of consumers who register a referral from i is $1/\phi\left(\frac{k}{N};n\right)$ for $k>0$. Referrer i's optimal choice of k_i is obtained by equalizing the marginal benefit $\left(1-a\right)\left(1-G\left(p\right)\right)r/\phi\left(\frac{k}{N};n\right)$ and the marginal cost $\rho\left(k_i\right)$ of the k_ith referral. For any $r>r_0\equiv\frac{\rho(0)}{(1-a)(1-G(p))}$, the unique equilibrium k^E satisfies $r\left(1-a\right)\left(1-G\left(p\right)\right)=\rho\left(k^E\right)\phi\left(\frac{k^E}{N};n\right)$. Since $\rho\left(k^E\right)\phi\left(\frac{k^E}{N};n\right)$ is increasing in k^E and $\left(1-a\right)\left(1-G\left(p\right)\right)r>\rho\left(0\right)\lim_{\frac{k}{N-1}\rightarrow0}\phi\left(\frac{k}{N};n\right)=\rho\left(0\right)$ for $r>r_0$, there exists a unique $k^E>0$.■