Endogenous Party Structure *

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Abstract

This paper proposes a model of two-party representative democracy on a single-dimensional political space, in which voters choose their parties in order to influence the parties' choices of representative. After two candidates are selected as the median of each party's support group, Nature determines the candidates' relative likability (valence). Based on the candidates' political positions and relative likability, voters vote for the preferable candidate without being tied to their party's choice. We show that (1) there exists a nontrivial equilibrium under natural conditions,

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and (2) the equilibrium party border and the ex ante probabilities of the two-party candidates winning are sensitive to the distribution of voters. In particular, we show that if a party has a more concentrated subgroup, then the party tends to alienate its centrally located voters, and the party's probability of winning the final election is reduced. Even if voter distribution is symmetric, an extremist party (from either side) can emerge as voters become more politically divided.

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1 Introduction

In a two-party electoral system, office-motivated parties set their policy platforms to attract the majority of voters in order to get elected. Downs (1957) and Black (1958) have shown that if the policy space is one-dimensional then both parties choose the median voter's "bliss point" as their party platform. Although this theoretical result is a nice justification for a two-party system, we do not observe this outcome in US politics. In the real world, we observe that candidates whose positions are quite far from the median voter's are quite often elected in party primaries, especially in parties with a strong subgroup with extreme positions. In 2004, moderate Republican senator Arlen Specter faced a tough challenge from the Right in the Republican primary election; but once Specter defeated the challenge by a narrow margin, he was comfortably reelected in the general election with great support from moderate central voters. During his reelection bid in 2006, moderate Democratic senator Joe Lieberman lost the Democratic Party primary election but won the general election as a third-party candidate.

These examples show that party primary elections by party members play an important role in determining party candidates. Wittman (1983), Calvert (1985), and Roemer (2001) show that policy divergence can occur if party members are policy motivated and voting outcomes are uncertain (valence models). Although the result that parties' levels of policy orientation explain the level of equilibrium policy divergence is quite reasonable, one problem still remains. How did each party's policy orientation evolve? A party's policy orientation should be determined by its constituents, but who the party's constituents are is also affected by the two parties' policy orientation.[1]

[1]A recent experience in California describes the importance of a party's policy orientation in deter-

In this paper, we formulate a two-party model in which the party-separating threshold political positions and parties' policy orientation (party candidates or policies) are endogenously determined. We assume that voters are strategic in choosing their parties, foreseeing their influence on the choice of candidates (party constituency affects the party's policy orientation). Then, we introduce uncertainty in voting outcomes following Wittman (1983). Specifically, we assume that each candidate has some chance of winning, owing to the uncertainty of the election. As is often seen in the real world, the candidates' campaign and debate performances can change the voting outcome.[2] Some voters may prefer the candidate from the opposite party even if their political position is very far from the candidate's position.[3] With such uncertainty in the voting outcome, even if an extreme candidate runs as the party candidate, she may win the final election if she

mining the party threshold (Fiorina, Abraham, and Pope 2011, pp. 210-211). In the 1994 election, the California Republican Party won its governorship in a landslide and won four of the six other statewide races for state office, and Republicans defeated four Democratic House incumbents. However, thereafter, the California Republican Party was taken over by its extreme social conservative elements, nominating hard-core conservatives with limited appeal to the moderates in primary elections; and in 2002 Democrats won all the statewide races for the first time in California history. In less than a decade, California changed its hue from dark red to dark blue.

[2]For example, recall the loss of incumbent George Allen, a Republican, in the 2006 Virginia Senate race, and the victory by Scott Brown, a Republican, in the 2010 Massachusetts Senate race to replace late Edward M. "Ted" Kennedy, a Democrat who had been senator for more than forty years. These shocks were clearly not idiosyncratic: the shocks can be quite dramatic and devastating.

[3]Such uncertainty in voting outcomes can be generated by having common shocks to voters' utilities (nonidiosyncratic shocks). This line of modeling is called the "valence" model of politics (see Schofield 2004, Bernhardt, Krasa, and Polborn 2008). Persson and Tabelini (2000) and Roemer (2001) also discuss voting models with common shocks on voters' utilities.

happens to be judged much more likable than the moderate candidate, although such an event would occur only with very low probability. Suppose that an extreme candidate is chosen by a party through the influence of a strong, extreme subgroup in that party. Then, the moderate potential supporters of the party are alienated, as the party does not reflect their voice in choosing the party candidate. If they participate in the other party, which has more diverse support groups, they may be able to play a more significant role in choosing that party's candidate. As a result, the party threshold shifts accordingly, and the more diverse party selects a more moderate candidate, while the party supported by an extreme group selects a more extreme candidate. This is a self-sustaining outcome—an equilibrium. Obviously, the diverse party's candidate's political position is closer to the median voter's position, and she has a higher probability of getting elected.

In the above story, one important missing element is how a candidate is elected in a party primary. The mechanism of choosing a primary winner is very important to a complete description of a two-party system with primary elections. We will adopt a bold simplification of how party primaries work in order to concentrate on the voters' party choice problem: we assume that each party chooses a median-positioned candidate of the party's constituents. This simplifying assumption, used by Besley and Coate (2003), is useful in illustrating how party orientation (here policy) and party constituents interact.[4] We also discuss the relaxation of this assumption in the conclusion.

Our game goes as follows. In stage 1, voters choose their parties by calculating their

[4]In their recent paper, Krasa and Polborn (2015) present an interesting multidistrict model with a single-dimensional policy space, in which the candidate's position is determined in a party primary in each district. In a special case, their model can justify our party-median candidate assumption.

expected utility from the final election from joining each party (with small groups of other voters). By the voters' party choice, the two party candidates are assumed to be selected as the median voter of each party. In stage 2, Nature plays, and the two candidates' relative likability (valence) is determined randomly. In stage 3, voters cast their ballots for the preferable candidate given the two candidates' positions and likability. A voter's party affiliation does not bind her voting behavior, and she votes sincerely. The final voting outcome is the equilibrium outcome of this voting game. Our solution concept, political equilibrium, is a subgame perfect equilibrium, except that we allow for small coalitional deviations instead of each voter's unilateral deviation in the party-choice stage (stage 1). We will not simply adopt Nash behavior, since we assume that each voter is negligible. In this framework, unilateral deviations cannot affect the parties' candidate-selection processes, so any partition of voters can be a Nash equilibrium. To avoid this difficulty, we consider small coalitional deviations and define a "political equilibrium" as a partition of voters from which any arbitrarily small coalitional deviations are unprofitable.

We will first characterize our political equilibrium, and find that our equilibrium is consistent with voters' party sorting. Using this property, we provide sufficient conditions for the existence of a political equilibrium (Theorem 1). Then, we move on to investigate how the party threshold is affected by the distribution of voters over policy space. If a party's support group is concentrated on the extreme side while the other party's support group is more spread out, then the party tends to lose the moderates' support, since moderates tend to choose the more diverse party that makes it easier for voters to have a voice. This effect is illustrated in Example 1 with an asymmetric voter distribution. In an example with a tri-peaked symmetric voter distribution (Example 2: a step function with peaks at the left extreme, the center, and the right extreme), we conduct a comparative static exercise to analyze what happens when voters are more politically divided. When the voter distribution is uniform, there is a unique symmetric equilibrium. However, as the population of the moderate left and right decreases gradually, two other asymmetric equilibria suddenly appear. Such asymmetric equilibria have the feature of having one party composed mostly of extreme voters and the other party composed of the rest of the voters, including the centrist group: the former party chooses an extreme candidate with a low probability of winning (but still a chance of winning if common shock is strongly in his favor), while the latter chooses a moderately oppositely biased candidate with a high chance of winning. Voters who are happy with the extreme candidate despite her low chance of winning continue to support the extremist party. However, another oppositely biased equilibrium is also self-sustainable. Thus, if voters are deeply divided politically, there will be multiple quite asymmetric equilibria.[5]

Three articles are most closely related to our paper. Feddersen (1992) constructs a model in which voters choose political positions and calls a group of voters who choose the same political position a party. In the sense that voters choose their party strategically, our model is closest to Feddersen (1992), since voters are assumed to be strategic players in his model as well as ours. However, there are also a number of differences between the two approaches. Feddersen's model is deterministic, allows an arbitrary number of parties, and allows a multidimensional policy space. In contrast, uncertainty plays an

[5]Although we cannot explain why the party line shifted dramatically in California, we can say that both dark red and dark blue are consistent with voters' party choice behavior as long as voters are politically divided.

essential role in our model, and we restrict our attention to the two-party case on a single-issue space. In our model, a party's political position (the candidate's position) is determined by aggregating the party supporters' political positions (via the party's median voter's policy). Extending the Wittman (1983) model, Roemer (2001, Chapter 5) endogenizes the party threshold by assuming that voters sort into parties by comparing their (deterministic) utility levels from two candidates' policies, which are determined by strategic interactions by the two party-median voters. In our model, voters compare the expected utility levels of joining each party. In this sense, voters in our model are more farsighted and strategic in their party choice, although primary elections are greatly simplified in our model. Our Example 2 will bring out the difference between these two approaches. Gomberg, Marhuenda, and Ortuño-Ortín (2004) also consider a two-party model, which endogenizes voters' party affiliation that determines each party's candidate's position. They prove the existence of a strong Nash equilibrium allowing for multidimensional policy space, while assuming that the policy outcome is determined by the two parties' policy positions weighted by the size of each party's support group, following the spirit of Alesina and Rosenthal (2000).

Our model is a static model, and we do not discuss the causality of events. Fiorina, Abraham, and Pope (2011) argue that each party's elite activists tend to have rather extreme views, and they influence primary election outcomes, resulting in greater polarization of policy. Levendusky (2009) stresses the role that party elites play in sorting voters into the two parties by clarifying each party's political positions.[6] Sorting of voters

[6]In the 1960s, voters were not sorted to Democrats and Republicans by their political positions (Southern states were the stronghold of conservative democrats), but by the 1980s the conservatives

can aggravate the polarization of the party candidates even further. Levendusky (2009) provides empirical evidence that supports his hypothesis. However, it would be very difficult to construct a formal game-theoretical model with many players (party elites, voters, party candidates, etc.) that describes the dynamic evolution of party policies and voter sorting, since we need to specify our model precisely through specific assumptions on how rational party elites and voters are and what information they possess when they choose their actions. The results and predictions will be very sensitive to specific setups and assumptions.

In section 2, we present our model. In section 3, we define political equilibrium and investigate its properties. Using these properties, we provide some insights into how the party threshold is affected by the distribution of voters over their political positions. In section 4, we show by two examples when the equilibrium is biased and when there are multiple equilibria. The main observations from the examples are: if one party has a stronger extreme subgroup, then that party loses some of its centrist supporters; and if the voters are more polarized, then there tend to be asymmetric equilibria in which one party consists of mostly extremists while the other party has both centrists and extremists as its supporters. In section 5, we conclude with a brief discussion of how relaxing our assumptions affects our results.

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sorted to the Republicans while the liberals sorted into the Democrats. Levendusky (2009) asserts that party elites clarified party/ideology mapping, resulting in voter sorting.

2 The Model

2.1 Policy Space and Voters

There is a one-dimensional policy space, and a continuum of citizens, namely voters, is distributed over the interval $[0, 1]$. There are two parties, and all voters belong to one of them. The party names themselves do not matter, but for convenience, *we let the party whose supporters' median political position (party median) is smaller (larger) than the other party's party median be party L (party R).* Each party selects a candidate who represents the party, and each voter casts a vote for his or her favorite candidate. Following the citizen-candidate models by Osborne and Slivinski (1996) and Besley and Coate (1997), we assume that the winner becomes the policy maker who implements her own preferred policy.

Each voter cares about the policy chosen by the elected representative and cares about the representative's likability. We assume that the candidates' likability is a random variable that is initially unknown to the voters but is revealed after the candidate starts the campaign (before voting). Note that this random shock is not an idiosyncratic shock across voters but is common to all voters, and thus affects the voting outcome.[7] At the actual voting stage, some voters might prefer the candidate of the opposite party. Voters are distributed continuously on $[0, 1]$ with density function $g(\theta)$ with $g(\theta) \geq \underline{g}$ for all voter

[7] It is well known that each candidate takes the median voter's position if there is no uncertainty, following the median voter theorem. We generate uncertainty in voting by using a valence model, which suits our purposes. Krasa and Polborn (2014) have an interesting simple model to introduce uncertainty in voting.

types $\theta \in [0,1]$ for some $\underline{g} > 0$. Type θ voters are expected-utility maximizers with the following von Neumann-Morgenstern utility function is

$$u(x;\theta,\epsilon) = -|x-\theta| + \epsilon$$

when party L's candidate with policy position $x \in [0,1]$ wins, and

$$u(y;\theta) = -\left|y-\theta\right|,$$

when party R's candidate with policy position $y \in [0,1]$ wins, where $\epsilon \in \mathbb{R}$ denote a realization of a random variable that describes party L's candidate's relative likability advantage over party R's candidate. We will assume that random variable ϵ is uniformly distributed with density function

$$f(\epsilon) = \begin{cases} \frac{1}{2a} & \text{if } \epsilon \in [-a,a] \\ 0 & \text{otherwise} \end{cases}$$

We assume $a \geq 1$ to assure that any candidates have positive winning probabilities even in an extreme voter distribution. Clearly, the expected value of ϵ is zero, and a positive realization ϵ means that party L's candidate is more likable, while a negative realization means that she is less likable.

2.2 Allocations, Party Candidates, and Coalitional Deviations

Voters choose their party affiliations, and as a result, voters' party membership distributions are determined. An **allocation** is a list $G = (g_L, g_R)$, where $g_L : [0,1] \to \mathbb{R}_+$ and $g_R : [0,1] \to \mathbb{R}_+$ are marginal membership distributions of party L and party R, respectively, such that $g_L(\theta) + g_R(\theta) = g(\theta)$ holds for all $\theta \in [0,1]$.

With the two parties' membership distributions, each party's candidate is determined. We assume that each party candidate is the median voter of party members elected, following Besley and Coate (2003).[8] Let $x(G)$ be implicitly defined by

$$\int_0^{x(G)} g_L(\theta)d\theta = \int_{x(G)}^1 g_L(\theta)d\theta \tag{1}$$

and $y(G)$ is defined in the same way.

Since we are interested in how moderate voters' party choice is affected by the presence of a strong subgroup with extreme positions, we will focus on centrally located voters who contemplate which party to belong to. We will simply assume that any voter whose political position is more extreme than the median of a party would join the party. An allocation $G = (g_L, g_R)$ is an **admissible allocation** if and only if (i) $g_L(\theta) = g(\theta)$ for all $\theta \in [0, x(G)]$, and (ii) $g_R(\theta) = g(\theta)$ for all $\theta \in [y(G), 1]$. *We will focus on admissible allocations exclusively throughout the paper.*[9]

Consider admissible allocations $G = (g_L, g_R)$ and $G' = (g'_L, g'_R)$. A **coalitional deviation** from G to G' is a mapping $\gamma' : [0, 1] \to \mathbb{R}$ such that (i) $\gamma'(\theta) = g'_L(\theta) - g_L(\theta)$ for all $\theta \in [0, 1]$, and (ii) $\gamma'(\theta) = 0$ for all $\theta \notin (x(G), y(G))$. That is, γ' describes the net movements from party L to party R, and coalitional deviation is allowed only from interval $(x(G), y(G))$. We interpret $\int_0^1 |\gamma'(\theta)| \, d\theta$ as **the size of coalitional deviation** γ' naturally. Coalitional deviation γ' is considered as the one from party R (L) to party L (R) if and only if $\int_0^1 \gamma'(\theta)d\theta > 0 (< 0)$. We say that coalitional deviation γ' from G

[8]More generally, we can introduce a (membership-based) party's policy choice function following Gomberg et al. (2004). We chose our assumption to make the analysis more concrete.

[9]We can drop this restriction by introducing arbitrarily small psychological cost for voters in $[0, x]$ and $[y, 1]$ to belong to less politically aligned parties. Appendix D details this.

to G' is **profitable** if and only if $Eu(x(G'), y(G'), \theta) > Eu(x(G), y(G), \theta)$ for all θ with $|\gamma'(\theta)| > 0$, where $Eu(x, y, \theta)$ is type θ voter's expected utility when x and y are the candidates of parties L and R, respectively.

2.3 Timing of Events and Equilibrium Concept

The voting game goes as follows:

1. Voters choose their party affiliations (each voter must choose one of the two parties).
2. Nature plays to determine x's likability ϵ (valence).
3. Observing ϵ, voters vote for x or y sincerely based on their preference only (not constrained by their party affiliations).

We define our equilibrium by applying backward-induction logic. In stage 1, voters choose their party affiliation by foreseeing $x(G)$ and $y(G)$, and the resulting probabilities of winning of these candidates are determined by the median voter's preference only (by taking sincere voting in stage 3).

Definition 1 *An admissible allocation G is a* ***political equilibrium*** *if and only if (i) voters play weakly dominant strategies (sincere voting) in stage 3, (ii) voters are backward-induction-rational expected utility maximizers, and (iii) there is a coalition size limit $\bar{\Delta} > 0$ such that there are no profitable coalitional deviations of which coalition size is less than $\bar{\Delta}$ in stage 1.*

Regarding small deviations, Osborne and Tourky (2008) also use a similar deviation named "ϵ-club" and define the "small club Nash equilibrium." However, note that Osborne and Tourky (2008) use the ϵ-clubs deviations at the voting stage just to let voters vote sincerely (with unilateral deviations, voters are indifferent between candidates). In contrast, we assume that voters form small coalitions to influence the party candidates' positions. Thus, their equilibrium concept is very different from ours.

3 The Main Analysis

In this section, we will first analyze stages 3 and 2, and the expected utility of each voter in stage 1. We will show that every political equilibrium is a sorting allocation, and proceed to prove the existence of political equilibrium. We also analyze its properties.

3.1 Stages 3 and 2

First, note that voters' behavior is not determined by the party they belong to. There is absolutely no commitment: voters consider only the candidates' political positions and their likability when deciding whom to vote for. We assume that all voters vote sincerely (weakly dominant strategies). Let us focus on the median voter θ_{med} defined implicitly by $\int_0^{\theta_{med}} g(\theta)d\theta = \int_{\theta_{med}}^1 g(\theta)d\theta$. Then *the level of likability (valence) $\epsilon(x, y)$, which makes the median voters θ_{med} indifferent between both candidates ($-|y-\theta_{med}| = |x-\theta_{med}|+\epsilon(x, y)$),* is written as follows:

$$\epsilon(x, y) \equiv -|y - \theta_{med}| + |x - \theta_{med}| = 2\theta_{med} - x - y, \qquad (2)$$

since $x \leq \theta_{med} \leq y$ by definition.

Assuming the simple majority voting at the voting stage, we have the following lemma (for the proof, see Appendix A):

Lemma 1 *If $\epsilon > \epsilon(x, y)$, then x is the winner. If $\epsilon < \epsilon(x, y)$, then y is the winner.*

Since ϵ is a random variable drawn from a probability distribution with density function f, once x and y are determined, $1 - F(\epsilon(x, y)) = \frac{1}{2} - \frac{\epsilon(x,y)}{2a}$ and $F(\epsilon(x, y)) = \frac{1}{2} + \frac{\epsilon(x,y)}{2a}$ are the winning probabilities of candidates x and y, respectively, from this lemma. Taking these probabilities and the political positions of both candidates into account, voters choose their parties. Since $\epsilon(x, y) = 2\theta_{med} - x - y$, a direct implication of the above lemma is that x has a higher (lower) chance of winning if $\theta_{med} < (>) \frac{x+y}{2}$ (see Figure 1).

3.2 Expected Utility by Voters in Stage 1

At stage 1, all voters choose either the party L or the party R. Note that since every voter is negligible, each voter's party choice has absolutely no impact on the party's representative selection. The expected utility of a voter of type θ when two candidates are x and y is

$$\begin{aligned}
Eu(x, y; \theta) &= \int_{-\infty}^{\epsilon(x,y)} f(\epsilon)(-|y - \theta|)d\epsilon + \int_{\epsilon(x,y)}^{+\infty} f(\epsilon)(-|x - \theta| + \epsilon)d\epsilon \\
&= \underbrace{F(\epsilon(x, y))}_{\text{prob. } y \text{ winning}} \times \underbrace{(-|y - \theta|)}_{\text{utility from } y \text{ winning}} + \underbrace{(1 - F(\epsilon(x, y)))}_{\text{prob. } x \text{ winning}} \times \underbrace{(-|x - \theta|)}_{\text{utility from } x\text{winning}} \\
&+ \underbrace{\int_{\epsilon(x,y)}^{+\infty} \epsilon f(\epsilon)d\epsilon}_{\text{ave. of } \epsilon \text{ when } x \text{ wins}} \qquad (3) \\
&= \left(\frac{1}{2} + \frac{\epsilon(x, y)}{2a}\right)(-|y - \theta|) + \left(\frac{1}{2} - \frac{\epsilon(x, y)}{2a}\right)(-|x - \theta|) + \frac{a}{4} - \frac{\epsilon(x, y)^2}{4a},
\end{aligned}$$

where $\epsilon(x, y) = 2\theta_{med} - x - y$, since $\theta_{med} \in (x, y)$ always holds.[10] Note that if we replace x and y by $x(G)$ and $y(G)$, respectively, then this expected utility formula $Eu(x(G), y(G); \theta)$

[10]Suppose that $\theta_{med} \leq x < y$. Then, since x and y are the medians of parties L and R, we reach a contradiction. The case where $x < y \leq \theta_{med}$ follows the same logic. Thus, $\theta_{med} \in (x, y)$ must hold.

describes voter θ's expected payoff under allocation G, rationally expecting what happens in stages 2 and 3 given sincere voting (weakly dominant strategy) in stage 3.

3.3 Deviation Incentives for Small Coalitions from the Moderate Group

Here, we analyze deviation incentives for small coalitions from an allocation $G = (g_L, g_R)$. We will concentrate on coalitional deviations from the central interval $(x(G), y(G))$. The following lemma states that the identities of voters who switch their parties in interval $(x(G), y(G))$ are irrelevant for each voter's expected utility as long as the size of the coalition is small and the coalition is from the interval $(x(G), y(G))$: what matters is the total mass of voters who switch their parties.

Lemma 2 *Suppose that coalitional deviations γ' and γ'' bringing admissible allocations G' and G'' from an admissible allocation $G = (g_L, g_R)$, respectively, satisfy $\int_0^1 \gamma'(\theta)d\theta = \int_0^1 \gamma''(\theta)d\theta$. Then, $(x(G'), y(G')) = (x(G''), y(G''))$ holds.*

This lemma allows us to focus on the mass of the net movements of voters. Let Δ be the size of the net movements of the voters who move from party R to party L among the moderate voter group in $(x(G), y(G))$, and let $G_{+\Delta}$ be the allocation generated from any coalitional deviation γ' satisfying $\int_{x(G)}^{y(G)} \gamma'(\theta)d\theta = \Delta > 0$: i.e., the net movements of voters are from party R to party L. If the net movements of voters are from party L to party R, we denote the new allocation by $G_{-\Delta}$. In the following, we first focus on $G_{+\Delta}$. We consider a small-sized coalitional deviation $\Delta \leq \bar{\Delta}$. In this case, the coalitional deviation reduces the population of party R and increases that of party L by Δ. That is, starting from allocation $G = (g_L, g_R)$, if a mass Δ of party R members located to the right of median $x(G)$ joins party L, the new median voter type $x'(G_{+\Delta})$ of party L and $y'(G_{+\Delta})$ of party R are determined by the following equations:

$$\int_0^{x'(G_{+\Delta})} g_L(\theta)d\theta = \int_{x'(G_{+\Delta})}^1 g_L(\theta)d\theta + \Delta$$

$$\int_0^{y'(G_{+\Delta})} g_R(\theta)d\theta - \Delta = \int_{y'(G_{+\Delta})}^1 g_R(\theta)d\theta,$$

respectively, using the definitions of the party medians (1). We are considering a small coalitional deviation $\Delta \leq \bar{\Delta}$, so we will take $\Delta \to 0$. By totally differentiating them, we have

$$\frac{dx'}{d\Delta} = \lim_{\Delta \to 0} \frac{x(G_{+\Delta}) - x(G)}{\Delta} = \frac{1}{2g_L(x)}, \tag{4}$$

and similarly, we have

$$\frac{dy'}{d\Delta} = \frac{1}{2g_R(y)}.$$

These derivatives represent that, by a small coalitional deviation, both $x(G)$ and $y(G)$ move to the right. Thus, type θ's expected payoff is affected by such a deviation through changes in x and y. Using (3), we can write the impact of a small coalitional deviation with size Δ from the interval $(x(G), y(G))$ as

$$\begin{aligned}
&\lim_{\Delta \to 0} \frac{1}{\Delta} \left[Eu(x(G_{+\Delta}), y(G_{+\Delta}); \theta) - Eu(x(G), y(G); \theta) \right] \\
&= \frac{dEu(x(G), y(G); \theta)}{d\Delta} \\
&= \frac{1}{2} \left[-\frac{1}{g_R(y(G))} \left(\frac{1}{2} - \frac{x(G) + y(G)}{2a} \right) + \frac{1}{g_L(x(G))} \left(\frac{1}{2} + \frac{x(G) + y(G)}{2a} \right) \right. \\
&\left. -\frac{\theta}{a} \left(\frac{1}{g_L(x(G))} + \frac{1}{g_R(y(G))} \right) \right].
\end{aligned} \tag{5}$$

The first two terms in the brackets of (5) are changes in the expected utility that both candidates bring by moving to the right. Since every voter has a linear utility, these changes are common among all voters. In other words, they do not depend on voters' types. The last term in the bracket is a change in the expected utility that is brought about by the changes in the winning probability of y and in the average likability of candidate x.

Note that θ shows up only in the last term in the brackets of (5), and $\frac{dEu(x(G),y(G);\theta)}{d\Delta}$ is strictly decreasing in $\theta \in (x, y)$. This implies that for all θ, $\frac{dEu(x(G),y(G);\theta)}{d\Delta} \geq 0$ if and only if $\frac{dEu(x(G),y(G);\theta')}{d\Delta} > 0$ for all $\theta' \in (x(G), \theta)$. Similarly, for all θ, $\frac{dEu(x(G),y(G);\theta)}{d\Delta} \leq 0$ if and only if $\frac{dEu(x(G),y(G);\theta')}{d\Delta} < 0$ for all $\theta' \in (\theta, y(G))$. Thus, a small coalition will have an incentive to deviate from an admissible allocation G, unless it is a **sorting allocation**, which is an allocation $G^{\tilde{\theta}} = (g_L^{\tilde{\theta}}, g_R^{\tilde{\theta}})$ with a **party threshold** $\tilde{\theta} \in [0, 1]$ such that (i) $g_L^{\tilde{\theta}}(\theta) = g(\theta)$ and $g_R^{\tilde{\theta}}(\theta) = 0$ for all $\theta \in [0, \tilde{\theta})$, and (ii) $g_L^{\tilde{\theta}}(\theta) = 0$ and $g_R^{\tilde{\theta}}(\theta) = g(\theta)$ for all $\theta \in (\tilde{\theta}, 1]$. Moreover, to be a political equilibrium, it is necessary to have $\frac{dEu(x(G),y(G);\tilde{\theta})}{d\Delta} = 0$.

Proposition 1 *Every political equilibrium is a sorting allocation. A sorting allocation $G^{\tilde{\theta}}$ is a political equilibrium only if* $\frac{dEu(x(G^{\tilde{\theta}}),y(G^{\tilde{\theta}});\tilde{\theta})}{d\Delta} = 0$.

Note that the above condition is just a necessary condition, since we have considered only special type of coalitional deviations. Now, we will characterize political equilibrium by considering all possible deviations. Starting from a sorting allocation $G^{\tilde{\theta}}$ satisfying $\frac{dEu(x(G^{\tilde{\theta}}),y(G^{\tilde{\theta}});\tilde{\theta})}{d\Delta} = 0$, we will check whether or not $G^{\tilde{\theta}}$ is immune to any coalitional deviations with size Δ. Again, since $\frac{dEu(x(G),y(G);\theta)}{d\Delta}$ is strictly decreasing in $\theta \in (x, y)$, we need to focus only on the following two coalitional deviations by Lemma 2. Let $\gamma_{+\Delta}^{\tilde{\theta}} : [0, 1] \to \mathbb{R}$ be such that $\gamma_{+\Delta}^{\tilde{\theta}}(\theta) = g(\theta)$ for all $\theta \in (\tilde{\theta}, \tilde{\theta} + \delta_{+\Delta}^{\tilde{\theta}}]$, and $\gamma_{+\Delta}^{\tilde{\theta}}(\theta) = 0$, otherwise, where $\int_{\tilde{\theta}}^{\tilde{\theta}+\delta_{+\Delta}^{\tilde{\theta}}} g(\theta)d\theta = \Delta$. Similarly, let $\gamma_{-\Delta}^{\tilde{\theta}} : [0, 1] \to \mathbb{R}$ be such that $\gamma_{-\Delta}^{\tilde{\theta}}(\theta) = -g(\theta)$ for all $\theta \in (\tilde{\theta} - \delta^{\tilde{\theta}}_{-\Delta}, \tilde{\theta}]$, and $\gamma^{\tilde{\theta}}_{-\Delta}(\theta) = 0$, otherwise, where $\int_{\tilde{\theta}-\delta^{\tilde{\theta}}_{-\Delta}}^{\tilde{\theta}} g(\theta)d\theta = \Delta$. Any other coalitional deviation with size Δ must have larger support than $\gamma^{\tilde{\theta}}_{+\Delta}$ or $\gamma^{\tilde{\theta}}_{-\Delta}$. We have a simple characterization of political equilibrium.

Proposition 2 *A sorting allocation $G^{\tilde{\theta}}$ is a political equilibrium if and only if (i) $\frac{dEu(x(\tilde{\theta}),y(\tilde{\theta});\tilde{\theta})}{d\Delta} =$ 0, and (ii) for all $\Delta \leq \bar{\Delta}$ small enough, (a) $Eu(x(G^{\tilde{\theta}}), y(G^{\tilde{\theta}}); \theta) \geq Eu(x(G^{\tilde{\theta}}_{+\Delta}), y(G^{\tilde{\theta}}_{+\Delta}); \theta)$ for all $\theta \in (\tilde{\theta}, \tilde{\theta} + \delta^{\tilde{\theta}}_{+\Delta})$, and (b) $Eu(x(G^{\tilde{\theta}}), y(G^{\tilde{\theta}}); \theta) \geq Eu(x(G^{\tilde{\theta}}_{-\Delta}), y(G^{\tilde{\theta}}_{-\Delta}); \theta)$ for all $\theta \in (\tilde{\theta} - \delta^{\tilde{\theta}}_{-\Delta}, \tilde{\theta})$.*

Condition (ii) says that $G^{\tilde{\theta}}$ is immune to size Δ coalitional deviations from interval $(\tilde{\theta}, y(G^{\tilde{\theta}}))$ if and only if voter $\tilde{\theta} + \delta^{\tilde{\theta}}_{+\Delta}$ has no incentive to join size Δ coalitions (recall Lemma 2). Symmetrically, condition (ii) says that $G^{\tilde{\theta}}$ is immune to size Δ coalitional deviations from interval $(x(G^{\tilde{\theta}}), \tilde{\theta})$ if and only if voter $\tilde{\theta} - \delta^{\tilde{\theta}}_{-\Delta}$ has no incentive to join size Δ coalitions. In the next section, we rewrite conditions (i) and (ii) to obtain a more tractable characterization of political equilibrium to obtain an existence theorem.

3.4 Existence of Political Equilibrium

Here, we will provide sufficient conditions for the existence of a political equilibrium. To do so, we need to characterize political equilibria in a tractable form. A sorting allocation is described completely by its threshold $\tilde{\theta}$, and each candidate's position also determined by $\tilde{\theta}$. With an abuse of notation, we will also denote each candidate as a function of $\tilde{\theta}$: i.e. $x = x(\tilde{\theta})$ and $y = y(\tilde{\theta})$ in the following sections when we focus on a change in the threshold $\tilde{\theta}$. Given this sorting allocation, x and y can be written as $x = x(\tilde{\theta})$ and

$y = y(\tilde{\theta})$, and the condition (5) becomes

$$\frac{dEu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)}{d\Delta} \tag{6}$$
$$= \frac{1}{2}\left[-\frac{1}{g(y(\tilde{\theta}))}\left(\frac{1}{2} - \frac{x(\tilde{\theta}) + y(\tilde{\theta})}{2a}\right) + \frac{1}{g(x(\tilde{\theta}))}\left(\frac{1}{2} + \frac{x(\tilde{\theta}) + y(\tilde{\theta})}{2a}\right) - \frac{\theta}{a}\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right)\right].$$

Setting $\theta = \tilde{\theta}$ in (6), we can define function $\varphi : [0,1] \to \mathbb{R}$ such that

$$\varphi(\tilde{\theta}) = 2 \times \frac{dEu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta})}{d\Delta}$$
$$= -\frac{1}{g(y(\tilde{\theta}))}\left(\frac{1}{2} - \frac{x(\tilde{\theta}) + y(\tilde{\theta})}{2a}\right) + \frac{1}{g(x(\tilde{\theta}))}\left(\frac{1}{2} + \frac{x(\tilde{\theta}) + y(\tilde{\theta})}{2a}\right) - \frac{\tilde{\theta}}{a}\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right) \tag{7}$$
$$= \frac{1}{2a}\left(x(\tilde{\theta}) + y(\tilde{\theta}) - 2\tilde{\theta}\right)\left(\frac{1}{g(y(\tilde{\theta}))} + \frac{1}{g(x(\tilde{\theta}))}\right) + \frac{1}{2}\left(\frac{1}{g(x(\tilde{\theta}))} - \frac{1}{g(y(\tilde{\theta}))}\right)$$

This function will prove to be useful in subsequent analysis. An immediate consequence of introducing $\varphi(\tilde{\theta})$ is that condition (i) of Proposition 2 is equivalent to $\varphi(\tilde{\theta}) = 0$. Thus, to characterize political equilibrium by using φ, we need to connect condition (ii) of Proposition 2 with φ.

Note that $\varphi(\tilde{\theta})$ denotes the change of the border type $\tilde{\theta}$'s expected utility when the party threshold $\tilde{\theta}$ moves, but it is not the change of the expected utility of any particular type of voters. This is because the evaluating type $\tilde{\theta}$ itself is also changing as the party threshold $\tilde{\theta}$ changes. To evaluate the expected utility change of some type θ, we need to adjust the formula to use the φ function to evaluate the expected utility change of each player when the party threshold $\tilde{\theta}$ changes. With cumbersome calculations, we obtain the following characterization of political equilibrium.

Proposition 3 *Suppose that g is continuous. Suppose that (1) voters in intervals $[0, x]$ and $[y, 1]$ belong to parties L and R, respectively, and (2) coalitional deviations are from* *interval* $(x(G), y(G))$. *Then, a sorting allocation with threshold* $\tilde{\theta}$ *is a political equilibrium if (i)* $\varphi(\tilde{\theta}) = 0$ *and (ii)* $\varphi'(\tilde{\theta}) - \frac{1}{a}\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right) < 0$. *On the other hand, a sorting allocation with threshold* $\tilde{\theta}$ *is a political equilibrium only if (i)* $\varphi(\tilde{\theta}) = 0$ *and (ii')* $\varphi'(\tilde{\theta}) - \frac{1}{a}\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right) \leq 0$.

Proposition 3 says that $\varphi(\tilde{\theta}) = 0$ is necessary, but we need an additional condition (ii) (this corresponds to condition (ii) in Proposition 2). Now, we can find a simple sufficient condition for a sorting allocation to be a political equilibrium.

Corollary 1 *Suppose that* g *is continuous. A sorting allocation with threshold* $\tilde{\theta}$ *is a political equilibrium if (i)* $\varphi(\tilde{\theta}) = 0$ *and (ii)* $\varphi'(\tilde{\theta}) \leq 0$.

With this corollary, it is easy to see that there exists a political equilibrium if $\varphi(0) > 0$ and $\varphi(1) < 0$, which are assured under the following mild conditions.[11]

Theorem 1 *If* g *is continuous with* $g(\theta) > 0$ *for all* $\theta \in [0, 1]$, *then* $g(0) \leq g(\theta_{med})$ *and* $g(1) \leq g(\theta_{med})$ *are sufficient for the existence of political equilibrium with interior* $\tilde{\theta}^*$.

3.5 Party Threshold and the Median Position

In this subsection, we investigate how the distribution of voter types is important to determining the equilibrium party structure by using our characterization of a sorting political equilibrium, which includes Proposition 3 or Corollary 1. We start by comparing the

[11]Note that we are assuming that there are always two parties even in the case of $\tilde{\theta} = 0$ or 1. Here, we are considering the case where the minority party is extremely small ($\tilde{\theta} = \epsilon$ or $\tilde{\theta} = 1 - \epsilon$ for ϵ very small). Taking the limit, we have $\lim_{\epsilon \to 0} \varphi(\epsilon) = \varphi(0)$ and $\lim_{\epsilon \to 0} \varphi(1 - \epsilon) = \varphi(1)$.

equilibrium party threshold $\tilde{\theta}$ with the traditional "median voter" θ_{med}. We will consider the condition where the median type becomes the threshold of a two-party structure.

It is still hard to tell in general how the sign of $\varphi(\tilde{\theta})$ changes as $\tilde{\theta}$ goes up, but we can decompose the effects. Rewriting $\varphi(\tilde{\theta})$, we have

$$\frac{2ag(x(\tilde{\theta}))g(y(\tilde{\theta}))}{g(x(\tilde{\theta}))+g(y(\tilde{\theta}))} \times \varphi(\tilde{\theta}) = \left(x(\tilde{\theta}) + y(\tilde{\theta}) - 2\tilde{\theta}\right) + \ a \times \left(\frac{g(y(\tilde{\theta})) - g(x(\tilde{\theta}))}{g(x(\tilde{\theta}))+g(y(\tilde{\theta}))}\right) \gtreqless 0. \quad (8)$$

We will focus on the sign of $\varphi(\theta_{med})$: As long as $\varphi(0) > 0$ and $\varphi(1) < 0$, there is a sorting political equilibrium with $\tilde{\theta}^* \gtreqless \theta_{med}$, if $\varphi(\theta_{med}) \gtreqless 0$ holds, or equivalently,

$$\underbrace{(x(\theta_{med}) + y(\theta_{med}) - 2\theta_{med})}_{A} + \ a \times \underbrace{\left(\frac{g(y(\theta_{med})) - g(x(\theta_{med}))}{g(x(\theta_{med})) + g(y(\theta_{med}))}\right)}_{B} \gtreqless 0. \quad (9)$$

Since $x(\theta_{med}) + y(\theta_{med}) - 2\theta_{med} = (y(\theta_{med}) - \theta_{med})) - (\theta_{med} - x(\theta_{med}))$, term A being positive means that party L's candidate is closer to the median voter than party R's candidate, and thus party L has a higher probability of winning. Term B being positive means (more or less) that party R's candidate's political position is harder to move than party L's candidate's position: thus, party L is more responsive to voters' party choice (a smaller $g(x(\theta_{med}))$ means a greater responsiveness of $x(\theta_{med})$ for the same mass of voters joining party L). Note that as a, the uncertainty in likability level, becomes larger (smaller), term B (A) becomes the dominant force in determination of the sign of $\varphi(\theta_{med})$.

Now, we will determine whether symmetric voters' distribution assures the existence of equilibrium with $\tilde{\theta}^* = \theta_{med}$. At a glance, if $g(\theta)$ is symmetric, both terms A and B become zero at θ_{med} and the party threshold at the median θ_{med} would be a sorting equilibrium. However, it turns out that it is not sufficient to have symmetric g, though the additional required condition is often satisfied.

Proposition 4 *Suppose that the conditions in Theorem 1 are met, and that g is continuously differentiable and symmetric. Then, there is a political equilibrium with $\tilde{\theta}^* = \theta_{med} = \frac{1}{2}$ if and only if*

$$\frac{g(\theta_{med})}{2g(x(\theta_{med}))^2}\left(\frac{2}{a} - \frac{g'(x(\theta_{med}))}{g(x(\theta_{med}))}\right) - \frac{4}{g(x(\theta_{med}))a} \leq 0.$$

The above proposition tells us that even if g is symmetric, there may not be a political equilibrium with $\tilde{\theta}^* = \theta_{med} = \frac{1}{2}$. If the necessary and sufficient conditions are violated, then condition (ii) of Proposition 3 is violated although condition (i) of it is satisfied at $\tilde{\theta}^* = \theta_{med} = \frac{1}{2}$. In such a situation, there are only asymmetric equilibria.[12] More generally, if g is not symmetric, then $\tilde{\theta}$ and θ_{med} need not coincide with each other. They can coincide, but only in very special situations.

3.6 Uniqueness of Political Equilibrium

Now, let us consider the issue of uniqueness of political equilibrium. In the previous subsection, we compared the party threshold of an equilibrium with the median voter's position. However, there may be another equilibrium that has different characteristics. To ensure uniqueness, we need to know the global shape of φ function. A sufficient condition for uniqueness is that $\varphi(\tilde{\theta})$ is monotonically decreasing, which is assured when both of the two terms in (8) are decreasing in $\tilde{\theta}$. In the following simple case, we can assure that.

If the density g is concave, it is **single peaked** at some $\theta^p \in [0, 1]$. With a concave and rather flat density function g, we can guarantee uniqueness of equilibrium.

[12]See the second to last case (Figure 6) in Example 2 (see Appendix B). The symmetric allocation is unstable and is not a political equilibrium.

Proposition 5 *Suppose that the conditions of Theorem 1 are met and that g is continuously differentiable. Then, there is a unique equilibrium if (i) g is concave with a peak at θ^p, and (ii) $g(\theta^p) \leq 2g(\theta)$ for all $\theta \in [0, 1]$.*

With uniqueness of equilibrium, we can use inequality (9) to see how the equilibrium party threshold $\tilde{\theta}^*$ differs from θ_{med}. Now, suppose that $\theta^p < \theta_{med}$, which implies that party L has a more extreme group (see Figure 2). In this case, the winning probability effect (term A) tends to be positive ($\theta_{med} - x(\theta_{med}) < y(\theta_{med}) - \theta_{med}$: candidate x has better chance to win) while the policy responsiveness effect (term B) is negative ($g(x(\theta_{med})) < g(y(\theta_{med})$: candidate y's position is more responsive). Here, the value of a matters in determining the party threshold $\tilde{\theta}^*$. Party L expands ($\theta_{med} < \tilde{\theta}^*$) if uncertainty in likability is low (a is small), while it shrinks ($\tilde{\theta}^* < \theta_{med}$) if uncertainty is high (a is large). When uncertainty is high, far right candidate y has a good chance to win, and the centrally located voters try to bring y closer to them by participating party R.

4 The Party Structure in a Political Equilibrium

In this section, we analyze two parametric examples to illustrate how the party structure is determined by the distribution of voters. For simplicity, we will allow discontinuity of g and analyze a minimally asymmetric or nonuniform voter distributions: density functions g will be step functions. Given the discontinuity of the g function, the φ function has kinks or discontinuities, but we can easily approximate it by a continuous function using the standard procedure. The conditions of Theorem 1 are all satisfied after an approximation of g in the following examples.

In the first example, we consider the case where the voters' distribution in the left area is denser than the right area and the median is on the left side, and discontinuity occurs only at the median. We show that party L has a denser voter distribution and a shorter tail, and that there is a unique equilibrium in which party L loses some of its moderate supporters.

Example 1. Consider the case where $g(\theta)$ is a step function:

$$g(\theta) = \begin{cases} \frac{1}{2\theta_{med}} & \text{if } \theta \leq \theta_{med} \\ \frac{1}{2(1-\theta_{med})} & \text{if } \theta > \theta_{med} \end{cases}$$

Without loss of generality, we assume $\theta_{med} \leq \frac{1}{2}$ so that $\frac{1}{2\theta_{med}} \geq \frac{1}{2(1-\theta_{med})}$. In this case, we have a unique political equilibrium with

$$\tilde{\theta}^* = \frac{2\theta_{med}((2a+1)\theta_{med} - a)}{4\theta_{med} - 1} < \theta_{med}.$$

Thus, the party threshold is unambiguously biased: although the winning probability effect favors party L ($\theta_{med} - x(\theta_{med})$ is smaller than $y(\theta_{med})$), the policy responsive effect also favors party L ($g(x(\theta_{med})) > g(y(\theta_{med}))$). In this case, the latter effect dominates the former, and party L loses its moderate support group. The details of the figure and the calculations of $g(\theta)$ and $\varphi(\tilde{\theta})$ are given in Appendix B and Figure 3.[13] $\square$

Example 1 also shows that as long as g is relatively flat, the equilibrium is unique even if the voter distribution is asymmetric although the party threshold is biased.

In the following example with symmetric voter distribution, we show that there can be multiple equilibria. In the following, we consider a symmetric voter distribution g to show

[13] Although g is discontinuous at θ_{med}, $\varphi(\tilde{\theta})$ is not discontinuous but only kinks at θ_{med} because $x \leq \theta_{med} \leq y$ always hold and x nor y strides over a step at θ_{med}.

that there can be multiple equilibria if g is not single peaked. There are three core groups: Extreme Left (EL), Center (C), and Extreme Right (ER). We assume that Moderate Left (ML) and Moderate Right (MR) are distributed along a wider political range and are less concentrated than EL, C, and ER. This distribution describes a political situation of three political groups with distinct political positions. If these groups become stronger, multiple equilibria easily emerge even if voter distribution is symmetric. There can be a pair of equilibria that are symmetric to each other: one in which party L is composed mostly of group EL with an extreme policy while party R has groups C and ER with a rather moderately Right policy, and the other in which party R is composed mostly of group ER with an extreme policy while party L has groups C and EL with a rather moderately Left policy.

Example 2. Let us consider the following symmetric voter distribution described by a step function ($0 < b \leq 1$).

$$g(\theta) = \begin{cases} 3 - 2b & \text{for all } \theta \in \underbrace{[0, \frac{1}{9}]}_{EL} \cup \underbrace{[\frac{4}{9}, \frac{5}{9}]}_{C} \cup \underbrace{[\frac{8}{9}, 1]}_{ER} \\ b & \text{for all } \theta \in \underbrace{(\frac{1}{9}, \frac{4}{9})}_{ML} \cup \underbrace{(\frac{5}{9}, \frac{8}{9})}_{MR} \end{cases}$$

When $b = 1$, this example degenerates to uniformly distributed g. As b decreases from unity, the voters' distribution becomes more and more politically divided, although we assume that there are still plenty of members in the centrist group. If b is still large, the equilibrium is still unique (see Figure 4), while two asymmetric equilibria show up if b goes down sufficiently (see Figure 5). We provide the full analysis of this example in Appendix B. □

This example shows that if there are core extreme groups (if voters are divided politically), political equilibria can be significantly biased and a political party may represent an extreme core group by alienating the center-ground voters even if the voters' distribution is symmetric.[14] The existence of multiple equilibria means that even if a political environment $g(\theta)$ does not change, the political outcome can be different. That is, when voters are politically divided, if some large enough exogenous shock occurs, then the party supporters' allocation can jump from one political equilibrium to another.

5 Conclusion

In this paper, we considered a two-party representative democracy and investigated how the distribution of voters' policy positions on a one-dimensional policy issue space affects the party threshold and the probability of each party's winning. We introduced a common shock that affects each voter's utility, in contrast with the standard idiosyncratic shocks in the probabilistic voting model. Focusing on centrally located voters' party choice, we also introduced a new equilibrium concept of political equilibrium, which is immune to any small coalitional deviations near the party threshold, in contrast with Nash equilibrium and strong equilibrium. We showed that voters' distribution intrinsically affects the party

[14] This example starkly contrasts our political equilibrium notion with Roemer's equilibrium notion of an endogenous party structure (Roemer 2001, Chapter 5). In our model, the voters' party choice is determined by a comparison of expected utilities from joining the L and R parties, but Roemer assumes that $\tilde{\theta}$ is determined by a comparison of two parties' policies. For example, if $x = \frac{1}{18}$ and $y = \frac{5}{9}$, the party threshold is the middle point of the two: $\tilde{\theta} = \frac{11}{36}$. Thus, our biased equilibrium cannot be supported as an equilibrium. In fact, in this example, the Roemer equilibrium must be symmetric $\tilde{\theta} = \frac{1}{2}$.

threshold in the political equilibrium. In addition, we showed that when voters are divided into three political positions, multiple equilibria appear as the divisions grow deeper. Especially if voters are deeply divided, symmetric equilibrium disappears even though the distribution is symmetric. In each asymmetric equilibrium, the minority candidate becomes more extreme, and the other becomes more moderate. These multiple equilibria emerging from deeply divided voters can be interpreted as the political instability we see when elections swing extremely between Left and Right.

In future research, we may consider the following two extensions. First, it would be interesting to think about how to make each party's supporters select their candidate strategically in the original Besley-Coate model (Besley and Coate 1997). One way is to assume that given the party threshold, each voter tries to find her ideal candidate for the party (depending on her policy position and her candidate's chances of winning). It may be possible for us to drop our simple median voter assumption in order to show the existence of equilibrium. We can consider the following game. In primary elections, each voter announces her ideal policy position (taking winning probabilities and her true bliss point) given the other party's candidate position, and the median of announced positions becomes the party's candidate position. With this party decision rule, the candidates' position profile is determined as a Nash equilibrium. In this game, we can show that the best response curve of each party is more moderate than the party median (bounded above by the party's true median position), and the equilibrium outcome is weakly more moderate than the naive primary elections we considered in this paper. However, the characterization of equilibrium under general assumptions can be very difficult.

Second, we used a static model in this paper. Although a static approach has advantages, it also has drawbacks — we need to treat both candidates symmetrically, and we cannot introduce incumbents and challengers into the model. Bernhardt et al. (2011) consider a two-party repeated election model with single-dimensional policy space, in which candidates are distinguished by their ideology and valence (likability).[15] Voters decide who to vote for, observing the incumbent's valence and policy (but not her ideology) while knowing only the challenger's party (whose position is drawn randomly from the party's support). Analyzing the unique symmetric stationary equilibrium when voters are distributed symmetrically, they show that a high-valence incumbent chooses a more moderate policy while a low-valence incumbent chooses a more extreme policy. Their results are very interesting, and the underlying model is similar; thus it might seem natural to extend their model by endogenizing party structure. However, their analysis is quite sophisticated, and introducing asymmetric voter distribution is already a very hard problem. To overcome the difficulty, we may simply assume static expectation dynamics (Kramer 1977; Ferejohn, Fiorina, and Packel 1980; Ferejohn, McKelvey, and Packel 1984; Kollman, Miller, and Page 1992; and, in particular, Bender, Diermeier, Siegel, and Ting 2011). Voters know the incumbent's likability (valence) but do not know how likable a challenger is going to be compared with whoever wins in the other party's primary election. The incumbent's policy and valence level are intact, and the party threshold can be determined by the previous election. Suppose that a party occupies the office and there is an incumbent candidate. The challenging party chooses its candidate as noted in the previous paragraph. It may be interesting to see how the challenging party reacts to

[15]Duggan (2000) is the first to analyze the now standard repeated election model, but without valence. He proves the existence and uniqueness of a stationary equilibrium for symmetrically distributed voters.

likable and unlikable incumbents, and how the dynamics of candidate profiles emerge.

Appendix A: Proofs

Proof of Lemma 1 Each candidate is a median type of each party, $x \leq \theta_{med} \leq y$. Assume that ϵ makes type $\hat{\theta} \in [x, y]$ indifferent between x and y. Then, for all $\theta \in [x, y)$ such that $\theta < \hat{\theta}$, and for all $\bar{\theta} \in [0, x)$,

$$\begin{aligned}
0 = -|y-\hat{\theta}| + |\hat{\theta}-x| - \epsilon = -(y-\hat{\theta}) + (\hat{\theta}-x) - \epsilon = 2\hat{\theta} - x - y - \epsilon \\
&> 2\theta - x - y - \epsilon = h(x, y; \theta) - \epsilon \\
&\geq 2x - x - y - \epsilon \\
&= x - y - \epsilon = -(y-\bar{\theta}) + (x-\bar{\theta}) - \epsilon = -|y-\bar{\theta}| + |\bar{\theta}-x| - \epsilon.
\end{aligned}$$

Thus, all voters of $\theta \in [0, \hat{\theta})$ prefer x to y, since $h(x, y; \theta)$, which is the relative evaluation of y to x is negative; that is, all voters of $\theta < \hat{\theta}$ type vote for x when ϵ. Here, if $\epsilon > \epsilon(x, y)$, then, from

$$\begin{aligned}
0 &= -|y-\hat{\theta}| + |\hat{\theta}-x| - \epsilon = 2\hat{\theta} - x - y - \epsilon \\
&< 2\hat{\theta} - x - y - \epsilon(x, y) \\
&= 2\hat{\theta} - x - y - (-|y-\theta_{med}| + |\theta_{med}-x| - \epsilon) = 2(\hat{\theta} - \theta_{med}),
\end{aligned}$$

we have $\hat{\theta} > \theta_{med}$. Hence, x gets a majority and wins when $\epsilon > \epsilon(x, y)$.

Similarly, if $\epsilon < \epsilon(x, y)$, $\hat{\theta} < \theta_{med}$ and every type $\theta > \hat{\theta}$ vote for y, and y wins. $\square$

Proof of Lemma 2 Since deviations γ' and γ'' are from the interval $(x(G), y(G))$, we have $\gamma'(\theta) = \gamma''(\theta) = 0$ for all $\theta \in [0, x(G)] \cup [y(G), 1]$. First assume $\int_0^1 \gamma'(\theta)d\theta = \int_0^1 \gamma''(\theta)d\theta =$

$\Delta > 0$: i.e., the net voter movements are from party R to party L. Since G' and G'' are also admissible, by the definitions of $x(G')$ and $y(G')$, we have

$$\int_0^{y(G')} \left(g_R(\theta) - \gamma'(\theta)\right) d\theta = \int_0^{y(G')} g_R(\theta) d\theta - \Delta = \int_{y(G')}^1 g(\theta) d\theta.$$

and

$$\int_0^{y(G'')} \left(g_R(\theta) - \gamma'(\theta)\right) d\theta = \int_0^{y(G'')} g_R(\theta) d\theta - \Delta = \int_{y(G'')}^1 g(\theta) d\theta.$$

Thus, $y(G') = y(G'')$ must follow. Given this, we have

$$\int_0^{y(G')} \left(g_L(\theta) + \gamma'(\theta)\right) d\theta = \int_0^{y(G'')} \left(g_L(\theta) + \gamma''(\theta)\right) d\theta = \int_0^{y(G)} g_L(\theta) d\theta + \Delta$$

holds, since $y(G') = y(G'') > y(G)$ and G is admissible. Since G' and G'' are admissible, for G'', we have

$$\int_0^{x(G')} g(\theta) d\theta = \frac{1}{2}\left[\int_0^{y(G)} g_L(\theta) d\theta + \Delta\right]$$

and

$$\int_0^{x(G'')} g(\theta) d\theta = \frac{1}{2}\left[\int_0^{y(G)} g_L(\theta) d\theta + \Delta\right].$$

Thus, we have $x(G') = x(G'')$ as well. We can treat the case of $\int_0^1 \gamma'(\theta) d\theta = \int_0^1 \gamma''(\theta) d\theta = -\Delta < 0$ symmetrically. We have completed the proof. $\square$

Proof of Proposition 3 Proof of Proposition 3 is provided by using the following lemmas. First, we obtain the formula of expected utility change when party threshold $\tilde{\theta}$ shifts to $\tilde{\theta} + \delta$. Those deviations can be expressed by shifting the party threshold $\tilde{\theta}$ by δ. In the following lemma, we provide the difference in the expected utility of type θ when the threshold changes from $\tilde{\theta}$ to $\tilde{\theta} + \delta$ or to $\tilde{\theta} - \delta$.

Lemma 3 *Consider sorting allocations described by $\tilde{\theta}$ and $\tilde{\theta}+\delta$ such that $\delta > 0$ and that δ is sufficiently small. Then, we have*

$$\begin{aligned}
&Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta);\theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta});\theta) \\
&= \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{g(\theta')}{2}\left[\varphi(\theta') - \frac{\theta-\theta'}{a}\left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))}\right)\right] d\theta'.
\end{aligned}$$

As a consequence, $Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta);\theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta});\theta)$ is decreasing in θ for all $\theta \in (\tilde{\theta}, y(\tilde{\theta}))$. Similarly, consider sorting allocations described by $\tilde{\theta}$ and $\tilde{\theta}-\delta$. Then, we have

$$\begin{aligned}
&Eu(x(\tilde{\theta}-\delta), y(\tilde{\theta}-\delta);\theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta});\theta) \\
&= -\int_{\tilde{\theta}-\delta}^{\tilde{\theta}} \frac{g(\theta')}{2}\left[\varphi(\theta') - \frac{\theta-\theta'}{a}\left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))}\right)\right] d\theta'.
\end{aligned}$$

As a consequence, $Eu(x(\tilde{\theta}-\delta), y(\tilde{\theta}-\delta);\theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta});\theta)$ is increasing in θ for all $\theta \in (x(\tilde{\theta}), \tilde{\theta})$.

Proof of Lemma 3 Differentiating $Eu(x(\tilde{\theta}), y(\tilde{\theta});\theta)$ with respect to $\tilde{\theta}$, we obtain:

$$\begin{aligned}
&\frac{dEu(x(\tilde{\theta}), y(\tilde{\theta});\theta)}{d\tilde{\theta}} \\
&= \frac{g(\tilde{\theta})}{2}\left[-\frac{1}{g(y)}\left(\frac{1}{2} - \frac{x+y}{2a}\right) + \frac{1}{g(x)}\left(\frac{1}{2} + \frac{x+y}{2a}\right) - \frac{\theta}{a}\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)\right] \\
&= \frac{g(\tilde{\theta})}{2}\left[-\frac{1}{g(y)}\left(\frac{1}{2} - \frac{x+y}{2a}\right) + \frac{1}{g(x)}\left(\frac{1}{2} + \frac{x+y}{2a}\right) - \frac{\tilde{\theta}}{a}\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)\right. \\
&\qquad\qquad \left. - \frac{\theta-\tilde{\theta}}{a}\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)\right] \\
&= \frac{g(\tilde{\theta})}{2}\left[\varphi(\tilde{\theta}) - \frac{\theta-\tilde{\theta}}{a}\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)\right].
\end{aligned}$$

This implies that for small $\delta > 0$, we have

$$
\begin{aligned}
& Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta); \theta) \\
& = Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta) + \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{dEu(x(\theta'), y(\theta'); \theta)}{d\theta'} d\theta' \\
& = Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta) + \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - \frac{\theta - \tilde{\theta}}{a}\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)\right] d\theta'.
\end{aligned}
$$

θ appears only in the brackets as $\theta - \theta'$, so that the second term in this expression is decreasing in θ. Hence, $Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$ is decreasing in θ. The latter half of the statement in lemma 3 can be shown by a symmetric argument. $\square$

By applying the first-order Taylor expansion, we can approximate the utility change of the critical coalition member's utility in the below lemma when a coalition γ_R^δ deviates.

Lemma 4 *Suppose that $\varphi(\tilde{\theta}) = 0$ and that f and g are differentiable functions. Then, for sufficiently small $\delta > 0$, $Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta); \tilde{\theta}+\delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\delta)$ is approximated as*

$$
\begin{aligned}
& Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta); \tilde{\theta}+\delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\delta) \\
& = \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - \frac{\tilde{\theta}+\delta-\theta'}{a}\left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))}\right)\right] d\theta' \\
& \simeq \frac{\delta^2 g(\tilde{\theta})}{4} \left[\varphi'(\tilde{\theta}) - \frac{1}{a}\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right)\right].
\end{aligned}
$$

Proof of Lemma 4 First, we will approximate $Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta); \tilde{\theta}+\delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+$

δ) by using the first-order Taylor expansion.

$$\begin{aligned}
&Eu(x(\tilde{\theta}+\delta), y(\tilde{\theta}+\delta); \tilde{\theta}+\delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\delta) \\
&= \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - \frac{\tilde{\theta}+\delta-\theta'}{a} \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) \right] d\theta' \\
&= \frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} g(\theta')\varphi(\theta') d\theta' \\
&+ \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{\tilde{\theta}+\delta-\theta'}{a} \left(-\frac{g(\theta')}{2} \right) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) d\theta'.
\end{aligned}$$

Noting $\varphi(\tilde{\theta}) = 0$, the first term is approximated as

$$\begin{aligned}
\frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \varphi(\theta') g(\theta') d\theta' &\simeq \frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} (\varphi(\tilde{\theta}) g(\tilde{\theta}) + (\varphi'(\tilde{\theta}) g(\tilde{\theta}) + \varphi(\tilde{\theta}) g'(\tilde{\theta}))(\theta' - \tilde{\theta})) d\theta' \\
&= \frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \varphi'(\tilde{\theta}) g(\tilde{\theta}) (\theta' - \tilde{\theta}) d\theta' \\
&= \frac{1}{2} \varphi'(\tilde{\theta}) g(\tilde{\theta}) \left[\frac{(\theta' - \tilde{\theta})^2}{2} \right]_{\tilde{\theta}}^{\tilde{\theta}+\delta} \\
&= \frac{\delta^2}{4} \varphi'(\tilde{\theta}) g(\tilde{\theta}).
\end{aligned}$$

To calculate the second term, first note that

$$\frac{d}{d\theta'} \epsilon(x(\theta'), y(\theta'))) = -\frac{g(\theta')}{2} \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right).$$

Thus, partially integrating the second term, we obtain

$$\begin{aligned}
&\int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{\tilde{\theta}+\delta-\theta'}{a} \left(-\frac{g(\theta')}{2} \right) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) d\theta' \\
&= \int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{\tilde{\theta}+\delta-\theta'}{a} \frac{d}{d\theta'} \epsilon(x(\theta'), y(\theta')) d\theta' \\
&= \underbrace{\left[\frac{\tilde{\theta}+\delta-\theta'}{a} \epsilon(x(\theta'), y(\theta')) \right]_{\tilde{\theta}}^{\tilde{\theta}+\delta}}_{A} + \underbrace{\int_{\tilde{\theta}}^{\tilde{\theta}+\delta} \frac{\epsilon(x(\theta'), y(\theta')))}{a} d\theta'}_{B}.
\end{aligned}$$

Now, term A is rewritten as

$$
\begin{aligned}
&\frac{1}{a}\left[\left(\tilde{\theta}+\delta-\theta'\right)\delta(x(\theta'),y(\theta'))\right]_{\tilde{\theta}}^{\tilde{\theta}+\delta} \\
&=\frac{1}{a}\left[\left(\tilde{\theta}+\delta-\left(\tilde{\theta}+\delta\right)\right)\epsilon(x(\tilde{\theta}+\delta),y(\tilde{\theta}+\delta))-\left(\tilde{\theta}+\delta-\tilde{\theta}\right)\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))\right] \\
&=-\frac{\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))}{a}\delta.
\end{aligned}
$$

Since $\epsilon(x(\theta'),y(\theta')) \simeq \epsilon(x(\tilde{\theta}),y(\tilde{\theta}))+\frac{d\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))}{d\theta'}\left(\theta'-\tilde{\theta}\right)$, by substituting $\frac{d\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))}{d\tilde{\theta}} = -\frac{g(\tilde{\theta})}{2}\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right)$ into this approximation, term B can be approximated as

$$
\begin{aligned}
&\int_{\tilde{\theta}}^{\tilde{\theta}+\delta}\frac{\epsilon(x(\theta'),y(\theta'))}{a}d\theta' \\
&\simeq \frac{\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))}{a}\int_{\tilde{\theta}}^{\tilde{\theta}+\delta}d\theta'+\frac{1}{a}\frac{d\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))}{d\theta'}\int_{\tilde{\theta}}^{\tilde{\theta}+\delta}(\theta'-\tilde{\theta})d\theta' \\
&=\frac{\epsilon(x(\tilde{\theta}),y(\tilde{\theta}))}{a}\delta+\frac{1}{a}\left(-\frac{g(\tilde{\theta})}{2}\right)\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right)\frac{\delta^2}{2}.
\end{aligned}
$$

Thus, the second term is $A+B=-\frac{g(\tilde{\theta})}{2a}\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right)\frac{\delta^2}{2}$. Hence, we have the approximation formula:

$$
\begin{aligned}
&Eu(x(\theta'),y(\theta');\tilde{\theta}+\delta)-Eu(x(\tilde{\theta}),y(\tilde{\theta});\tilde{\theta}+\delta) \\
&\simeq \frac{\delta^2}{4}\varphi'(\tilde{\theta})g(\tilde{\theta})-\frac{g(\tilde{\theta})}{2a}\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right)\frac{\delta^2}{2} \\
&=\frac{\delta^2 g(\tilde{\theta})}{4}\left[\varphi'(\tilde{\theta})-\frac{1}{a}\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right)\right].
\end{aligned}
$$

We have completed the proof. □

Proposition 3 Conditions (ii) and (iii) in Proposition 3 directly follow from Lemma 4.

□

Proof of Corollary 1 Since g is a density function, their values are nonnegative. Thus, from Lemma 4, we get the conclusion directly. $\square$

Proof of Theorem 1 Noting that $x(0) = 0$, $y(0) = \theta_{med}$, $x(1) = \theta_{med}$, and $y(1) = 1$, we obtain

$$\varphi(0) = \frac{1}{2a}\left[\frac{1}{g(\theta_{med})} \times (-a + \theta_{med}) + \frac{1}{g(0)} \times (a + \theta_{med})\right],$$
$$\varphi(1) = \frac{1}{2a}\left[\frac{1}{g(1)} \times (-a - (1 - \theta_{med})) + \frac{1}{g(\theta_{med})} \times (a - (1 - \theta_{med}))\right].$$

Thus, $g(\theta_{med}) \geq g(0)$ and $g(\theta_{med}) \geq g(1)$ are sufficient for $\varphi(0) > 0$ and $\varphi(1) < 0$, respectively. This implies that we can assure the existence of political equilibrium under these conditions. $\square$

Proof of Proposition 4 Let $\tilde{\theta} = \theta_{med} = \frac{1}{2}$. Then, by symmetry of g, we have $\theta_{med} - x(\theta_{med}) = y(\theta_{med}) - \theta_{med}$, $g(x(\theta_{med})) = g(y(\theta_{med}))$ and $g'(x) = -g'(y)$. Thus, $x(\theta_{med}) + y(\theta_{med}) = 1$ is obtained. Since f is symmetric, $1 - F(0) = F(0) = \frac{1}{2}$. Then,

$$\begin{aligned}
\varphi\left(\frac{1}{2}\right) &= -\frac{1}{g(y)}\left(\frac{1}{2} - \frac{x+y}{2a}\right) + \frac{1}{g(x)}\left(\frac{1}{2} + \frac{x+y}{2a}\right) - \frac{\frac{1}{2}}{a}\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right). \\
&= \frac{1}{g(x)}\left(-\frac{1}{2} + \frac{1}{2a} + \frac{1}{2} + \frac{1}{2a}\right) - \frac{1}{a}\frac{1}{g(x)} \\
&= 0
\end{aligned}$$

Thus, when $\tilde{\theta} = \theta_{med}$, $\varphi(\theta_{med}) = 0$. In addition, by using the above facts, we have

$$\varphi'(\tilde{\theta}) = \frac{g(\theta_{med})}{2g(x)^2}\left[\frac{2}{a} - \frac{g'(x)}{g(x)}\right] - \frac{2}{g(x)a}.$$

Moreover, the necessary condition in Proposition 3,

$$\varphi'(\tilde{\theta}) - \frac{1}{a}\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right) \leq 0$$

is equivalent to

$$\frac{g(\theta_{med})}{2g(x)^2}\left(\frac{2}{a}-\frac{g'(x)}{g(x)}\right)-\frac{4}{g(x)a}\leq 0.$$

Hence, if this condition is satisfied, there is a political equilibrium with $\tilde{\theta}=\theta_{med}$. $\square$

Proof of Proposition 5. We first prove the following lemma.

Lemma 5 *Suppose that the conditions of Theorem 1 are met and that g is continuously differentiable. Then, there is a unique equilibrium if we have*

1. $\dfrac{g(\tilde{\theta})}{g(x(\tilde{\theta}))}+\dfrac{g(\tilde{\theta})}{g(y(\tilde{\theta}))}\leq 4$, *and*

2. $g'(y(\tilde{\theta}))\dfrac{g(x(\tilde{\theta}))}{g(y(\tilde{\theta}))}\leq g'(x(\tilde{\theta}))\dfrac{g(y(\tilde{\theta}))}{g(x(\tilde{\theta}))}$ *for all* $\tilde{\theta}\in[0,1]$.

Proof of Lemma 5. If both the first and second terms (A and B, respectively) in (8) are non-increasing in $\tilde{\theta}$, then we have $\varphi'(\tilde{\theta})\leq 0$ for all $\tilde{\theta}$. First we analyze the first term A. Since $x(\tilde{\theta})$ and $y(\tilde{\theta})$ are the solutions of $2G(x(\tilde{\theta}))=G(\tilde{\theta})$, and $1-2G(y(\tilde{\theta}))=1-G(\tilde{\theta})$, respectively, we obtain

$$\frac{dA}{d\tilde{\theta}}=\frac{g(\tilde{\theta})}{2g(x(\tilde{\theta}))}+\frac{g(\tilde{\theta})}{2g(y(\tilde{\theta}))}-2=\frac{1}{2}\left[\frac{g(\tilde{\theta})}{g(x(\tilde{\theta}))}+\frac{g(\tilde{\theta})}{g(y(\tilde{\theta}))}-4\right].$$

Second, let's analyze the behavior of the second term B. Differentiating B with respect to $\tilde{\theta}$ we obtain

$$\begin{aligned}\frac{dB}{d\tilde{\theta}}&=\frac{\left(g'(y)\frac{dy}{d\tilde{\theta}}-g'(x)\frac{dx}{d\tilde{\theta}}\right)(g(x)+g(y))-(g(y)-g(x))\left(g'(x)\frac{dx}{d\tilde{\theta}}+g'(y)\frac{dy}{d\tilde{\theta}}\right)}{(g(x)+g(y))^2}\\&=\frac{\left(g'(y)\frac{g(\tilde{\theta})}{g(y)}-g'(x)\frac{g(\tilde{\theta})}{g(x)}\right)(g(x)+g(y))-(g(y)-g(x))\left(g'(x)\frac{g(\tilde{\theta})}{g(x)}+g'(y)\frac{g(\tilde{\theta})}{g(y)}\right)}{(g(x)+g(y))^2}\\&=\frac{2g'(y)\frac{g(\tilde{\theta})}{g(y)}g(x)-2g'(x)\frac{g(\tilde{\theta})}{g(x)}g(y)}{(g(x)+g(y))^2}=\frac{2g(\tilde{\theta})}{(g(x)+g(y))^2}\left[g'(y)\frac{g(x)}{g(y)}-g'(x)\frac{g(y)}{g(x)}\right].\end{aligned}$$

$\square$

Condition 1 of Lemma 5 corresponds to $dA/d\tilde{\theta} \leq 0$, and condition 2 corresponds to $dB/d\tilde{\theta} \leq 0$. It is easy to see that the conditions are satisfied as long as g does not fluctuate much (a flat density: for example, if $\max_\theta g(\theta) \leq 2 \times \min_\theta g(\theta)$, then condition 1 is surely satisfied). This condition on $g'(x)$ and $g'(y)$ is satisfied if (a) g is concave or (b) $g'(\theta)$ does not change much within interval $[0, \theta^p)$ and interval $(\theta^p, 1]$. Since (a) implies single-peakedness and $\max_\theta g(\theta) = g(\theta^p)$, we have completed the proof.$\square$

Appendix B: Examples 1 and 2

Example 1

We can explicitly calculate the φ function. Since φ is a step function and is discontinuous at θ_{med}, we have two cases to calculate: (I) the case of $\tilde{\theta} \leq \theta_{med}$ and (II) the case of $\tilde{\theta} > \theta_{med}$. Noting that each candidate satisfies $x \leq \theta_{med} \leq y$ under any sorting political equilibria, the two cases are given below.

(I) The case of $\tilde{\theta} \leq \theta_{med}$. Two candidates are

$$x(\tilde{\theta}) = \frac{\tilde{\theta}}{2} \quad \text{and} \quad y(\tilde{\theta}) = \theta_{med} + \frac{1 - \theta_{med}}{2\theta_{med}}\tilde{\theta}.$$

In this case, calculating $\varphi(\tilde{\theta})$, $\varphi \gtreqless 0$ holds if and only if

$$2a \cdot \varphi(\tilde{\theta}) = 2\left(\theta_{med} + \frac{1 - 4\theta_{med}}{2\theta_{med}}\tilde{\theta}\right) + 2a\left(2\theta_{med} - 1\right) \gtreqless 0.$$

For $\varphi(\tilde{\theta}^*) = 0$ to hold, we have

$$\tilde{\theta}^* = \frac{2\theta_{med}((2a+1)\theta_{med} - a)}{4\theta_{med} - 1}.$$

To satisfy $\tilde{\theta}^* \leq \theta_{med}$, we must have $\theta_{med} > \frac{1}{4}$ since

$$\theta_{med} - \tilde{\theta}^* = \frac{(2a-1)(1-2\theta_{med})\theta_{med}}{4\theta_{med} - 1}$$

and $\theta_{med} \leq \frac{1}{2}$ and $a > \frac{1}{2}$. Indeed, if $\theta_{med} > \frac{1}{4}$ then the sufficient condition of the sorting political equilibrium at $\tilde{\theta}^*$ is satisfied (Corollary 1), since we have $2a \cdot \varphi'(\tilde{\theta}) = \frac{1-4\theta_{med}}{\theta_{med}} < 0$. We also need $a \leq \frac{\theta_{med}}{1-2\theta_{med}}$, since $\varphi(0) > 0$ must hold in order to have $\varphi(\tilde{\theta}^*) = 0$.

(II) The case of $\tilde{\theta} > \theta_{med}$. As well as (I), two candidates are

$$x(\tilde{\theta}) = \frac{\theta_{med}(1-2\theta_{med})}{2(1-\theta_{med})} + \frac{\theta_{med}}{2(1-\theta_{med})}\tilde{\theta} \quad \text{and} \quad y(\tilde{\theta}) = \frac{1+\tilde{\theta}}{2}.$$

In this case, calculating $\varphi(\tilde{\theta})$, $\varphi(\tilde{\theta}) \gtreqless 0$ holds if and only if

$$2a \cdot \varphi(\tilde{\theta}) = 2\left(\frac{1}{2} + \frac{\theta_{med}(1-2\theta_{med})}{2(1-\theta_{med})} + \frac{4\theta_{med}-3}{2(1-\theta_{med})}\tilde{\theta}\right) + 2a(2\theta_{med} - 1) \gtreqless 0.$$

Noting that $\varphi(\tilde{\theta})$ function is not discontinuous at θ_{med} but just kinks because of $x \leq \theta_{med} \leq y$.[16] See Figure 3.

For $\varphi(\tilde{\theta}^{**}) = 0$ as well as the case (I), we have

$$\tilde{\theta}^{**} = \frac{-(2+4a)\theta_{med}^2 + 6a\theta_{med} - 2a + 1}{3 - 4\theta_{med}}.$$

Since

$$\tilde{\theta}^{**} - \theta_{med} = \frac{(2a-1)\left[-(1-\theta_{med})^2 - \theta_{med}\right]}{3-4\theta_{med}} < 0,$$

[16] On the other hand, on the function $g(\theta)$ in Example 2, x and y stride over some steps. Thus $\varphi(\tilde{\theta})$ is discontinuous at several points.

there is no party threshold $\tilde{\theta}^{**}$ satisfying $\varphi'(\tilde{\theta}^{**}) = 0$ in this range, which implies that there is no political equilibrium in this range.

In conclusion, there is a unique equilibrium, and the equilibrium party threshold satisfies $\tilde{\theta}^{*} < \theta_{med}$. This implies that the party with the shorter tail (or higher density: here party L) loses some of its moderate supporters in any political equilibrium. $\square$

Example 2

Since everything is symmetric, we can focus on the cases of $\tilde{\theta} \in [0, \frac{1}{2}]$. We will investigate what will happen on $x(\tilde{\theta})$ and $y(\tilde{\theta})$ (thus including $\varphi(\tilde{\theta})$), as $\tilde{\theta}$ increases from 0 to $\frac{1}{2}$. The following tables summarize the relevant information. We have three cases:

Table 1. x, y and $2a\cdot\varphi$ where $g(\theta)$ is the step function.

Case 1. $\frac{3}{5} < b \le 1$ ($x(\tilde{\theta})$ enters into interval $(\frac{1}{9}, \frac{4}{9})$ before $\tilde{\theta}$ enters into interval $(\frac{4}{9}, \frac{1}{2})$, implying $y(\frac{1}{2}) \in (\frac{5}{9}, \frac{8}{9})$)

	$x(\tilde{\theta})$	$y(\tilde{\theta})$	$2a\cdot\varphi(\tilde{\theta})$
$0 \le \tilde{\theta} \le \frac{1}{9}$	$\frac{\tilde{\theta}}{2}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde{\theta}\right)$
$\frac{1}{9} < \tilde{\theta} < \frac{1}{3b} - \frac{1}{9}$	$\frac{1-b}{6(3-2b)} + \frac{b}{2(3-2b)}\tilde{\theta}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{3-b}{b(3-2b)}\left(\frac{10-7b}{6(3-2b)} - \frac{9-7b}{2(3-2b)}\tilde{\theta}\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{1}{3b} - \frac{1}{9} \le \tilde{\theta} < \frac{4}{9}$	$\frac{1}{6} - \frac{1}{6b} + \frac{\tilde{\theta}}{2}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{2}{b}\left(\frac{2}{3} - \frac{1}{6b} - \tilde{\theta}\right)$
$\frac{4}{9} \le \tilde{\theta} \le \frac{1}{2}$	$\frac{5}{6} - \frac{5}{6b} + \frac{3-2b}{2b}\tilde{\theta}$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde{\theta}$	$\frac{2}{b}\left(2 - \frac{3}{2b} + \frac{3-4b}{b}\tilde{\theta}\right)$

Case 2. $\frac{3}{8} < b \le \frac{3}{5}$ ($x(\tilde{\theta})$ enters into interval $(\frac{1}{9}, \frac{4}{9})$ after $\tilde{\theta}$ enters into interval $(\frac{4}{9}, \frac{1}{2})$, implying $y(\frac{1}{2}) \in (\frac{5}{9}, \frac{8}{9})$)

	$x(\tilde{\theta})$	$y(\tilde{\theta})$	$2a\cdot\varphi(\tilde{\theta})$
$0 \le \tilde{\theta} \le \frac{1}{9}$	$\frac{\tilde{\theta}}{2}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde{\theta}\right)$
$\frac{1}{9} < \tilde{\theta} < \frac{4}{9}$	$\frac{1-b}{6(3-2b)} + \frac{b}{2(3-2b)}\tilde{\theta}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{3-b}{b(3-2b)}\left(\frac{10-7b}{6(3-2b)} - \frac{9-7b}{2(3-2b)}\tilde{\theta}\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{4}{9} \le \tilde{\theta} \le \frac{5}{3(3-2b)} - \frac{13b}{9(3-2b)}$	$\frac{-1+b}{2(3-2b)} + \frac{\tilde{\theta}}{2}$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde{\theta}$	$\frac{3-b}{b(3-2b)}\left(\frac{18-11b}{6(3-2b)} - \frac{2}{3b} + \frac{3-5b}{2b}\tilde{\theta}\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{5}{3(3-2b)} - \frac{13b}{9(3-2b)} < \tilde{\theta} \le \frac{1}{2}$	$\frac{5}{6} - \frac{5}{6b} + \frac{3-2b}{2b}\tilde{\theta}$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde{\theta}$	$\frac{2}{b}\left(2 - \frac{3}{2b} + \frac{3-4b}{b}\tilde{\theta}\right)$

Case 3. $b \le \frac{3}{8}$ ($x(\tilde{\theta})$ does not enter into interval $(\frac{1}{9}, \frac{4}{9})$ implying $y(\frac{1}{2}) \in [\frac{8}{9}, 1]$ by symmetry)

	$x(\tilde{\theta})$	$y(\tilde{\theta})$	$2a\cdot\varphi(\tilde{\theta})$
$0 \le \tilde{\theta} \le \frac{1}{9}$	$\frac{\tilde{\theta}}{2}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde{\theta}\right)$
$\frac{1}{9} < \tilde{\theta} < \frac{4}{9}$	$\frac{1-b}{6(3-2b)} + \frac{b}{2(3-2b)}\tilde{\theta}$	$\frac{1}{2} + \frac{\tilde{\theta}}{2}$	$\frac{3-b}{b(3-2b)}\left(\frac{10-7b}{6(3-2b)} - \frac{9-7b}{2(3-2b)}\tilde{\theta}\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{4}{9} \le \tilde{\theta} < \frac{4}{3(3-2b)} - \frac{5b}{9(3-2b)}$	$\frac{-1+b}{2(3-2b)} + \frac{\tilde{\theta}}{2}$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde{\theta}$	$\frac{3-b}{b(3-2b)}\left(\frac{18-11b}{6(3-2b)} - \frac{2}{3b} + \frac{3-5b}{2b}\tilde{\theta}\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{4}{3(3-2b)} - \frac{5b}{9(3-2b)} \le \tilde{\theta} \le \frac{1}{2}$	$\frac{-1+b}{2(3-2b)} + \frac{\tilde{\theta}}{2}$	$\frac{4-3b}{2(3-2b)} + \frac{\tilde{\theta}}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde{\theta}\right)$

As mentioned above, when $b = 1$, $g(\theta)$ is a uniform distribution. Then the equilibrium is unique and symmetric, $\tilde{\theta} = \frac{1}{2}$. Even if b becomes only a little smaller than 1, $\varphi(\tilde{\theta})$ becomes discontinuous at $\tilde{\theta} = \frac{1}{9}$ in Case 1 of Table 1 and at $\frac{8}{9}$ from the symmetry, and shifts below because of the discontinuity of $g(\theta)$ at $\frac{1}{9}$ and $\frac{8}{9}$; see Figure 4. As b gets smaller, this shift gets larger; then two asymmetric equilibria appear in $(\frac{1}{9}, \frac{4}{9})$ and $(\frac{5}{9}, \frac{8}{9})$ in addition to the symmetric equilibrium; see Figure 4. As b gets increasingly smaller, these asymmetric equilibria approach $\frac{1}{9}$ and $\frac{8}{9}$, respectively, and finally stick to them.[17] With the case of deeply divided voters, in each asymmetric equilibrium, one party will be formed by all extremists ($\{EL\}$ and $\{ER\}$, respectively) and few moderates, while the other party will be formed by the rest ($\{L, C, R, ER\}$ and $\{EL, L, C, R\}$, respectively), and their candidates are extremist and moderately biased centrist. As a result, in one equilibrium $x(\tilde{\theta})$ and $y(\tilde{\theta})$ are around $\frac{1}{18}$ and $\frac{5}{9}$, and in the other they are around $\frac{4}{9}$ and $\frac{17}{18}$), respectively.[18]

Appendix C: Necessity for "Admissible" Allocations

In the main text, we confined our attention to admissible allocations in order to focus on centrally located voters' party choice. This restriction excludes the possibility of extreme

[17] In $b < \frac{1}{2}$ where b is in Case 2, the symmetric equilibrium disappears although $\varphi(\frac{1}{2}) = 0$ since the condition of Proposition 3 cannot be met (in this example, the condition in Proposition 3 becomes $g(\theta_{med}) \leq 4g(x(\theta_{med}))$ for any a), so that there are only two asymmetric equilibria; see Figure 5.

[18] When b gets even smaller and approaches Case 1, $x(\frac{1}{2})$ is in $(0, \frac{1}{9})$ at $\tilde{\theta} = \frac{1}{2}$. Then, although $b < \frac{1}{2}$, the condition of Proposition 3 is met again because of $g(x(\theta_{med})) = g(\theta_{med})$, so that the symmetric equilibrium appears again. Since the asymmetric equilibria still exist, there are again three equilibria; see Figure 5.

voters' switching their parties. In this appendix, we show that unless we can exclude voters at intervals $[0, x)$ and $(y, 1]$ from joining coalitional deviations, there may not be a political equilibrium. We provide a simple example to illustrate this point. As we have seen, coalitional deviations from intervals $[0, x)$ and $(y, 1]$ are sufficient to upset the immunity to the coalitions; by combining voters in (x, y) and $(y, 1]$, we can create an even simpler and robust example.

Example 3. Assume that g is uniform $g(\theta) = 1$ for all $\theta \in [0, 1]$, and that f is very widely spread (for example, $f(\epsilon) = \frac{1}{2a}$ for all $\epsilon \in [-a, a]$ with a large number a. In this case, whoever the two candidates x and y are, their chances of winning are always almost $\frac{1}{2}$ and $\frac{1}{2}$, respectively. Now, since everything is symmetric, a natural candidate for an equilibrium is a symmetric allocation $g_L(\theta) = g(\theta)$ for all $\theta < \frac{1}{2}$ and $g_R(\theta) = g(\theta)$ for all $\theta > \frac{1}{2}$. In this case, $x = \frac{1}{4}$ and $y = \frac{3}{4}$. Can this be immune to a coalitional deviation far from the party threshold? We denote a coalitional deviation as γ. Consider a deviation from party R to L: $\gamma(\theta) = g(\theta)$ for all $\theta \in (\frac{3}{4} - \delta, \frac{3}{4} - \frac{1}{2}\delta) \cup (\frac{3}{4} + \frac{1}{2}\delta, \frac{3}{4} + \delta]$ where $\delta > 0$ is a small positive number. That is, after the deviation, there is no impact on party R's candidate: $y' = \frac{3}{4}$. However, clearly x' is closer to θ_{med} after the deviation. Given a widespread f, the chances of x' and y' to win are still almost $\frac{1}{2}$ and $\frac{1}{2}$. Then, deviators in γ have a closer candidate from L who wins with probability $\frac{1}{2}$, so they are all better off. $\square$

Although this example may appear extreme, the force of the coalitions is robust in our model. However, in fact, voters with an extreme political position tend to have a strong, sometimes even fanatical, belief in their position. It seems unnatural for voters who are even farther right than the median of party R to move to party L, while moderate voters around the party threshold do not move.

Note that this difficulty is not due to our linear demand assumption. Consider a strictly convex utility (Osborne 1995) case: $u(p_k; \theta, \epsilon) = -v(|p_k - \theta|) + \epsilon$, where $v'(\cdot) < 0$ and $v''(\cdot) > 0$ and $k \in C$ is a winner. The convex utility function means that voters who are farther away from candidates do not take much interest in them. With such a utility function, one may think that extreme left or right voters — voters far to the left (right) of the median of party L (R) — have no incentive to switch parties, and we may be able to drop the assumption of an admissible allocation. It is perhaps true that such a convex cost function reduces the incentive to switch parties, but it would not totally resolve the problem, since a voter with an extreme position may be made better off by her party's candidate becoming more moderate and gaining a higher chance of winning even if the voter does not care about the other party's candidate's position. It all depends on the relative magnitudes of two effects: dissatisfaction with her party's candidate's position becoming more moderate and satisfaction with the candidate's increase in winning probability.

Appendix D: Psychological Costs for Voters in $[0, x)$ and $(y, 1]$

Here, by introducing arbitrarily small psychological costs to extreme voters, we can show that every political equilibrium is an admissible allocation. As formally explained in Appendix C (Example 3), if we allow voters from intervals $[0, x(g_L))$ and $(y(g_R), 1]$ to join coalitional deviations, we will face a nonexistence problem of political equilibrium. The logic of the example is roughly as follows: Imagine that, among party R supporters, a small group of supporters to the right of y together with a group to the left of y with the same measure switch their party to L. Then, the party median of party R is kept at y, while x moves right. If there is a large amount of uncertainty in the election result (ϵ has a high variance), then party R supporters may appreciate having closer party L as candidate.[19] This is unfortunate, but if we introduce an arbitrarily small psychological cost of joining the party whose position is not aligned with the voters with relatively extreme positions, then we can rule out the possibilities of these voters joining more distanced parties.

Definition 2 *Let $\delta > 0$ be a small number.* ***The psychological cost*** *of joining party i, $\Phi(\theta, i; x, y)$, is a function in $\theta, x,$ and y defined as follows: $\Phi(\theta, i; x, y) = \Phi > 0$ if $i = R$ and $\theta \leq x(G) + \delta$, or $i = L$ and $y(G) - \delta \leq \theta$, and $\Phi(\theta, i; x, y) = 0$, otherwise.*

This assumption says that if a voter's political position is more extreme than the median of a party that is closer to her position, then it is psychologically costly for her to join the other party: she feels some stress as if she is sinning against her convictions. Note that Φ can be set arbitrarily small. As long as $\Phi > 0$, a voter in interval $[0, x(G)]$ would not belong to party R: if she moves back to party R alone then she can save psychological cost Φ without affecting the candidates' positions. Thus, in every political equilibrium, voters at intervals $[0, x(G) + \delta]$ and $[y(G) - \delta, 1]$ belong to parties L and R, respectively.

[19] If the variance of ϵ is small, party R supporters do not necessarily appreciate closer x since this implies that candidate y has a lower chance of winning the election.

Noticing that a coalition size is bounded above and the changes in candidates' positions are bounded above, we can see that the coalition members' utility gains by switching party are also bounded above. This implies that in the presence of psychological costs, voters in these intervals would never deviate. These results are summarized as follows.

Proposition 6 *In the presence of psychological costs, (i) in any political equilibrium, voters in interval* $[0, x(G)]$ *always belong to party L, while voters in interval* $[y(G), 1]$ *always belong to party R for any value of* $\Phi > 0$*. Moreover, (ii) for any value of* $\Phi > 0$*, there is a coalition size limit* $\bar{\Delta}(\Phi) > 0$ *such that there is no profitable coalitional deviation formed by voters in intervals* $[0, x(G)]$ *and* $[y(G), 1]$*, of which size is less than* $\bar{\Delta}(\Phi)$*.*

Here, we provide the proof for Proposition 6. The statement in the first half is provided in the main text, so we concentrate on the second half. First notice that the sorting result (Proposition 1) can be stated as follows: every political equilibrium must be a sorting allocation at interval $(x(G), y(G))$ in order to be immune to coalitional deviations formed by voters at interval $(x(G), y(G))$. The first half of Proposition 1 says that if there is a psychological cost $\Phi > 0$ then, in every political equilibrium, voters at intervals $[0, x(G)+\delta]$ and $[y(G) - \delta, 1]$ belong to parties L and R, respectively. Thus, the sorting allocation is the only candidate for political equilibria in the presence of psychological cost $\Phi > 0$. This implies that there is a party threshold $\tilde{\theta}^*$ such that $x(G) + \delta \le \tilde{\theta}^* \le y(G) - \delta$, since δ is arbitrarily small. By this result, we can ensure that if a small enough coalition deviates from interval $[0, x(G)]$, then the new median $x(G_\Delta)$ still satisfies $x(G) < x(G_\Delta) < x(G)+\delta$. Thus, we can approximate each coalition's deviation incentive.

Let us start with a coalitional deviation γ' with a (small) size $\Delta > 0$ that belongs to the interval $[0, x(G)]$, moving from party L to party R. Using the definitions of the party medians (1), the new median voter type $x'(G_\Delta)$ of party L is determined by

$$\int_0^{x'(G_\Delta)} g(\theta)d\theta - \Delta = \int_{x'(G_\Delta)}^1 g(\theta)d\theta$$

and $y'(G_\Delta)$ of party R is by

$$\int_0^{y'(G_\Delta)} g(\theta)d\theta + \Delta = \int_{y'(G_\Delta)}^1 g(\theta)d\theta$$

Here, we used the fact that the original allocation is a sorting allocation. Since we are considering a small coalitional deviation, we will take $\Delta \to 0$. By totally differentiating them, we have $g(x')dx' - d\Delta = -g(x')dx'$, or

$$\frac{dx'}{d\Delta} = \frac{1}{2g(x)}, \tag{10}$$

and similarly, we have

$$\frac{dy'}{d\Delta} = -\frac{1}{2g(y)}.$$

These derivatives represent that, by the small coalitional deviation γ', x moves to the right while y moves to the left. Thus, type θ's expected payoff is affected by such a deviation through changes in x and y. Using (3), we can write the impact of the coalitional deviation γ' from the interval $[0, x(G)]$ on voters $\theta \in [0, x(G)]$ as

$$\begin{aligned}
\frac{dEu(x(G), y(G); \theta)}{d\Delta} &= \frac{1}{4a}\left[-\frac{1}{g(y)}(y - x - a) + \frac{1}{g(x)}(4\theta_{med} - 3x - y - a)\right] \\
&= \frac{1}{4a}\left(\frac{1}{g(x)} - \frac{1}{g(y)}\right)(y - x - a) + \frac{1}{2ag(x)}(2\theta_{med} - x - y) \\
&< \frac{1}{4a\underline{g}}(y - x - a) + \frac{1}{2a\underline{g}}(2\theta_{med} - x - y) < \frac{3 - a}{4a\underline{g}}
\end{aligned}$$

That is, $\Delta Eu(x(G), y(G); \theta) < \frac{3-a}{4a\underline{g}}\Delta \leq \frac{3-a}{4a\underline{g}}\bar{\Delta}$ holds. If $\Phi > \frac{3-a}{4a\underline{g}}\bar{\Delta}$ holds, then there is no incentive for such a coalition to deviate. Let

$$\bar{\Delta}(\Phi) = \frac{3-a}{4a\underline{g}}\Phi > 0$$

By setting this to be the coalition size limit, we can ensure that there is no coalition of size smaller than $\bar{\Delta}(\Phi)$ that can deviate from interval $[0, x(G)]$. In exactly the same way, we can show that there is no coalition of size smaller than $\bar{\Delta}(\Phi)$ that can deviate from interval $[y(G), 1]$.$\square$

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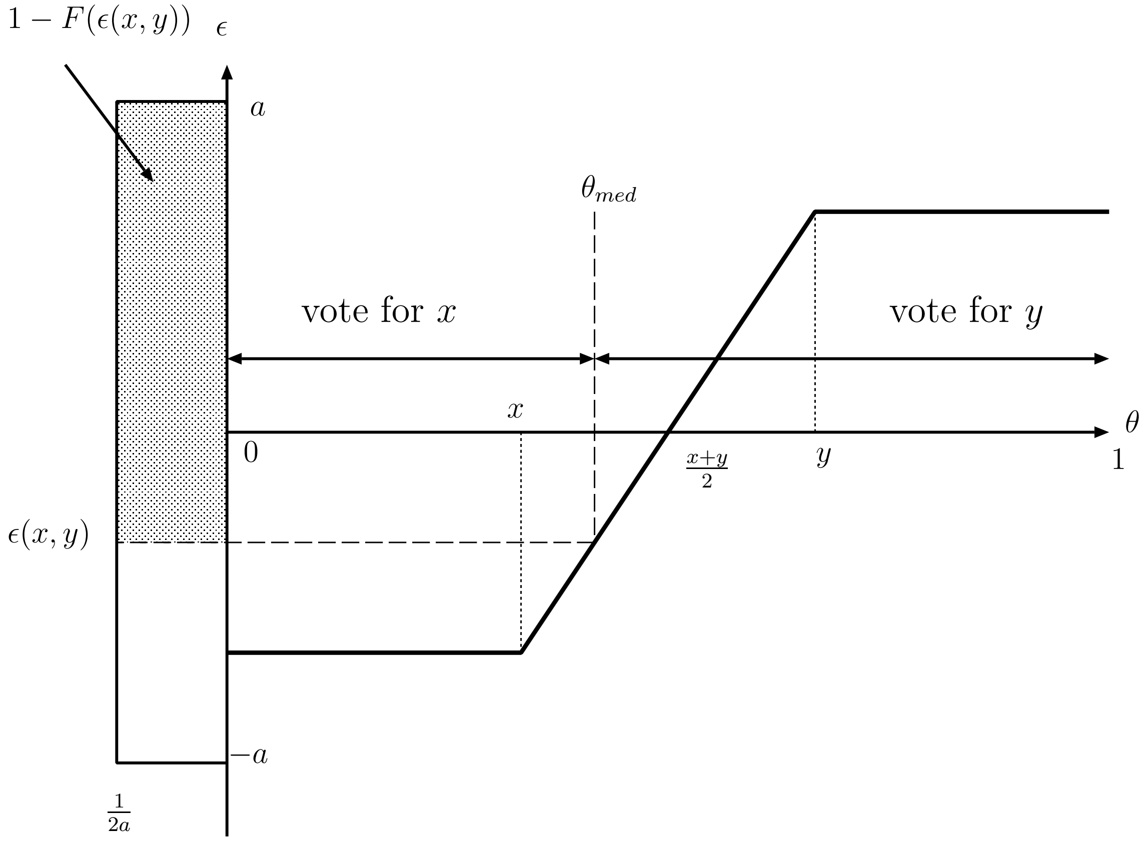


Figure 1: Voting at $\epsilon = \epsilon(x, y)$

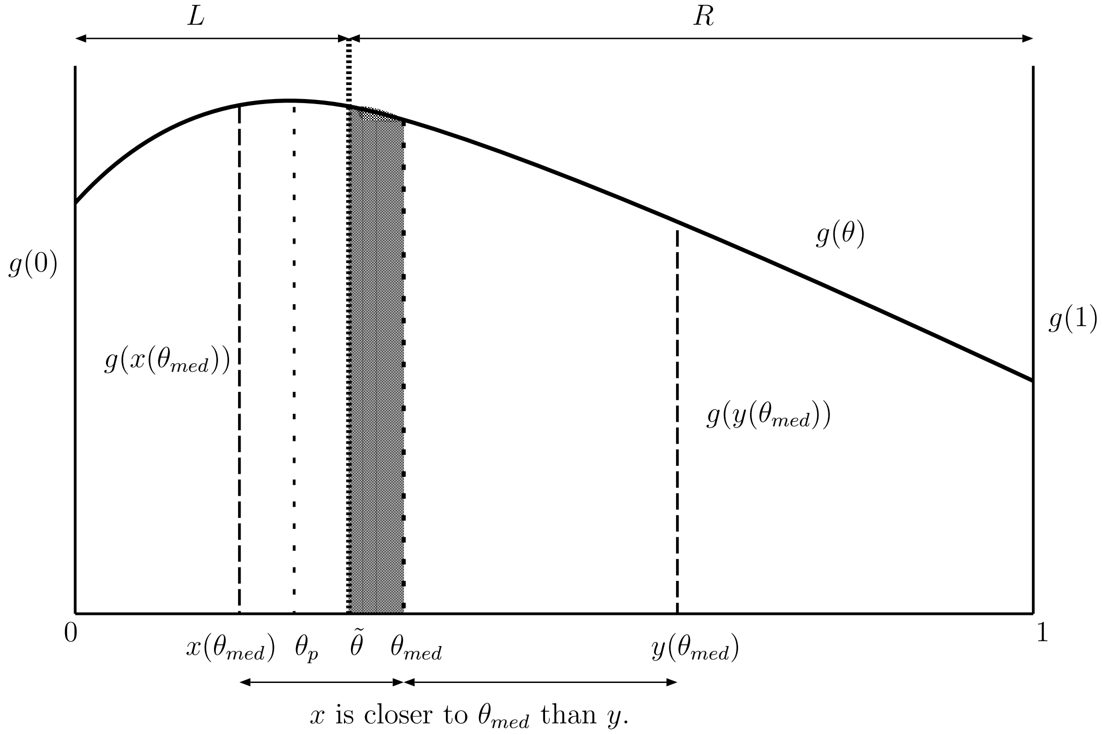


Figure 2: Single-peaked voter distribution with a biased peak to the left. Voters in the shaded area participate in party R when uncertainty in likability is high.

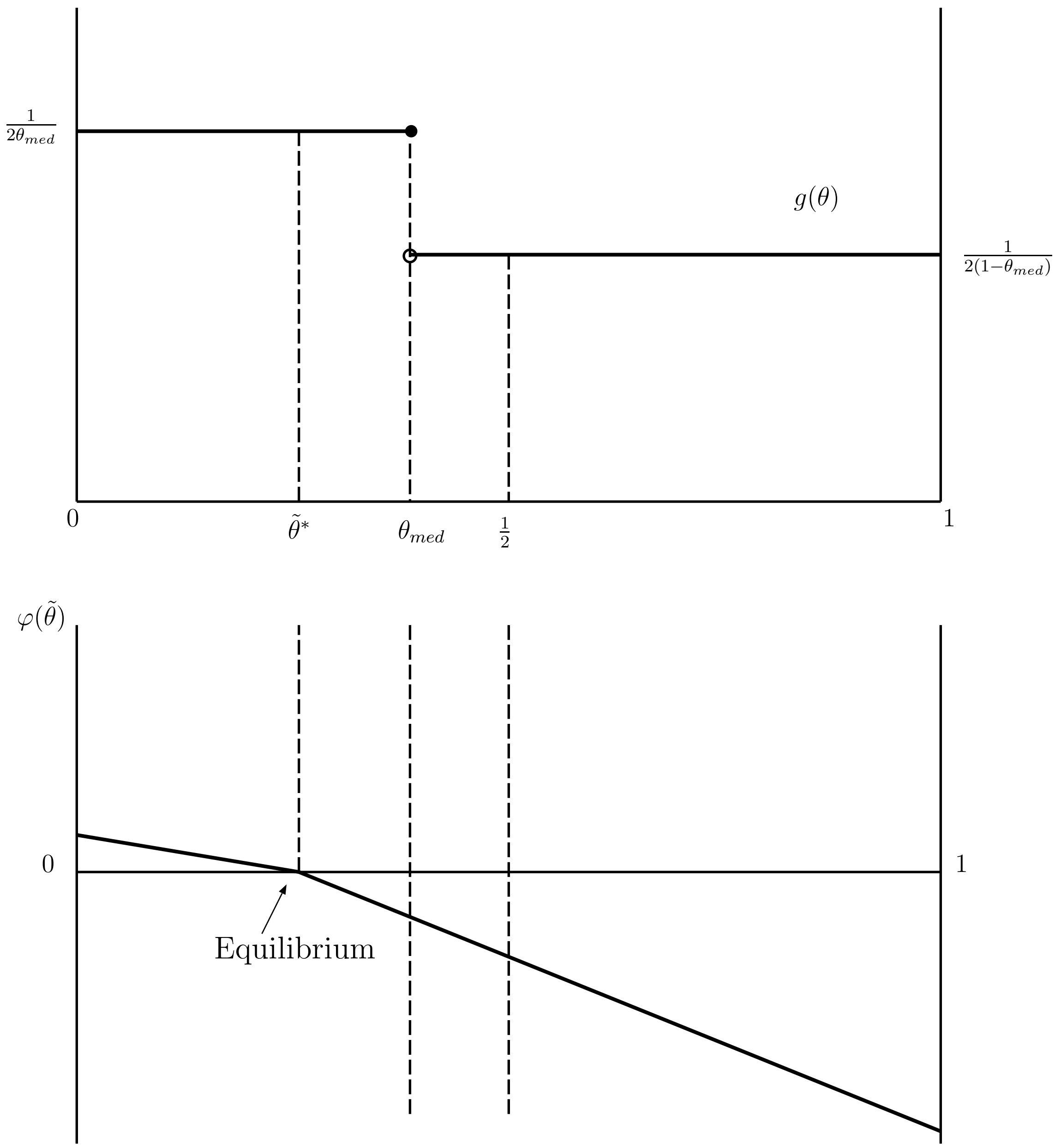


Figure 3: Example 1

The left voters are distributed more densely in a narrow interval than the right. In this case, there is a unique equilibrium at a point where $\varphi(\theta)$ is kinked.

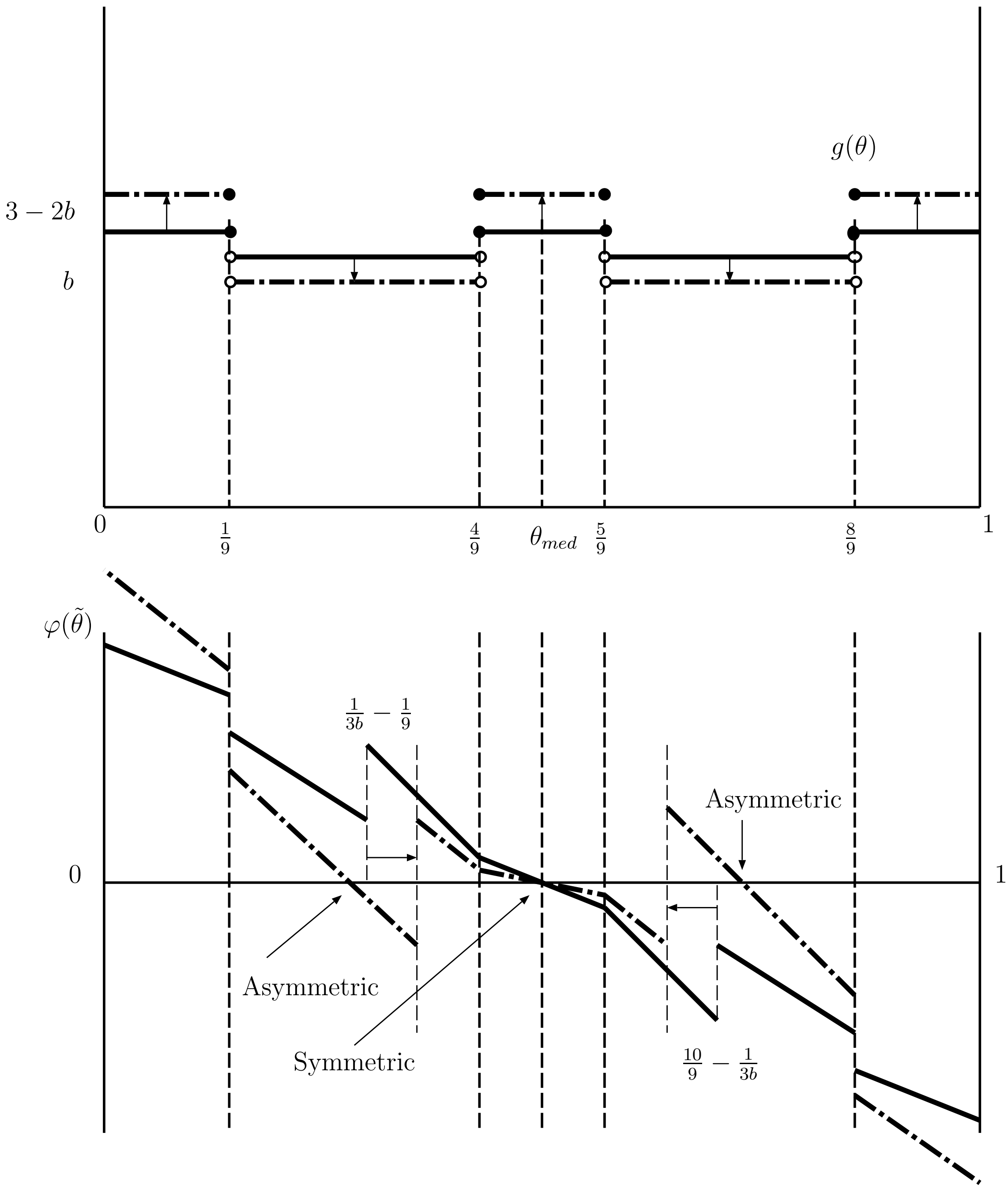
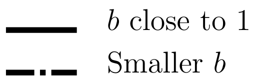


Figure 4: Case 1 of Example 2; $\frac{3}{5} < b \leq 1$



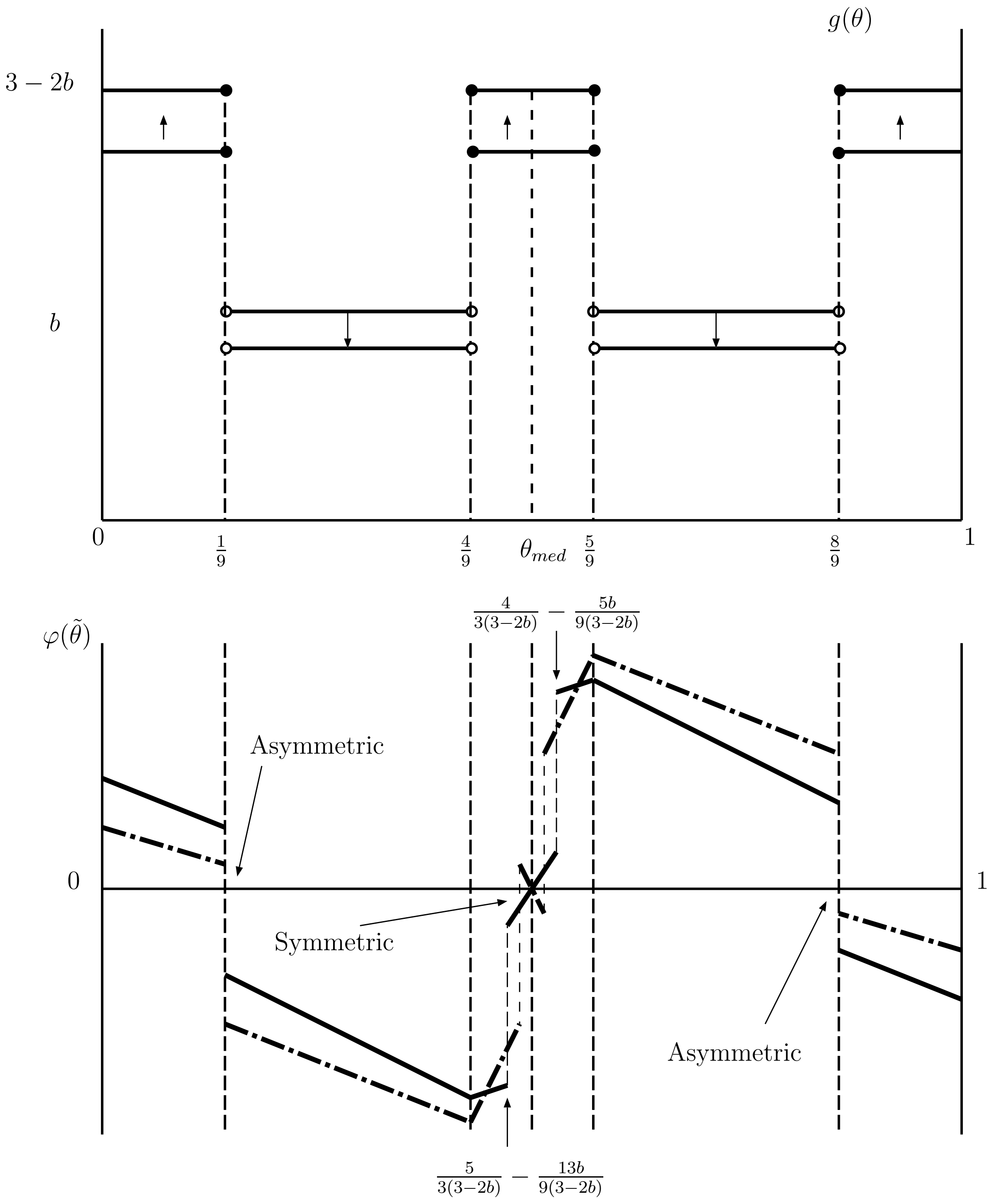


Figure 5: Case 2 and 3 of Example 2; Even Smaller b

Case 2; No symmetric equilibrium (Upward Sloping)

Case 3; With Symmetric equilibrium (Downward Sloping)