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Risk aversion in the small and in the large: Calibration results for betweenness functionals

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Abstract A reasonable level of risk aversion with respect to small gambles leads to a high, and absurd, level of risk aversion with respect to large gambles. This was demonstrated by Rabin (Econometrica 68:1281–1292, 2000) for expected utility theory. Later, Safra and Segal (Econometrica 76:1143–1166, 2008) extended this result by showing that similar arguments apply to many non-expected utility theories, provided they are Gâteaux differentiable. In this paper we drop the differentiability assumption and by restricting attention to betweenness theories we show that much weaker conditions are sufficient for the derivation of similar calibration results.

Keywords Risk aversion · Calibration results · Betweenness functionals

The hypothesis that in risky environments decision makers evaluate actions by considering possible final wealth levels is widely used in economics. But this hypothesis leads to the following conclusion: A reasonable level of risk aversion with respect to small gambles leads to a high, even absurd, level of risk aversion with respect to large gambles. For example, if a risk averse

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decision maker is rejecting the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all positive wealth levels below 300,000, then at wealth level 290,000 he will also reject the lottery $(-10{,}000, \frac{1}{2}; 5{,}503{,}790, \frac{1}{2})$. This was demonstrated by Rabin (2000) for the case of expected utility theory (see also Epstein 1992), but later, Safra and Segal (2008) extended this result by showing that similar arguments apply to many non-expected utility theories. To understand these arguments, consider the following two properties (D1 and D2) and the two subsequent assumptions (B3 and B4):

D1 Actions are evaluated by considering possible *final* wealth levels.
D2 Risk aversion: If lottery Y is a mean preserving spread of lottery X, then X is preferred to Y.

B3 Rejection of a small actuarially favorable lottery in a certain range. For example, rejection of the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all wealth levels below 300,000.
B4 Acceptance of large, very favorable lotteries. For example, acceptance of the lottery $(-10{,}000, \frac{1}{2}; 5{,}000{,}000, \frac{1}{2})$.

Rabin (2000) showed that within expected utility theory, properties D1 and D2 imply a tension between B3 and B4. If for given small and relatively close $\ell < g$, the decision maker rejects $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels $x \in [a, b]$, then he also rejects $(-L, \frac{1}{2}; G, \frac{1}{2})$ at x^* for some L, G, and $x^* \in [a, b]$ where G is huge while L is not (see above example).

These results seem to question the very foundations of expected utility theory. The fact that decision makers are risk averse with respect to small lotteries is well established and recorded in many experiments (see references below). But it is also evident that decision makers are not shy of taking large and very favorable risks. In fact, stock markets are a standing proof for people's willingness to take such risks.

In Safra and Segal (2008) we proved that this surprising result is not restricted to expected utility theory, but applies to many general non-expected utility theories as well. Assuming some degree of smoothness of preferences, we showed that D1 and D2 imply a tension between B4 and a modified version of B3.[1]

Violations of B4 seem foolish and unrealistic. It is of course possible to find vNM utility functions that will reject the lottery $(-10{,}000, \frac{1}{2}; 5{,}000{,}000, \frac{1}{2})$—for example, $u(x) = (x + 10000)^\alpha$ for $0 < \alpha < 0.11$. But rejecting such lotteries means that the decision maker will be willing to pay more than \$6000 to insure himself against a 10% chance of losing \$10000. It seems very unlikely that such people exist. It is possible that the problem is with the risk aversion assumptions (D2 and B3), and people do not reject all risks, or at least, small risks.[2] But many experiments show that decision makers actually do reject

[1]This version of B3 requires that for every lottery X, the decision maker prefers X to an even chance lottery of playing $X - \ell$ and $X + g$.
[2]See Foster and Hart (2007).

small, actuarially favorable risks (see, for example, Rabin and Thaler 2001). Moreover, risk aversion is a standard assumption in Economics and Finance. This leads to the conclusion that D1 must be violated.

Several recent models try to drop the final-wealth assumption—for example, gain-loss models such as prospect theory (Kahneman and Tversky 1979; Tversky and Kahneman 1992) and its offsprings. This is probably the right conclusion from the analysis of this and our previous paper: Behavior is governed by more than one preference relation, and decision makers use different preferences in different situations. It should however be realized that abandoning the final-wealth assumption carries a serious theoretical cost as such models typically violate either completeness or transitivity.

Our previous analysis applies only to Gâteaux differentiable models. There are many such models, including (the differentiable versions of) betweenness (Dekel 1986; Chew 1983), quadratic utility (Machina 1982; Chew et al. 1991), and rank-dependent utility (Quiggin 1982). But although Gâteaux differentiability is weaker than Fréchet differentiability (used by Machina 1982), it is still strong enough to rule out some axiomatic models like Chew's (1989) semi-weighted utility theory, Gul's (1991) disappointment aversion theory (see Appendix 2), and Smorodinsky's (2000) fair solution. Moreover, Gâteaux differentiability may be too restrictive from a behavioral point of view. It requires two properties.

1. The representation function V is differentiable with respect to α along the segment $(1-\alpha)F+\alpha F'$ for all distributions F and F'.
2. For a fixed F, the derivative with respect to α of $V((1-\alpha)F+\alpha F')$ at $\alpha=0$ is linear in F'. In particular, the derivatives in two opposite directions have the same absolute value and different signs.

Even when the functional V is either quasi-concave or quasi-convex, the first requirement is satisfied, for given F and F', only for *almost* all α. Moreover, the second requirement seems unnecessarily strong for the analysis of most economic applications, as in many cases it is sufficient to assume that only indifference sets are well behaved. And as has been demonstrated by Dekel (1986), preferences with well behaved indifference sets need not be Gâteaux differentiable. Moreover, differentiability assumptions are problematic as they can seldom be verified by empirical methods.

This paper shows conditions under which the tension between risk aversion in the small and in the large exists even without the differentiability assumptions. It thus makes the implications of Safra and Segal (2008) even stronger. Specifically, we show that when preferences satisfy betweenness, that is, if $F\sim G$ implies that for all $\alpha\in[0,1]$, $F\sim\alpha F+(1-\alpha)G\sim G$, then the Gâteaux differentiability requirement can be dropped. Moreover, the modified version of B3 can be significantly weakened. The extension of expected utility to betweenness preferences is of special interest since all indifference sets of betweenness functionals are still hyperplanes (though not necessarily parallel to each other). In Section 1 we provide definitions and state two facts that are taken from Safra and Segal (2008). In Section 2.1 we prove two theorems for betweenness preferences satisfying Machina's (1982) behavioral H1 and H2 assumptions (or their opposites), without assuming Gâteaux differentiability. Assumptions H1 and H2 require some smoothness of local utilities. In Section 2.2 we use a modified version of assumption B3 that is much weaker than that of Safra and Segal (2008) and prove a theorem for betweenness functionals that does not require any differentiability assumptions.

1 Preliminaries

We assume throughout that preferences over distributions are representable by a functional V which is risk averse with respect to mean-preserving spreads, monotonically increasing with respect to first order stochastic dominance, and continuous with respect to the topology of weak convergence. Denote the set of all such functionals by $\mathcal{V}$.

According to the context, utility functionals are defined over lotteries (of the form $X = (x_1, p_1; \ldots; x_n, p_n)$) or over cumulative distribution functions (denoted F, H). Degenerate cumulative distribution functions are denoted δ_x.

Betweenness functionals in $\mathcal{V}$ (Chew 1983, 1989; Fishburn 1983; Dekel 1986) are characterized by linear indifference sets. That is, if F and H are in the indifference set $\mathcal{I}$, then for all $\alpha \in (0, 1)$, so is $\alpha F + (1 - \alpha)H$. Formally:

Definition 1 V satisfies betweenness if for all F and H satisfying $V(F) = V(H)$ and for all $\alpha \in (0, 1)$, $V(F) = V(\alpha F + (1 - \alpha)H) = V(H)$.

The fact that indifference sets of the betweenness functional are hyperplanes implies that for all F, the indifference set through F can also be viewed as an indifference set of an expected utility functional with vNM utility $u(\cdot; F)$. Following Machina (1982), this function is called the local utility of V at F.

In Safra and Segal we strengthen Rabin's (2000) results and proved the following facts (see Proofs of Theorems 1 and 2 in Safra and Segal (2008)).

Fact 1 *Let $V \in \mathcal{V}$ be expected utility. Let $g > \ell > 0$ and $G > b - a$, and let*

$$L > \left[(\ell + g) \frac{1 - \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}}}{1 - \frac{\ell}{g}} + (G + a - b) \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \right] \frac{(1-p)}{p}. \quad (1)$$

If for all $x \in [a, b]$,

$$V(x, 1) \geqslant V\left(x - \ell, \frac{1}{2}; x + g, \frac{1}{2}\right)$$

then

$$V(a, 1) \geqslant V(a - L, p; a + G, 1 - p).$$

That is, a rejection of the small lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all $x \in [a, b]$ implies a rejection of the large lottery $(-L, p; G, 1-p)$ at a. For example, a rejection of $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all wealth levels between 100,000 and 140,000 implies a rejection at wealth level 100,000 of the lottery $(-5{,}035, \frac{1}{2}; 10{,}000{,}000, \frac{1}{2})$.

Fact 2 *Let $V \in \mathcal{V}$ be expected utility. Let $0 < \ell < g < L$ and let $b - a = L + g$. If for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then for all p and G satisfying*

$$G < \frac{p}{1-p}(\ell + g)\frac{\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} - 1}{\frac{g}{\ell} - 1}, \tag{2}$$

it follows that

$$V(b, 1) \geqslant V(b - L, p; b + G, 1 - p).$$

For example, a rejection of $(-100, \frac{1}{2}; 110, \frac{1}{2})$ at all wealth levels between 150,000 and 200,000 implies a rejection, at wealth level 200,000, of the lottery $(-50{,}000, 6.6 \cdot 10^{-6}; 100{,}000{,}000, 1 - 6.6 \cdot 10^{-6})$.

2 Results

2.1 Hypotheses 1 and 2

Machina (1982) introduced the following assumptions.

Definition 2 The functional $V \in \mathcal{V}$ satisfies Hypothesis 1 (H1) if for any distribution F, $-\frac{u''(x;F)}{u'(x;F)}$ is a nonincreasing function of x.

The functional $V \in \mathcal{V}$ satisfies Hypothesis 2 (H2) if for all F and H such that F dominates H by first order stochastic dominance and for all x

$$-\frac{u''(x; F)}{u'(x; F)} \geqslant -\frac{u''(x; H)}{u'(x; H)}$$

In the sequel, we will also use the opposite of these assumptions, denoted ¬H1 and ¬H2. ¬H1 says that for any distribution F, $-\frac{u''(x;F)}{u'(x;F)}$ is a nondecreasing function of x while ¬H2 says that if F dominates H by first order stochastic dominance, then for all x

$$-\frac{u''(x; F)}{u'(x; F)} \leqslant -\frac{u''(x; H)}{u'(x; H)}$$

Hypotheses 1 and 2 (and their opposites) assume that local utilities are differentiable, but they do not imply Gâteaux differentiability. We provide an example in Appendix 2. The results of this subsection imply that betweenness functionals are susceptible to Rabin-type criticism whenever they satisfy Hypotheses 1 or 2, or their opposites, ¬H1 or ¬H2.

Theorem 1 *Let $V \in \mathcal{V}$ be a betweenness functional satisfying ¬H1 or ¬H2. Let $g > \ell > 0$ and $G > b - a$, and let L satisfy inequality (1). If for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then*

$$V(a, 1) > V(a - L, p; a + G, 1 - p).$$

Theorem 2 *Let $V \in \mathcal{V}$ be a betweenness functional satisfying H1 or H2. Let $0 < \ell < g < L$, let $b - a = L + g$, and let p and G satisfy inequality (2). If for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then*

$$V(b, 1) \geqslant V(b - L, p; b + G, 1 - p).$$

In other words, calibration results like those of Facts 1 and 2 apply not only to expected utility theory, but also to betweenness functions satisfying Hypotheses 1, 2, or their opposites.

2.2 Bi-stochastic B3

Hypotheses H1 and H2 (and their opposites) require twice differentiable local utility functions.[3] It is not clear what are the *behavioral* implications of this differentiability assumption. Moreover, there are several known models, like the rank dependent and the disappointment aversion theories, whose local utilities are not differentiable. Our next aim is to obtain results that do not depend on this form of differentiability.

In this subsection we consider a stochastic version of B3 where the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at both deterministic and stochastic wealth levels. This modified version of assumption B3 is much weaker than the one used in Safra and Segal (2008), as the lottery X here can have two outcomes at most.

Definition 3 (Bi-Stochastic B3): The functional V satisfies (ℓ, g) bi-stochastic B3 on $[a, b]$ if for all $X = (x, p; y, 1 - p)$ with support in $[a, b]$,

$$V(X) > V\left(x - \ell, \frac{p}{2}; x + g, \frac{p}{2}; y - \ell, \frac{1 - p}{2}; y + g, \frac{1 - p}{2}\right).$$

The lottery X in Definition 3 serves as background risk—risk to which the binary lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is added. Guiso et al. (1996) find that when investors face greater background risk or uninsurable income risk, they reduce

[3]Note that differentiability of the local utilities and differentiability of the functional are two different issues. The rank dependent model (Quiggin 1982) is Gâteaux differentiable but its local utilities are not, hence it does not satisfy H1, H2, or their opposites. At the end of Appendix 2 we show that H1 and H2 do not imply even Gâteaux differentiability.

their holdings of risky assets. Hochguertel (2003) finds that "higher subjective uncertainty about incomes obtaining in five years from now is associated with safer and more liquid portfolios." Paiella and Guiso (2001) too provide some data showing that decision makers are more likely to reject a given lottery in the presence of background risk.[4] Accordingly, if rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is likely when added to non-stochastic wealth, it is even more likely when added to a lottery. In other words, (ℓ, g) bi-stochastic B3 is as acceptable as the behavioral assumption that decision makers reject the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ used by Rabin. Note that for expected utility functionals the stochastic version of B3 is equivalent to the deterministic one: A rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all deterministic wealth levels implies its rejection at all stochastic wealth levels.

The next theorem provides calibration results when bi-stochastic B3 replaces Hypotheses 1 and 2 and their opposites.

Theorem 3 *Let $V \in \mathcal{V}$ be a betweenness functional. Let $0 < \ell < g$, $G > c$, and let L satisfy Eq. 1 for $b - a = c$. Also, let $\bar{p}$ and $\bar{G}$ satisfy Eq. 2 for $b - a = \bar{L} + g$. If V satisfies (ℓ, g) bi-stochastic B3 on $[w - g - \bar{L}, w + c]$ then either*

1. $V(w, 1) > V(w - L, p; w + G, 1 - p)$*; or*
2. $V(w, 1) > V(w - g - \bar{L}, \bar{p}; w - g + \bar{G}, 1 - \bar{p})$.

In other words, a betweenness functional satisfying bi-stochastic B3 is bound to reject lotteries from at least one of the two Facts of Section 1.

The conditions of Theorem 3 are satisfied, for example, by Gul's (1991) disappointment aversion theory, which is a special case of betweenness. According to this theory the value of the lottery $p = (x_1, p_1; \ldots; x_n, p_n)$ is given by

$$V(p) = \frac{\alpha}{[1 + (1 - \alpha)\beta]} \sum_x q(x)u(x) + \left[1 - \frac{\alpha}{[1 + (1 - \alpha)\beta]}\right] \sum_x r(x)u(x)$$

for some $\beta \in (-1, \infty)$ and where $p = \alpha q + (1 - \alpha)r$ for a lottery q with positive probabilities on outcomes that are better than p and a lottery r with positive probabilities on outcomes that are worse than p. Risk aversion behavior is obtained whenever u is concave and $\beta \geqslant 0$ (1991, Th. 3). This functional is not Gâteaux differentiable (see Appendix 2).

This functional satisfies (ℓ, g) bi-stochastic B3 if for every x, $u(x) > \frac{1}{2}u(x - \ell) + \frac{1}{2}u(x + g)$. Starting from $X = (x_1, p; x_2, 1 - p) \sim \delta_w$, the local utility at X is more concave than u, and therefore it too rejects the (ℓ, g) randomness that is added to w. By betweenness, this randomness is also rejected by the functional V.

[4] Rabin and Thaler (2001) on the other hand seem to claim that a rejection of a small lottery is likely only when the decision maker is unaware of the fact that he is exposed to many other risks.

3 Concluding remarks

As mentioned above, a natural conclusion from our results is that the final-wealth assumption should be dropped. Hence, decision makers' behavior must be governed by more than one preference relation and they need to use different preferences in different situations. This is in line with Cox and Sadiraj (2006) and Rubinstein (2006), who reached a similar conclusion for expected utility preferences. Our conclusion is strictly stronger as it holds for a much larger set of preferences (betweenness preferences, if no smoothness condition is required, and all preferences, if Gâteaux differentiability is allowed). Moreover, it carries serious implications for economic theory and finance, where the assumption of a unique and universal preference relation over final-wealth distributions is commonly used.

Appendix 1

Proof of Theorems 1 and 2 Let $\mathcal{I}$ be the indifference set of V through δ_w and let $u(\cdot;\delta_w)$ be one of the increasing vNM utilities obtained from $\mathcal{I}$ (note that all these utilities are related by positive affine transformations). Obviously, $\mathcal{I}$ is also an indifference set of the expected utility functional defined by $U(F) = \int u(x;\delta_w)dF(x)$. By monotonicity, V and U increase in the same direction relative to the indifference set $\mathcal{I}$.

We show first that $u(\cdot;\delta_w)$ is concave. Suppose not. Then there exist z and H such that $z = \mathrm{E}[H]$ and $U(H) > u(z;\delta_w)$. By monotonicity and continuity, there exist p and z' such that $(z-w)(z'-w) < 0$ and $(z, p; z', 1-p) \in \mathcal{I}$. As $U(H) > u(z;\delta_w)$, it follows that $(H, p; z', 1-p)$ is above $\mathcal{I}$ with respect to the expected utility functional U. Therefore, $(H, p; z', 1-p)$ is also above $\mathcal{I}$ with respect to V, which is a violation of risk aversion, since $(H, p; z', 1-p)$ is a mean preserving spread of $(z, p; z', 1-p)$.

By betweenness, $V(x, 1) > V(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$ implies that, for all $\varepsilon > 0$, $V(x, 1) > V(x, 1-\varepsilon; x-\ell, \frac{\varepsilon}{2}; x+g, \frac{\varepsilon}{2})$. Using the local utility at δ_x we obtain

$$u(x;\delta_x) > \frac{1}{2}u(x-\ell;\delta_x) + \frac{1}{2}u(x+g;\delta_x) \tag{3}$$

Theorem 1 *Assume* $\neg$*H1 or* $\neg$*H2. We show that for every* $x \in [a, b]$, $u(x;\delta_a) > \frac{1}{2}u(x-\ell;\delta_a) + \frac{1}{2}u(x+g;\delta_a)$. *Equation 3 holds for* $x = a$, *and* $\neg$*H1 implies*

$$u(x;\delta_a) > \frac{1}{2}u(x-\ell;\delta_a) + \frac{1}{2}u(x+g;\delta_a) \tag{4}$$

Also, since $x \geqslant a$, $\neg$*H2 implies that the local utility* $u(\cdot;\delta_a)$ *is more concave than the local utility* $u(\cdot;\delta_x)$ *and hence (4).*

By Fact 1, it now follows that the expected utility decision maker with the vNM function $u(\cdot;\delta_a)$ satisfies $u(a;\delta_a) > pu(a-L;\delta_a)+(1-p)u(a+G;\delta_a)$ and the lottery $(a-L, p; a+G, 1-p)$ lies below the indifference set $\mathcal{I}$. Therefore,

$$V(a,1) > V(a-L,p;a+G,1-p)$$

Theorem 2 *Assume H1 or H2. Here we show that for every $x \in [a,b]$, $u(x;\delta_b) > \frac{1}{2}u(x-\ell;\delta_b)+\frac{1}{2}u(x+g;\delta_b)$. Equation 3 holds in particular for $x = b$, and therefore H1 implies that for $x \leqslant b$,*

$$u(x;\delta_b) > \frac{1}{2}u(x-\ell;\delta_b)+\frac{1}{2}u(x+g;\delta_b) \tag{5}$$

Also, since $x \leqslant b$, H2 implies that the local utility $u(\cdot;\delta_b)$ is more concave than the local utility $u(\cdot;\delta_x)$. Hence starting at (3) and moving from δ_x to δ_b, H2 implies (5).

By Fact 2 it now follows that the expected utility decision maker with the vNM function $u(\cdot;\delta_b)$ satisfies $u(b;\delta_b) > pu(b-L;\delta_b)+(1-p)u(b+G;\delta_b)$ and the lottery $(b-L, p; b+G, 1-p)$ lies below the indifference set $\mathcal{I}$. Therefore,

$$V(b,1) > V(b-L,p;b+G,1-p)$$ □

Proof of Theorem 3 For every x and z such that $w+c \geqslant x > w > z \geqslant w-\bar{L}$ there is a probability q such that $X = (x, q; z, 1-q)$ satisfies $V(X) = V(w,1)$. By (ℓ, g) bi-stochastic B3,

$$V(x,q;z,1-q) > V\left(x-\ell, \frac{q}{2}; x+g, \frac{q}{2}; z-\ell, \frac{1-q}{2}; z+g, \frac{1-q}{2}\right) \tag{6}$$

Denote the distribution functions of the two lotteries in (6) by F and F', respectively. Betweenness implies that the local utilities at δ_w and at F are the same (up to a positive linear transformation). Hence, by using the local utility $u(\cdot;\delta_w)$, we obtain,

$$q\left[u(x;\delta_w)-\frac{1}{2}(u(x-\ell;\delta_w)+u(x+g;\delta_w))\right] \tag{7}$$
$$+(1-q)\left[u(z;\delta_w)-\frac{1}{2}(u(z-\ell;\delta_w)+u(z+g;\delta_w))\right] > 0$$

Suppose there are $w+c > x^* > w > z^* > w-\bar{L}$ such that $u(x^*;\delta_w) \leqslant \frac{1}{2}[u(x^*-\ell;\delta_w)+u(x^*+g;\delta_w)]$ and $u(z^*;\delta_w) \leqslant \frac{1}{2}[u(z^*-\ell;\delta_w)+u(z^*+g;\delta_w)]$. Then inequality (7) is reversed, and by betweenness, the inequality at (6) is reversed; a contradiction to bi-stochastic B3. Therefore, at least one of the following holds.

1. For all $x \in [w, w+c]$,

$$u(x;\delta_w) > \frac{1}{2}[u(x-\ell;\delta_w)+u(x+g;\delta_w)]$$

and, similarly to the Proof of Theorem 1 (replace a with w in (4)), betweenness implies that $V(w,1) > V(w-L, p; w+G, 1-p)$ and the first claim of the theorem is satisfied.

2. For all $x \in [w-\bar{L}, w]$,

$$u(x;\delta_w) > \frac{1}{2}[u(x-\ell;\delta_w) + u(x+g;\delta_w)]$$

and, using an argument similar to that of the Proof of Theorem 2 (replace b with w in (5)), betweenness implies $V(w,1) > V(w-g-\bar{L}, \bar{p}; w-g+\bar{G}, 1-\bar{p})$ and the second claim of the theorem is satisfied. □

Appendix 2

We first show that Gul's (1991) disappointment aversion functional is not Gâteaux differentiable.[5] Consider lotteries of the form $(x, p; y, 1-p-q; z, q)$ where $x < y < z$ are constant and a disappointment aversion functional $V(p,q)$ with $u(x) = 0$ and $u(z) = 1$. From (1) and (2) in Gul (1991) it follows that

$$V(p,q) = \begin{cases} \dfrac{(1-p-q)u(y)+q}{1+\beta p} & q \leqslant \dfrac{(1+\beta)pu(y)}{1-u(y)} \\[2ex] \dfrac{(1+\beta)(1-p-q)u(y)+q}{1+\beta-\beta q} & q > \dfrac{(1+\beta)pu(y)}{1-u(y)} \end{cases}$$

Along the line $p+q=1$ we obtain

$$V(p,q) = \frac{1-p}{1+\beta p}$$

hence along this line, V is differentiable. In particular, the derivatives at $\left(\frac{1-u(y)}{1+\beta u(y)}, 1-\frac{1-u(y)}{1+\beta u(y)}\right)$ in the directions $(1,0)$ and $(0,1)$ have the same absolute value and opposite signs. However, along the line $p+q=\frac{1}{2}$, we obtain

$$V(p,q) = \begin{cases} \dfrac{u(y)+1-2p}{2+2\beta p} & p \geqslant \dfrac{1-u(y)}{2+2\beta u(y)} \\[2ex] \dfrac{(1+\beta)u(y)+1-2p}{2+2\beta p+\beta} & p < \dfrac{1-u(y)}{2+2\beta u(y)} \end{cases}$$

It is easy to verify that the derivatives of V at $\left(\frac{1-u(y)}{2+2\beta u(y)}, \frac{1}{2}-\frac{1-u(y)}{2+2\beta u(y)}\right)$ in the directions $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ do not have the same absolute value. As V is differentiable along the line $p+q=1$ but not along the line $p+q=\frac{1}{2}$, it

[5] In Safra and Segal (1998) we wrongly claimed that when u is linear, Gul's functional is Gâteaux differentiable.

follows that neither V, nor any increasing transformation of V is Gâteaux differentiable.

Next we show that hypotheses H1, H2, and their opposites do not imply Gâteaux differentiability. It is easy to construct a weighted utility functional W (Chew 1983) satisfying H1 and H2. Define a functional V which is equal to W above indifference curve $\mathcal{I}$, and below $\mathcal{I}$ it is expected utility with $\mathcal{I}$ being one of its indifference curves. Similarly to the above analysis, this functional is not Gâteaux differentiable, but as expected utility and W satisfy H1 and H2, so does V.

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