Efficient Bilateral Trade*

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May 2014

Abstract

We re-examine the canonical question of Myerson and Satterthwaite (1983) whether two privately-informed parties, a buyer and a seller, can trade an indivisible good efficiently. We relax their assumption that utilities are quasi-linear; our main assumption instead is that the traded good is normal. We show that efficient trade is possible if agents’ utility functions are not too responsive to private information. In addition, we provide natural examples in which efficient trade is possible even though agents’ utility functions are highly responsive to their private information.

1 Introduction

Can a seller and a buyer of an indivisible good trade efficiently if they are privately informed about their values and if ex ante either of them might have the higher value for the good? The theorem of Myerson and Satterthwaite (1983), a central result of the theory of mechanism design, provides a negative answer to this question. Assuming that agents have quasi-linear utility functions, Myerson and Satterthwaite showed that no Bayesian incentive-compatible, individually rational, non-subsidized mechanism is ex-post efficient. The reason is that in an incentive compatible mechanism, each

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*We are still tweaking the exposition. For their comments, we would like to thank Simon Board, Erik Eyster, Hugo Hopenhayn, Moritz Meyer-ter-Vehn, Ichiro Obara, John Riley, Ronny Razin, Guofu Tan, Tom Wilkening, and the audiences at SWET and LSE. Garratt: FRBNY, Pycia: UCLA.
agent needs to be provided with rents commensurate with her type, and the
gains from efficient trade are not sufficient to cover the rents of all parties.
Their impossibility theorem had a large impact on the economics literature
and the practice of market design, and it offers a stark contrast to Coase’s
(1960) claim that markets lead to efficient outcomes when property rights
are unambiguously established and there are no transaction costs.1

Our results imply that this central mechanism design result hinges on the
assumption that agents have quasi-linear utilities. Instead of assuming that
preferences are quasilinear, we assume instead that the good traded is nor-
mal: a good is normal if each player’s reservation price for the good increases
with the agent’s wealth (see, for instance, Cook and Graham 1977).2 We also
impose a technical assumption on how the private information affects agents’
utility; with quasi-linearity the assumption reduces to requiring that higher
types have higher utility from the good.3 In an otherwise standard Myerson
and Satterthwaite’s environment, we prove that the efficient trade is gener-
ically possible as long as either agents’ utilities are not too dependent on
private information, or the asymmetry of information is not too large. We
also show that in simple examples, for instance, with shifted Cobb-Douglas
utilities, the efficient trade is always possible regardless of the information

1See, for instance, Milgrom’s (2004) discussion of the role the impossibility theorem
played in the FCC deliberations on the first US spectrum auctions, and Loertscher, Marx,
and Wilkening’s (2013) discussion of how the impossibility theorem led to the focus of
market design on primary markets.

2The normality condition for an indivisible good is a natural counterpart of normality
for divisible commodities. Cook and Graham require that each player’s reservation price
for the good strictly increases with the agent’s wealth. Thus their condition does not hold
under quasi linear utility. Our results however continue to hold for generic utility profiles
if we relax the Graham and Cook normality condition requiring only that each player’s
reservation price for the good weakly increases with the agent’s wealth. Such a relaxed
normality condition is satisfied by quasi-linear preferences.

3The mechanism we construct is efficient, individually rational, budget-balanced, and
makes truthful reporting a solution to the first-order condition of the agents’ maximization
problem whether or not our technical assumption is satisfied. The technical assumption is
only needed to ensure that the second-order condition of the agents’ maximization problem
is satisfied for all possible distributions of private information. The assumption postulates
that the elasticity of marginal utility of money with respect to private information is
constant and dominated by the elasticity of marginal utility of the good with respect
to private information. As noted above, when utilities are quasi-linear, this assumption
simply says that higher types have higher utility from consuming the good.
of the agents, and the extent of informational asymmetry. We prove these possibility results by constructing a Bayesian incentive-compatible and individually rational mechanism that is ex-post efficient and does not rely on any subsidies nor on budget breaking by third parties.

Why is efficient trade possible when the object is normal, but it cannot be attained with quasi-linear preferences? With quasi-linear preferences the only gains from trade are those of assigning the object to the highest value agent. With normal goods an additional source of efficiency gains opens up. Suppose the seller’s utility function over money and the object is the same as the buyer’s and suppose that seller’s and buyer’s money holdings are such that they derive the same utility from their endowments. Consider giving the good to the buyer with a small probability, while compensating the seller with a money transfer in the state when the seller keeps the good. Normality means that the money transfer needed to compensate the seller for the small probability of giving up the good is less than the money transfer the buyer is willing to make in return for the same probability of obtaining the good. Hence, such a lottery contract is Pareto improving.\footnote{We restrict attention in this discussion to small probabilities of transferring the good to the buyer in order to make sure the buyer has enough money to compensate the seller. For a more developed example of such gains from randomized trade in a setting with complete information, see Garratt (1999).} The change from quasilinear preferences to normal goods creates not only additional efficiency gains to trade but also affects agents’ informational rents: agents compete over how they share the additional gains from trade. Nevertheless, these additional gains from trade allow us to find contracts that are efficient, preferred by agents to no trade, and sufficiently unresponsive to private reports on types so as to not discourage truth telling.\footnote{Like Myerson and Satterthwaite we look at ex post efficiency in the sense that we evaluate the efficiency assuming that we know the agents’ types. With respect to the resolution of the lotteries, our contracts are efficient not only ex post but also ex ante.}

The question when efficient trade is possible is important and has been extensively studied. Two crucial assumptions have been recognized already by Myerson and Satterthwaite: their result requires that the distribution of types is continuous and that buyer’s value is always higher than the seller’s
value. Gresik and Sattthertwaite (1983) extended the impossibility result to symmetric settings with many agents while Makowski and Mezzetti (1994) showed that for an open set of asymmetric distributions trade can be efficient provided there are multiple buyers. Wilson (1985), Makowski and Ostroy (1989), Rustichini, Satterthwaite, and Williams (1994), Reny and Perry (2006), Cripps and Swinkels (2006), and many others established that trade is asymptotically efficient as the number of buyers and sellers becomes large. McAfee (1991) showed that efficient trade can be possible when the ex ante gains from trade are large and the two trading parties have access to an uniformed budget breaking third agent.

At the same time, we know of no successful attempt to go beyond Myerson and Sattthertwaite (1983) and demonstrate the possibility of fully efficient mechanisms in their original context of two agents, one buyer and one seller. What was demonstrated is the possibility of approximate efficiency in two contexts. Chatterjee and Samuelson (1983) showed that double-auctions are asymptotically efficient in the limit as the agents become infinitely risk-averse. And, McAfee and Reny (1992) showed that when private values are correlated (and a hazard rate assumption is satisfied) a judicious use of Cremer and McLean (1988) lotteries allows the parties to reduce their incentives to misreport so as to permit outcomes as close to efficiency as desired.

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6The case of discrete distributions is further studied by Matsuo (1989) and Kos and Manea (2008).

7This observation has been extended to settings with multiple buyers and sellers by Williams (1999) and Schweizer (2006).

8McAfee (1991) studied the problem of trading divisible goods; see (Riley 2012) for an analysis with indivisible goods.

9Relaxing the assumption that one of the agents is pre-assigned the role of a seller, and the other the role of a buyer, and working in the context of divisible goods, Cramton, Gibbons, and Klemperer (1987) show that trade may be efficient if initially both trading partners own some quantity of the good, and depending on the realization of types each of them might become a seller or a buyer.

10Chatterjee and Samuelson (1983) also showed that double auction is not efficient, and Chatterjee (1982) showed that no mechanism in a large class of incentive compatible mechanisms is efficient; these two papers are important precursors of Myerson and Satterthwaite's impossibility theorem. After Myerson and Satthertwaite, many authors provided alternative proofs of their impossibility result, see for instance Williams (1999) and Krishna and Perry (1998).

11For simplicity, we formulate our main theorem in the context of risk averse agents, but
While ours is likely the first paper to study optimal Bayesian incentive compatible bilateral trade mechanisms for an indivisible normal good, it follows a rich mechanism design literature studying wealth effects and risk aversion. Holt (1980), Matthews (1983), Maskin and Riley (1984), and most of the related literature, restrict attention to deterministic mechanisms such as the first-price or the second-price auctions. The exceptions are Garratt (1999) who shows that random mechanisms can dominate deterministic ones in a complete information setting, and Baisa (2013) who shows that such mechanisms can dominate second-price auctions in a setting with private information, and that efficiency, individual rationality, and the lack of subsidies is, in general, inconsistent with strategy-proofness.\footnote{Baisa studies a setting in which a seller wants to allocate a normal good to one of a finite number of buyers; unlike in our setting, in his setting the seller has no private information about the good. Baisa proves his impossibility claim by constructing an example of a profile of utility functions such that no strategy-proof, individually rational, non-subsidized mechanism allocates the good in an efficient way. Our example in Section 2 shows that in some settings efficient trade can be accomplished in strategy-proof way, and—for a large class of problems—we show in the Conclusion the generic impossibility of achieving efficient trade in an ex-post incentive compatible way. While we focus on bilateral trade, our analysis can be used to show that also in the allocation setting generically no ex-post incentive compatible mechanism is efficient.}

The rest of the paper is organized as follows. In Section 2 we study an example in which agents’ types are distributed independently on \([0,1]\) and agents’ utilities are linear in both money and consumption of the good. We present our model and assumptions in Section 3. Section 4 contains our main results. Suppose that the traded object is normal and that agents’ types impact their marginal utility of the good more than the marginal utility of money, and that either the utility functions are not too dependent on types, or that the supports of agents’ types are not too large. Then, for any we allow any level of risk aversion, including arbitrarily small risk aversion. Furthermore, as demonstrated by the example of Section 2, the underlying insight does not rely on risk aversion. Also, our results allow for both independence and correlation, and they do not rely on large bets in the spirit of Cremer and McLean: all lotteries we employ are bounded by agents’ wealth levels.
initial distribution of wealth, except possibly one, we construct a mechanism that is incentive-compatible, individually rational and efficient. Thus the example provided in Section 2 is not special. Furthermore, our main result implies that efficient trade is possible in problems arbitrarily close to Myerson and Satterthwaite’s quasilinear setting.

2 An Example

A seller is endowed with a good and money endowment $m_s$ while the buyer has money endowment $m_b$. Each agent privately knows his or her type $\theta \in [0, 1]$ and has a shifted Cobb-Douglas utility: $u(x, m; \theta) = (1 + \theta x) m$ where $m$ denotes the money the agent has and $x$ is a dummy variable taking values $x = 1$ or $x = 0$ depending on whether the agent has the good.\(^{13}\)

The seller’s type, denoted $c$ for cost of trade, is distributed according to an arbitrary distribution $F$ on $[0, 1]$. The buyer’s type, denoted $v$ for value, is distributed according to an arbitrary distribution $G$ on $[0, 1]$. Let $m_s$ be the seller’s money endowment, and $m_b$ be the buyer’s money endowment.

In this example there is a mechanism that generates efficient trade, is Bayesian incentive compatible, individually rational, and requires no subsidies. As shown below, the following mechanism $\phi$ satisfies these properties: $\phi$ allocates the good and the sum of the money endowments of both agents to the seller with probability $\frac{m_s}{m_s + m_b}$, and it allocates the good and the sum of money endowments to the buyer with the remaining probability, $\frac{m_b}{m_s + m_b}$.

Mechanism $\phi$ is obviously Bayesian incentive compatible because the allocation and transfers do not depend on the agents’ reports.\(^{14}\) To see that the mechanism is individually rational, notice that the mechanism gives the seller with type $c$ the expected utility of

$$\frac{m_s}{m_s + m_b} (1 + c) (m_s + m_b) = (1 + c) m_s,$$

and that this expected utility is equal to the utility of the seller if no trade

\(^{13}\)As usual, the utility of money derives from other goods the agent might purchase.

\(^{14}\)While mechanism $\phi$ is also dominant-strategy incentive compatible, our general results will only demonstrate the existence of a Bayesian incentive-compatible mechanism.
takes place. Similarly, the mechanism gives the buyer with type \( v \) the expected utility of

\[
\frac{m_b}{m_s + m_b} (1 + v) (m_s + m_b) = (1 + v) m_b,
\]

and this expected utility is larger than the utility of the buyer if no trade takes place (the latter utility is \( m_b \)).

Finally, to see that the mechanism is efficient notice that the Pareto frontier in this example consists of all randomizations among two outcomes: either the seller keeps the good and gets all the financial wealth, or the buyer gets the good and all the financial wealth. This can be immediately seen in Figure 1 (we provide an explicit argument in the appendix).\(^{15}\)

\[\text{Figure 1: Pareto Frontier in the Shifted Cobb-Douglas Example.}\]

\(^{15}\)An even simpler example obtains when agents’ have standard Cobb-Douglas utilities over the good and the money, \( u(x, m; \theta) = A(\theta) x^\alpha(\theta) m^\beta(\theta) \) for some functions \( A, \alpha, \beta \). With Cobb-Douglas utilities, the mechanism that allocates the good and all the money to the seller implements efficient trade. Of course, such an example is extreme since, without the indivisible good, agents have no use for money.
3 Model

We study trade between two agents, a buyer $b$ and a seller $s$. Each agent is endowed with some money holdings $m \in [0,M]$ where $M$ denotes the total money holdings of the two agents, and the seller is endowed with one unit of an indivisible good. Each agent’s utility depends on her privately known type $\theta$; we denote the seller’s type (cost) by $c \in [c,\bar{c}]$ and we denote the buyer’s type (value) by $v \in [v,\bar{v}]$.

The utilities of agents with types $c,v$ and money holdings $m$ is $u_s(x, m; c)$ and $u_b(x, m; v)$ where $x = 1$ if the agent has the good, and $x = 0$ if she does not. The utility is monotonic in having the good ($x$) and in money ($m$), it is strictly concave in money, and it is twice continuously differentiable in money and in type. For convenience, we extend the utility function notation to lotteries over the good $u(x, m, \theta) = xu(1, m, \theta) + (1 - x)u(0, m, \theta)$ for $x \in [0,1]$.

We assume that the good is normal in the standard sense (see Cook and Graham 1977): for any type $\theta$ and any relevant money levels $m, p, \epsilon > 0$, if $u(0, m; v) = u(1, m - p; v)$ then $u(0, m + \epsilon - p; v) < u(1, m + \epsilon; v)$. Normality captures the intuition that the more money an agent has, the more she is willing to pay for the good.

We also impose the following assumption on how agents’ utilities respond to their types. We assume that $\frac{\partial}{\partial m} \log \left( \frac{\partial}{\partial x} u(x, m, \theta) \right)$ does not depend on $m$, and that for any $x \in [0,1]$, $m \in [0,M]$, and any type $\theta$, we have

$$\frac{\partial}{\partial \theta} \log \left( \frac{\partial}{\partial x} u(x, m, \theta) \right) > \frac{\partial}{\partial \theta} \log \left( \frac{\partial}{\partial m} u(x, m, \theta) \right) \quad (1)$$

That is we want the type-elasticity of the marginal value of the good to exceed the type-elasticity of the marginal utility of money, and the latter elasticity to be constant in money. Both components of this assumption are automatically satisfied when utilities are quasi-linear in money, and higher types have higher utility from consuming the good. The assumption is also valid

\[\text{For simplicity of exposition we will mostly drop the superscript indicating whether } u \text{ is buyer’s utility or whether it is seller’s utility.}\]
satisfied when agents’ utilities are additively separable, \( u(x, m; \theta) = \theta x + v(m) \). Furthermore, this assumption is only needed for our analysis of the second-order condition of the mechanism we construct, where it is sufficient but not necessary. The necessary condition is more complex, and is provided in the analysis of the second-order condition.

The normality assumption allows us to determine some features of the Pareto frontier (See Figure 2).\(^{17}\) To describe the frontier, let us fix agents’ types \( c \) and \( v \). The frontier is the upper envelope of the curves \( C_S = \{(u(1, m, c), u(0, M - m, v)) : m \in [0, M]\} \) and \( C_B = \{(u(0, m, c), u(1, M - m, v)) : m \in [0, M]\} \). The curve \( C_S \) traces the utilities when the seller has the good, while the curve \( C_B \) traces the utilities when the buyer has the good. Since we assume that it is better to have the good than not to have it, the curve \( C_S \) starts higher than \( C_B \) on the axis of seller’s utilities (the vertical axis), and \( C_S \) ends lower than \( C_B \) on the axis of buyer’s utilities (the horizontal axis). In particular, the two curves need to intersect, and the normality assumption implies that \( C_S \) intersects \( C_B \) from above. In particular, this implies that there is exactly one intersection point.\(^{18}\)

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\(^{17}\)See Garratt (1999) for a more detailed discussion of the Pareto frontier for normal goods.

\(^{18}\)The uniqueness of the intersection point is the main implication of normality in the paper. Without normality the two curves could intersect multiple times.
As we move along the Pareto frontier from the seller’s most preferred point to the buyer’s most preferred point, we start on the curve $C_S$ and we end on the curve $C_B$. The point at which the curves $C_S$ and $C_B$ intersect cannot be part of the frontier because normality implies that at this point $C_S$ intersects $C_B$ strictly from above, and hence a randomization over any point just to the left of the intersection and any point just to the right of the intersection is strictly preferred to the intersection point by both trading parties. The frontier thus contains a flat part consisting of randomizations between two points: $S(c, v) \in C_S$ and $B(c, v) \in C_B$, where the seller strictly prefers $S$ to $B$ while the buyer strictly prefers $B$ to $S$. The strict concavity of $u$ in money implies that these two points are uniquely determined. We call them the critical points. Let $m^S(c, v)$ denote the seller’s money holdings at $S$ and $m^B(c, v)$ denote the buyer’s money holdings at $B$. In the example of Section 2, $m^S(c, v) = m^B(c, v) = M$.\(^{19}\)

The dependence of the points $B(c, v)$ and $S(c, v)$ on $(c, v)$ is continuously differentiable by the assumed regularity of $u$ and its strict concavity in money. We assume that these two points are either at the boundary (that is involve money levels of $M$ and 0) or they are both internal. In the appendix we show that in the internal case our assumptions imply that

\[
\frac{\partial}{\partial \theta} m^\theta(c, v) > 0 > \frac{\partial}{\partial \theta} m^\theta(c, v),
\]

(2)

(in the corner case the two inequalities become equalities). Thus, agents’ money holdings at the critical point of the Pareto frontier the agent prefers are decreasing in agent’s type, while the agent’s money holdings at the critical point the other agent prefers are increasing in agent’s own type.

### 4 Main Results

We now show that efficient trade is possible for general utility functions from our model section. We formulate our possibility result in two related ways.

\(^{19}\)While in this example the critical efficient levels of money holdings do not depend on the types, in general the critical levels of money holdings might depend on agents’ types, and hence on their reports.
First, efficient trade is possible if agents’ utilities are not too dependent on their types.

**Theorem 1.** Fix \( c^*, v^* \in [0, 1] \) and \( u^a (\cdot, \cdot; c^*) \) and \( u^b (\cdot, \cdot; v^*) \). For every profile of money endowments but one, there is \( \delta > 0 \) such that if

\[
\max_{\theta \in [0,1], m \in [0,M], x \in \{0,1\}} |u(x, m, \theta) - u(x, m, \theta^*)| < \delta, \tag{3}
\]

then there is an incentive-compatible, individually-rational, and budget-balanced mechanism that generates efficient trade.

As an immediate corollary we obtain

**Corollary 1.** Fix \( (c^*, v^*) \in (0,1)^2 \) and function \( u \). For any profile of money endowments but one, there are intervals \( (\underline{c}, \overline{c}) \ni c^* \) and \( (\underline{v}, \overline{v}) \ni v^* \) such that: if agents draw their types from arbitrary distributions on \( (\underline{c}, \overline{c}) \times (\underline{v}, \overline{v}) \), then there is an incentive-compatible mechanism, individually-rational, and budget-balanced mechanism that generates efficient trade.

This corollary obtains because by taking the intervals \( (\underline{c}, \overline{c}) \ni c^* \) and \( (\underline{v}, \overline{v}) \ni v^* \) to be sufficiently small, and re-scaling them to \( [0,1] \), we can ensure condition (3).

A special case of interest obtains when \( c^* = v^* \). In this case, the corollary takes the following simpler form:

**Corollary 2.** Fix any \( \theta^* \in (0,1) \) and function \( u \). For any profile of money endowments but one, there is an interval \( (\underline{\theta}, \overline{\theta}) \ni \theta^* \) such that for any distribution of agents’ types on \( (\underline{\theta}, \overline{\theta}) \times (\underline{\theta}, \overline{\theta}) \), there is an incentive-compatible, individually-rational, and budget-balanced mechanism that generates efficient trade.

Why is efficient trade possible when the object is normal, but it cannot be attained with quasi-linear preferences? With quasi-linear preferences the only gains from trade are those of assigning the object to the highest value agent. As we observed in the introduction, with normal goods an additional source of efficiency gains opens up. Suppose the seller’s utility function over
money and the object is the same as the buyer’s and suppose that seller’s and buyer’s money holdings are such that they derive the same utility from their endowments. Consider giving the good to the buyer with a small probability, while compensating the seller with a money transfer in the state when the seller keeps the good. Normality means that the money transfer needed to compensate the seller for the small probability of giving up the good is less than the money transfer the buyer is willing to make in return for the same probability of obtaining the good. Hence, such a lottery contract is Pareto improving. The change from quasilinear preferences to normal goods creates not only additional efficiency gains to trade but also affects agents’ informational rents: agents compete over how they share the additional gains from trade. Furthermore, unlike in the example of Section 2, in the above theorems the efficient allocation may depend on the agents’ types.

Before providing a formal proof, in the following example, we illustrate this dependence of optimal contracts on agents’ types, and the resulting need to elicit the types.

4.1 Example with Separable Utilities

We now look at separable utilities \( u(x, m; \theta) = \theta x + v(m) \), assuming that \( v(m) = \log(m) \), and define a range of types such that a mechanism exists which implements the efficient trade around a point \((c^*, v^*)\). As shown in the Appendix, this example has \( m^B(c, v) = \frac{Mc}{c+v} \) and \( m^S(c, v) = \frac{Mv}{c+v} \). Suppose \( M = 1 \) is endowed in such a way that for \( c = v = 2 \), the agents’ utilities are equal (that is \( m_s = .12 \) and \( m_b = .88 \)). Then, there is efficient trade for \( c, v \) distributed uniformly on \([1, 5]\). Incentive compatibility is ensured by Theorem 1. We need to check that for each \( c, v \in [1, 5] \) individual rationality (IR) restrictions are satisfied for \( \pi \in [0, 1] \). [Is this really all we need?]

The range of \( \pi \) that ensures IR is determined by the equations

\[
\pi \log\left(\frac{c}{c+v}\right) + (1 - \pi)(v + \log\left(\frac{c}{c+v}\right)) \geq \log(.88)
\]

and

\[
\pi(c + \log\left(\frac{v}{c+v}\right)) + (1 - \pi)\log\left(\frac{v}{c+v}\right) \geq \log(.12).
\]
which can be combined into the expression

\[
\frac{c + \log(0.12) - \log\left(\frac{v}{c+\pi}\right)}{c} \leq \pi \leq \frac{v + \log\left(\frac{c}{c+\pi}\right) - \log(0.88)}{v}
\]

[I have checked through numerical simulation that the above range is non empty and includes values in \([0, 1]\) for all \(c, v \in [1, 5]\). I am not sure if it is possible to show this analytically.]

### 4.2 Proof of Theorem 1

Fix a distribution of endowed money holdings other than \((m^S(c^*, v^*), M - m^S(c^*, v^*))\).

There are three cases depending on how much money the seller initially has.

In the first case, the seller’s endowed money holdings are strictly above \(m^S(c^*, v^*)\), then it remains so for utilities close to \(u^s, u^b\), and hence there is \(\delta > 0\) that guarantees that the no-trade mechanism is efficient and satisfies all our other requirements.

In the second case, the seller’s endowment is such that the seller’s utility is strictly below \(u(0, M - m^B(c^*, v^*)\), \(c^*\)). Then there is a point on the Pareto frontier of \((c^*, v^*)\) that strictly dominates the agents’ utility at the initial endowments (point \(F\) in Figure 3). At this point on the frontier the seller has no good and has money holdings \(m^S + t\) for some constant transfer \(t\), while the buyer has the good and money holdings \(m^B - t\). In particular, the pre-trade seller’s utility is strictly below \(u(0, m^S + t, c^*)\) and the pre-trade buyer’s utility is strictly below \(u(1, m^B - t, v^*)\). These bounds on agents’ pre-trade utility remain true for type profiles close to \((c^*, v^*)\). There is then \(\delta > 0\) that guarantees that the mechanism that allocates the good and money \(m^B - t\) to the buyer, and money \(m^S + t\) (without good) to the seller is Pareto efficient, individually rational, and does not require a subsidy. Furthermore, this mechanism is incentive compatible as it does not rely on agents reports.\(^{20}\)

\(^{20}\)A similar fixed-terms-of-trade mechanism delivers efficient trade whenever there exists a point on the Pareto frontier that is strictly preferred by both the buyer and seller to status quo, and strictly more favorable to the buyer than having the good and money \(m^B(c^*, v^*)\).
In the third case, the seller’s endowment is intermediate, that is the seller’s money holdings are strictly below \( m_S(c^*, v^*) \) but seller’s utility is weakly above \( u(0, M - m_B(c^*, v^*), c^*) \). This is the main case, and the reminder of the proof is devoted to its analysis.

In this third case, there is a point \( F = F(c^*, v^*) \) on the flat part of the frontier strictly between \( S(c^*, v^*) \) and \( B(c^*, v^*) \) (see Figure 4) that is strictly preferred by the buyer and the seller to the initial situation, \((u_s(1, m_S; c^*), u_b(0, m_B; v^*))\). The point \( F \) is determined by \( \pi(c^*, v^*) = \pi^* \in (0, 1) \) as follows: the utility pair \( F \) corresponds to the seller having the good and wealth \( m_S(c^*, v^*) \) with probability \( \pi^* \), and the buyer having the good and wealth \( m_B(c^*, v^*) \) with probability \( 1 - \pi^* \).

For small \( \delta \), the critical money holdings \( m^S(c, v) \) and \( m^B(c, v) \) are nearby \( m_S(c^*, v^*) \) and \( m_B(c^*, v^*) \), and there are \( \pi(c, v) \) nearby \( \pi^* \) such that the corresponding \( F(c, v) \) is strictly preferred by the agents to the initial situation. If both \( m^S(c, v) \) and \( m^B(c, v) \) are locally constant around \( (c^*, v^*) \) then a mechanism with fixed \( \pi(c, v) = \pi^* \) satisfies our postulates. Let us thus assume that at least one \( m^S(c, v) \) and \( m^B(c, v) \) varies in \( c, v \). By (2) and its qualifying discussion, one of the two functions have non-zero partials throughout the domain.
Figure 4: Individually Rational Part of the Pareto Frontier. The Case of Trade at Varying Prices.

The crux of the reminder of the argument is to show that \( \pi(c,v) \) can be defined in such a way that in a Bayesian Nash equilibrium of the budget-balanced mechanism that assigns the seller the good and money holdings \( m^S(c,v) \) with probability \( \pi(c,v) \) and assigns the good and money \( m^B(c,v) \) to the buyer with probability \( 1 - \pi(c,v) \), both agents report their true types. (Note that in this mechanism the buyer gets money \( M - m^S(c,v) \) when the seller gets the good, and the seller gets money \( M - m^B(c,v) \) when the buyer gets the good).

We thus need to find a function \( \pi \) such that for the seller

\[
\Pi^S(c,\hat{c}) = E_v \left( \pi(\hat{c}, v) u(1, m^S(\hat{c}, v), c) + (1 - \pi(\hat{c}, v)) u(0, M - m^B(\hat{c}, v), c) \right)
\]

is maximized at \( \hat{c} = c \), and similarly for the buyer,

\[
\Pi^B(v,\hat{v}) = E_c \left( \pi(c, \hat{v}) u(0, M - m^S(c, \hat{v}), v) + (1 - \pi(c, \hat{v})) u(1, m^B(c, \hat{v}), v) \right)
\]

is maximized at \( \hat{v} = v \). In order to guarantee this we will construct \( \pi \) such that the first order condition is satisfied for truthful reporting, and the second order condition is satisfied at all points satisfying the first order condition.
We will take \( \pi (c^*, v^*) \) to be \( \pi^* \in (0, 1) \) given above, thus guaranteeing that \( \pi (c, v) \in (0, 1) \) for small \( \delta > 0 \), and that the individual rationality is satisfied for small \( \delta > 0 \). As an aside, let us note that the individual rationality and the requirement that \( \pi \) is a probability constraints our mechanism to be well-behaved only locally around \( c^*, v^* \); the incentive compatibility conditions could be made to be globally satisfied.

4.2.1 The First Order Condition

Assuming truthful reporting by the other agent, the first order condition for the seller is

\[
0 = E_v \left\{ \frac{\partial}{\partial \hat{v}} \pi (\hat{c}, \hat{v}) u (1, m^S (\hat{c}, \hat{v}), c) + \pi (\hat{c}, \hat{v}) \left[ \frac{\partial}{\partial m} u (1, m^S (\hat{c}, \hat{v}), c) \right] \left[ \frac{\partial}{\partial \hat{c}} m^S (\hat{c}, \hat{v}) \right] \right\}
\]

and the first order condition for the buyer is

\[
0 = E_c \left\{ \frac{\partial}{\partial \hat{v}} \pi (c, \hat{v}) u (0, M - m^B (c, \hat{v}), c) - \pi (c, \hat{v}) \left[ \frac{\partial}{\partial \hat{c}} m^B (c, \hat{v}) \right] \left[ \frac{\partial}{\partial \hat{c}} m^S (c, \hat{v}) \right] \right\}
\]

We want \( \hat{c} = c \) to satisfy the seller’s first order condition and \( \hat{v} = v \) to satisfy the buyer’s first order condition, and hence the two conditions give us a system of PDE equations on \( \pi (c, v) \).\(^{21}\) These equations take the form

\[
E_v \left[ S_1 (c, v) \frac{\partial}{\partial c} \pi (c, v) + S_2 (c, v) \pi (c, v) \right] = \phi (c),
\]

\[
E_c \left[ B_1 (c, v) \frac{\partial}{\partial c} \pi (c, v) + B_2 (c, v) \pi (c, v) \right] = \psi (v),
\]

\(^{21}\)To ensure that the coefficient in front of \( \frac{\partial}{\partial c} \pi (c, v) \) is positive, we multiply the second equation by \( (-1) \) before calculating \( B_1, B_2, \) and \( \psi \) below.
where the coefficients in front of $\frac{\partial}{\partial c}\pi$ and $\frac{\partial}{\partial v}\pi$ are

$$S_1 (c, v) = u \left(1, m^S (c, v), c\right) - u \left(0, M - m^B (c, v), c\right) \quad > \quad 0,$$

$$B_1 (c, v) = u \left(1, m^B (c, v), v\right) - u \left(0, M - m^S (c, v), v\right) \quad > \quad 0,$$

the coefficients in front of $\pi$ are

$$S_2 (c, v) = \left[\frac{\partial}{\partial w} u \left(1, m^S (c, v), c\right) \right] \left[\frac{\partial}{\partial c} m^S (c, v) \right] + \left[\frac{\partial}{\partial w} u \left(0, M - m^B (c, v), c\right) \right] \left[\frac{\partial}{\partial c} m^B (c, v) \right],$$

$$B_2 (c, v) = \left[\frac{\partial}{\partial w} u \left(1, m^B (c, v), v\right) \right] \left[\frac{\partial}{\partial v} m^B (c, v) \right] + \left[\frac{\partial}{\partial w} u \left(0, M - m^S (c, v), v\right) \right] \left[\frac{\partial}{\partial v} m^S (c, v) \right],$$

and the functions $\phi, \psi$ are given by

$$\phi (c) = \quad E_v \left\{ \left[\frac{\partial}{\partial w} u \left(0, M - m^B (c, v), c\right) \right] \left[\frac{\partial}{\partial c} m^B (c, v) \right] \right\},$$

$$\psi (v) = \quad E_c \left\{ \left[\frac{\partial}{\partial w} u \left(1, m^B (c, v), v\right) \right] \left[\frac{\partial}{\partial v} m^B (c, v) \right] \right\}. $$

By assumption $u$ and its derivatives are continuously differentiable. The continuous differentiability of $m^S$ and $m^B$ follows from strict concavity of $u$ and the implicit function theorem (the implicit equations defining $m^S$ and $m^B$ are in the appendix).

The above averaged-out system of PDEs has a solution for any initial condition $\pi \left(c^*, v^*\right) = \pi^*$ by the following crucial lemma (proven in the appendix).

**Lemma 1.** Let $I$ be a bounded interval of positive length and let $F$ be a joint distribution of $(c, v)$ over domain $I^2 \subseteq \mathbb{R}^2$. Let $S_1 (\cdot, \cdot), S_2 (\cdot, \cdot)$ and $B_1 (\cdot, \cdot), B_2 (\cdot, \cdot)$ be functions defined on $I^2$, and $\phi, \psi$ be functions on $I$. Suppose that all these functions are continuously differentiable and that that $S_1, B_1 \neq 0$ for all arguments $(c, v)$. Then, the system of PDE equations

$$E_v \left[ S_1 (c, v) \frac{\partial}{\partial c} \pi (c, v) + S_2 (c, v) \pi (c, v) \right] = \phi (c),$$

(4)
has a solution \( \pi \) for any boundary condition \( \pi(c^*, v^*) = \pi^* \). Furthermore, as \( \frac{\delta}{\delta s_1}, \frac{\delta}{\delta t_1}, \frac{\phi}{\delta t_1}, \) and \( \frac{\psi}{\delta t_1} \) tend to zero, the derivatives of the solution \( \pi \) tend to zero as well.

Finally, notice that because \( \pi^* \in (0, 1) \), and because by taking \( \Delta \) to be sufficiently small we can guarantee that \( \pi \) is near flat, we can find a domain of types such that \( \pi \) takes values in \([0, 1]\) and satisfies the individual rationality conditions on the entire domain.

The above lemma is of independent interest. It tells us that for any marginal distributions of the linear PDE formulas from the lemma, we can find a function that implements these marginal distributions.\(^{22}\)

### 4.2.2 The Second Order Condition

The last thing to check is that the agents objectives satisfy the second-order condition at every point at which the first-order condition is satisfied so that truthful reporting is not only a solution of the first-order condition but also the optimal report. Let us thus check the second-order conditions for the seller; the buyer's problem is analogous.

Since at points at which the first-order condition is satisfied we have

\[
0 = \frac{d}{dc} \left( \frac{\partial}{\partial c} \Pi^S(c, c) \right) = \frac{\partial}{\partial c} \left( \frac{\partial}{\partial c} \Pi^S(c, c) \right) + \frac{\partial}{\partial c} \left( \frac{\partial}{\partial c} \Pi^S(c, c) \right),
\]

the second-order condition for the seller would be implied if we shown that

\[
\frac{\partial}{\partial c} \frac{\partial}{\partial c} \Pi^S(c, c) > 0.
\]

\(^{22}\)We have not been able to find this lemma in the literature on partial differential equations. The sufficient conditions for existence of solutions of non-averaged linear PDEs of Thomas (1934) and Mardare (2007) can easily tell us that the lemma is true if \( \frac{\partial}{\partial c} \frac{\partial}{\partial c} \Pi^S = \frac{\partial}{\partial c} \frac{\partial}{\partial c} \), which is satisfied for instance when the coefficients \( B_i, S_i \) are all constant, but they are not satisfied in the general case we consider here (which is not surprising as it is much easier to satisfy the PDE equations on average than it is to satisfy them pointwise).
A straightforward calculation shows that \( \frac{\partial}{\partial c} \frac{\partial}{\partial c} \Pi^S (c, c) \) equals

\[
E_v \left\{ \left[ \frac{\partial}{\partial c} \pi (c, v) \right] \left[ u_c (1, m^S (c, v), c) - u_c (0, M - m^B (c, v), c) \right] \right. \\
+ \pi (c, v) \left( \left[ \frac{\partial}{\partial m} u_c (1, m^S (c, v), c) \right] \left[ \frac{\partial}{\partial c} m^S (c, v) \right] + \left[ \frac{\partial}{\partial m} u_c (0, M - m^B (c, v), c) \right] \left[ \frac{\partial}{\partial c} m^B (c, v) \right] \right) \\
- \left. \left[ \frac{\partial}{\partial m} u_c (0, M - m^B (c, v), c) \right] \left[ \frac{\partial}{\partial c} m^B (c, v) \right] \right\}
\]

We can substitute in for \( \frac{\partial}{\partial c} \pi (c, v) \) from the first-order condition obtaining that \( \frac{\partial}{\partial c} \frac{\partial}{\partial c} \Pi^S (c, c) \) equals \( (1 - \pi (c, v)) \frac{\partial}{\partial c} m^B (c, v) \) times

\[
\left[ \frac{\partial}{\partial m} u_c (0, M - m^B (c, v), c) \right] \left[ u_c (1, m^S (c, v), c) - u_c (0, M - m^B (c, v), c) \right] \\
- \left[ \frac{\partial}{\partial m} u_c (0, M - m^B (c, v), c) \right] \left[ u (1, m^S (c, v), c) - u (0, M - m^B (c, v), c) \right]
\]

minus \( \pi (c, v) \frac{\partial}{\partial c} m^S (c, v) \) times

\[
\left[ \frac{\partial}{\partial m} u_c (1, m^S (c, v), c) \right] \left[ u_c (1, m^S (c, v), c) - u_c (0, M - m^B (c, v), c) \right] \\
- \left[ \frac{\partial}{\partial m} u_c (1, m^S (c, v), c) \right] \left[ u (1, m^S (c, v), c) - u (0, M - m^B (c, v), c) \right]
\]

By assumption we are considering the case when one of the partials \( \frac{\partial}{\partial c} m^B (c, v), \frac{\partial}{\partial c} m^S (c, v) \) is non-zero throughout the domain. Thus, (2) implies that the second order condition is satisfied provided both above displayed expressions are strictly positive. Since \( m^S \geq M - m^B \), the expressions are positive if

\[
\left[ \frac{\partial}{\partial m} u_c (0, m, c) \right] \left[ u_c (1, m', c) - u_c (0, m, c) \right] - \left[ \frac{\partial}{\partial m} u_c (0, m, c) \right] \left[ u (1, m', c) - u (0, m, c) \right] > 0
\]

and

\[
\left[ \frac{\partial}{\partial m} u_c (1, m', c) \right] \left[ u_c (1, m', c) - u_c (0, m, c) \right] - \left[ \frac{\partial}{\partial m} u_c (1, m', c) \right] \left[ u (1, m', c) - u (0, m, c) \right] > 0
\]
for all $m' \geq m$. We can re-express the two inequalities as

$$\frac{u_c(1, m', c) - u_c(0, m, c)}{u(1, m', c) - u(0, m, c)} > \frac{\partial}{\partial m} u_c(0, m, c)$$

and

$$\frac{u_c(1, m', c) - u_c(0, m, c)}{u(1, m', c) - u(0, m, c)} > \frac{\partial}{\partial m} u_c(1, m', c)$$

for all $m' \geq m$. These two inequalities are implied by our assumptions. Let us show it for the first of the two inequalities; the proof of the second one follows the same steps. Let us rewrite the left-hand side as

$$\frac{u_c(1, m', c) - u_c(0, m, c)}{u(1, m', c) - u(0, m, c)} = \left[ \frac{u_c(1, m', c) - u_c(1, m, c)}{u(1, m', c) - u(1, m, c)} \right] + \left[ \frac{u_c(1, m, c) - u_c(0, m, c)}{u(1, m', c) - u(0, m, c)} \right]$$

Now, the first inequality of (1) gives

$$\frac{u_c(1, m, c) - u_c(0, m, c)}{u(1, m, c) - u(0, m, c)} > \frac{\partial}{\partial m} u_c(0, m, c)$$

and the constancy of $\frac{u_c(m, 0, \tilde{m}, c)}{u_m(0, \tilde{m}, c)}$ in $\tilde{m}$ gives

$$\frac{\partial}{\partial \tilde{m}} u_c(0, \tilde{m}, c) = \frac{\partial}{\partial \tilde{m}} u(0, \tilde{m}, c)$$

Thus, the left-hand side is a ratio of sums such that the ratio of each summand in the nominator to the corresponding summand in denominator is weakly higher, and in one non-zero measure case strictly higher than the left-hand side above. This ends the proof of Theorem 1.

5 Conclusion

We focused on providing incentives for agents to truthfully reveal their cost/value information. It is natural to think that preferences are not ob-
servable and need to be elicited, while information such as the size of money holdings can be objectively verified. At the same time, in some environments, for instance in the example of Section 2, we can not only incentivize agents to reveal their value/cost of the good, we can also provide incentives for them to truthfully announce their money holdings, provided the cost of delivering more money than one has (in the event one is asked to do it) is appropriately high. This is so because—as long as the agent is able to deliver the money—each agent benefits from reporting higher money holdings rather than lower.

While we focused on Bayesian implementation, in the example of Section 2 the mechanism achieving efficient trade was strategy-proof. This is not true in general.\textsuperscript{23} Our analysis of the first order condition of agents’ optimization implies the following:

**Proposition 1.** *When the randomization interval is interior, \( m_S, m^B \in (0, M) \), and money endowments are such that efficiency requires randomization, then for generic utility function \( u \) no mechanism can implement efficient trade in an ex-post equilibrium.*

Finally, our results on efficient trade open the possibility that other problems might have efficient solutions in non-quasilinear settings. For instance, our results imply the possibility of efficient mechanism for two agents to make a binary decision, e.g. whether to provide a public good, when each of the agents favors a different decision and each has higher marginal utility of money if his preferred decision is taken.

**Appendix**

**Pareto Frontier in the Example of Section 2**

Fix \( c \) and \( v \) and take any allocation \((\pi, y, z)\) where \( \pi \) is the probability the seller gets the good, \( y \) is the expected financial allocation of the seller if he

\textsuperscript{23}In a related setting in which a seller wants to allocate a normal good to one of a finite number of buyers, and in which, unlike in our setting, the seller has no private information about the good, Baisa (2013) constructs an elegant example of a profile of utility functions such that no strategy-proof mechanism allocates the good in an efficient way.
gets the good, and $z$ is the expected financial allocation of the seller if he does not get the good. Denoting the total financial wealth by $M$, the seller’s utility is then

$$u^S = \pi (1 + c) y + (1 - \pi) z,$$

and the buyer’s utility is

$$u^B = \pi (M - y) + (1 - \pi) (1 + v) (M - z).$$

If $\pi = 0$ or $\pi = 1$ we are on the above postulated Pareto frontier. Consider $\pi \in (0, 1)$. We can then weakly improve for both of them by transferring the money to the buyer from the seller in the state when the buyer has the good, and transferring the money to the seller when he has the good. Let us do it so that for every dollar taken from the seller in the state he does not have the good, we allocate him $\frac{1 - \pi}{\pi}$ dollars in the state when he has the good, and let us continue transferring the money till either $z = 0$ or $y = M$. Consider the case we stop when the seller has no money left in the state he does not have the good that is when

$$M \geq y + \frac{1 - \pi}{\pi} z \quad (6)$$

(the case $y = M$ is symmetric). The utilities then are

$$u^{S1} = \pi (1 + c) \left( y + \frac{1 - \pi}{\pi} z \right),$$

and

$$u^{B1} = \pi \left( M - y - \frac{1 - \pi}{\pi} z \right) + (1 - \pi) (1 + v) M.$$

$$= (M - \pi y - (1 - \pi) z) + (1 - \pi) v M.$$

Notice that $u^{S1} \geq u^S$, and thus this transfer provides a Pareto improvement for the seller. Let us now further reallocate the money to the seller in the state he has the good till he has all the money in this state while compensating the buyer through lowering $\pi$ while keeping the seller’s utility con-
stant. Then, the new probability the seller has the good is 
\[ \bar{\pi} = \frac{\pi y + \frac{1 - \pi}{M} z}{M} \], and the buyer utility becomes

\[ u^{B2} = (1 - \pi') (1 + v) M = \left( 1 - \frac{\pi y + (1 - \pi) z}{M} \right) (1 + v) M \]

which is better than \( u^B \) because we are in the case where (6) is satisfied. Thus, both the seller and the buyer prefer a lottery from the postulated Pareto frontier to the initial allocation. To finish the argument notice that no two lotteries from the postulated frontier can be Pareto ranked.

**Equations on \( m^S \) and \( m^B \) and Derivation of 2**

When the critical points \( S \) and \( B \) are internal, the money levels are uniquely determined by the following equations

\[ \frac{\partial}{\partial m} \frac{u(1,m^S,c) - u(0,m^B,c)}{u(1,m^B,c) - u(0,M-m^S,c)} = \frac{\partial}{\partial m} \frac{u(0,m^S,c) - u(0,m^B,c)}{u(1,m^B,c) - u(0,M-m^S,c)}. \]

These equations express the fact that the randomization interval is tangent to the Pareto frontier at both critical points \( S \) and \( B \). For utility functions strictly concave in money, these equations uniquely determine \( m^S \) and \( m^B \).

Let us now derive the inequalities 2 that play a crucial role in our analysis of the second-order condition. We know from the above equalities that

\[ \frac{\partial}{\partial m} \frac{u(1,m^S,c)}{u(1,m^S,c) - u(0,M-m^B,c)} = \frac{\partial}{\partial m} \frac{u(0,m^S,c)}{u(1,m^B,c) - u(0,M-m^S,c)}. \]

Let us show that for \( m' \leq m \) (notice that \( M - m^B \leq m^S \)), the expression

\[ \frac{u_c(1,m,c) - u_c(0,m',c)}{u(1,m,c) - u(0,m',c)} \]

strictly decreases in \( c \). To prove this it is enough to show that

\[ \frac{u_{cm}(1,m,c)}{u_m(1,m,c)} > \frac{u_{cm}(1,m',c)}{u_m(1,m',c)}. \]
for $m' \leq m$. As in the analysis of SOC, let us rewrite the left-hand side as

$$\frac{u_c(1, m, c) - u_c(0, m', c)}{u(1, m, c) - u(0, m', c)} = \frac{[u_c(0, m, c) - u_c(0, m', c)] + [u_c(1, m, c) - u_c(0, m, c)]}{[u(0, m, c) - u(0, m', c)] + [u(1, m, c) - u(0, m, c)]}$$

$$= \frac{[u_c(1, m, c) - u_c(0, m, c)] + \int_{m'}^{m} u_{cm}(0, \tilde{m}, c)\, d\tilde{m}}{[u(1, m, c) - u(0, m, c)] + \int_{m'}^{m} u_{m}(0, \tilde{m}, c)\, d\tilde{m}}.$$

Now, (1) gives

$$\frac{u_c(1, m, c) - u_c(0, m, c)}{u(1, m, c) - u(0, m, c)} > \frac{\partial}{\partial m} u_c(0, m, c),$$

and the constancy of $\frac{u_{cm}(0, \tilde{m}, c)}{u_{m}(0, \tilde{m}, c)}$ in $\tilde{m}$ gives

$$\frac{\partial}{\partial \tilde{m}} u_c(0, \tilde{m}, c) = \frac{\partial}{\partial \tilde{m}} u(0, \tilde{m}, c).$$

Together this demonstrates that $\frac{\partial}{\partial c} u(1, m, c)$ strictly decreases in $c$.

Going back to our initial equality, as we increase $c$ while keeping money levels constant the left-hand side decreases, while the right hand side stays constant. Looking at the graph shows that to balance this out, the critical point shifts towards higher utility of the buyer. And, thus the seller’s money level $m^S$ decreases in this critical point. The analysis of other critical money levels is similar.

**The Analysis of the Example in Section 4.1**

The goal is to explicitly compute the optimum incentive compatible mechanism for an example with $u(0, m, \theta) = \log m$ and $u(1, m, \theta) = \theta + \log m$ for each agent.

Points S and B in figure 1 are defined by the equations

$$\frac{u^B_S(c, v), c}{u^B_S(c, v), c} = \frac{u^B_S(c, v), v}{u^B_S(c, v), v},$$

and

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\[ u_s(1, m^S(c, v), c) = u_s(0, M - m^B(c, v), c) + \frac{u'^B_s(c, v), c}{u'^B_s(c, v), v} \]
\[ \times [u_b(1, m^B(c, v), v) - u_b(0, M - m^S(c, v), v)]. \]

For this example, equation (7) becomes
\[ \frac{m^S(c, v)}{M - m^B(c, v)} = \frac{M - m^S(c, v)}{m^B(c, v)} \]
which implies
\[ m^S(c, v) = M - m^B(c, v). \] (9)

In other words, each player has equal income in each state. For this example, (8) becomes
\[ c + \log(m^S(c, v)) = \log(M - m^B(c, v)) + \frac{m^B(c, v)}{M - m^B(c, v)}[v + \log(m^B(c, v)) - \log(M - m^S(c, v))]. \]

Using (9), we get
\[ c = \frac{m^B(c, v)}{M - m^B(c, v)}v \]

or
\[ m^B(c, v) = \frac{Mc}{c + v} \]

and therefore,
\[ m^S(c, v) = \frac{Mv}{c + v}. \]

Next we use these expressions in the FOCs for truthful reporting (from
likewise for the buyer we have

0 = E_c \left[ \frac{\partial \pi(c, v)}{\partial v} \left( v + \log \left( \frac{Mc}{c + v} \right) \right) + \pi(c, v) \frac{c + v}{Mc} \left( -\frac{Mc}{(c + v)^2} \right) \right] - \frac{\partial \pi(c, v)}{\partial v} \left( v + \log \left( \frac{Mc}{c + v} \right) \right) + \pi(c, v) \frac{c + v}{Mc} \left( -\frac{Mc}{(c + v)^2} \right) \right] = E_c \left[ -\frac{\partial \pi(c, v)}{\partial v} v - \frac{1}{c + v} \right]

so our simultaneous system of partial differential equations is

0 = E_v \left[ \frac{\partial \pi(c, v)}{\partial c} c - \frac{1}{c + v} \right]

0 = E_c \left[ -\frac{\partial \pi(c, v)}{\partial v} v - \frac{1}{c + v} \right]

now, suppose that c, v are distributed independently and uniformly on (a, b) for some b > a ≥ 0. in terms of the notation of lemma 1, we have

\[ \psi(v) = -E_c \frac{1}{c + v} = - \int_a^b \frac{1}{c + v} \, dc = \frac{1}{b - a} \left[ -\log (b + v) + \log (a + v) \right], \]

\[ \phi(c) = E_v \frac{1}{c + v} = \int_a^b \frac{1}{c + v} \, dv = \frac{1}{b - a} \left[ \log (b + c) - \log (a + c) \right], \]
Following the construction in the proof of Lemma 1, we can set
\[ \Delta^s(c, v) = 1, \quad \Delta^b(c, v) = 1. \]

The function \( b(\cdot) \) is now given by the ODE
\[
\psi(v) = E_c \left[ B_1(c, v) \Delta^b(c, v) \right] b'(v) + E_c \left[ B_1(c, v) \frac{\partial}{\partial v} \Delta^b(c, v) + B_2(c, v) \Delta^b(c, v) \right] b(v) \\
= E_c [v] b'(v) = vb'(v),
\]
and thus
\[
b'(v) = \frac{-\log(b + v) + \log(a + v)}{(b - a)v}.
\]

Similarly
\[
s'(c) = \frac{\log(b + c) - \log(a + c)}{(b - a)c}.
\]

The domain on which this solution is valid is determined by the individual rationality requirement and the requirement that \( \pi \) takes values in \([0, 1]\).

**Proof of Lemma 1**

We will develop a constructive procedure to find proper randomization. As a preparation, consider the PDE
\[
S_1(c, v) \frac{\partial}{\partial c} \pi(c, v) + S_2(c, v) \pi(c, v) = 0, \tag{13}
\]
\[
B_1(c, v) \frac{\partial}{\partial v} \pi(c, v) + B_2(c, v) \pi(c, v) = 0, \tag{14}
\]

Considered separately, these equations are standard ODEs. They have solutions, and on a bounded domain we can assume that the solutions are positive. We can thus fix a solution \( \Delta^b > 0 \) to the first equation and a
solution $\Delta^s > 0$ to the second. Consider functions $b(\cdot)$ and $s(\cdot)$, and set

$$
\pi (c, v) = b (v) \Delta^b (c, v) + s (c) \Delta^s (c, v).
$$

Consider the second PDE equation from the lemma, and notice that the first summand above is zero for each $v$, and thus it is zero in expectation. Thus, the second equation reduces to

$$
\psi (v) = E_c \left[ B_1 (c, v) \frac{\partial}{\partial v} \left[ b (v) \Delta^b (c, v) \right] + B_2 (c, v) b (v) \Delta^b (c, v) \right] = E_c \left[ B_1 (c, v) \Delta^b (c, v) \right] b' (v) + E_c \left[ B_1 (c, v) \frac{\partial}{\partial v} \Delta^b (c, v) + B_2 (c, v) \Delta^b (c, v) \right] b (v)
$$

Since, $B_1 \Delta^b > 0$ this equation has solutions. Let $b$ be one such solution satisfying the initial condition $b (v^*) \Delta^b (c^*, v^*) = \frac{1}{2} \pi^*$. Similarly, we can find function $s$ for which the first PDE equation from the lemma is satisfied, and such that $s (c^*) \Delta^s (c^*, v^*) = \frac{1}{2} \pi^*$. Thus, for these two functions $b$ and $s$, the function $\pi$ defined above satisfies the system of PDE from the lemma, as well as the initial condition. The flatness claim of the lemma now follows because $\Delta^s$ and $\Delta^b$ are nearly flat (since the coefficient in front of the derivative is separated from zero, and the other parts of the equation are close to zero), and because $b$ and $s$ are nearly flat (for the same reason). QED

References


