ALGEBRAIC TOPOLOGY QUALIFYING EXAM

Write your answers on the test pages. Show all your work and explain all your reasoning. You may use any result from class or the course notes, as long as you state clearly what result you are using (including its hypotheses). Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 *points.*

Name: ______________________________

Date: June 8, 2017.

1. (10 points) Let T^2 be the real 2-dimensional torus and $f : S^1 \vee S^1 \to T^2$ be a continuous map. Does there exist a continuous map $g : T^2 \to S^1 \vee S^1$ such that $f \circ g$ is the identity map? Justify your answer.

2. (10 points) For $n \geq 2$, let S_n be the space obtained from a regular $(2n)$-gon by identifying the opposite sides with parallel orientations. Calculate the integral homology and cohomology groups of S_n.

3. (10 points) Show that if $f : \mathbb{RP}^{2n} \to X$ is a covering map of a CW-complex X, then f is a homeomorphism.

4. (10 points) Let M be a closed, connected, orientable real n-dimensional manifold and $f : S^n \to M$ a continuous map such that the induced morphism $f_* : H_n(S^n; \mathbb{Z}) \to H_n(M; \mathbb{Z})$ is non-trivial. Calculate $H_k(M; \mathbb{Q})$ for all k.

Differential Topology Qual

June 8, 2017

Write your answers on the test pages. Show all work and explain all reasoning. You may use results from the class or course notes. Please write legibly!! Oh, and don't forget to put your name on the exam.

Problem 1. Prove that $\mathbb{R}P^n$ admits a smooth structure.

Problem 2. Suppose Z is a smooth n-manifold. Suppose $X, Y \subset Z$ are submanifolds of dimensions k and l. What does it mean for X and Y to intersect transversally in Z? Prove that if they do intersect transversally then the intersection $X \cap Y$ is a submanifold of X, and calculate its dimension.

Problem 3. Suppose G is a Lie group. What is a left-invariant vector field on G? Prove that the tangent bundle TG admits a global frame.

Problem 4. Suppose ω is a closed 1-form on a smooth, path-connected manifold M. Prove that if ω is exact then

$$\int_{S^1} f^*\omega = 0$$

for any smooth map $f : S^1 \to M$. Next you will prove the converse. To do so, fix a point $x \in M$. Define a function $g : M \to \mathbb{R}$ by

$$g(y) = \int_\gamma \gamma^*(\omega)$$

where γ is a smooth path in M from x to y.

(a) Prove that g is well-defined; i.e., that $g(y)$ is independent of the path γ.

(b) Prove that $dg = \omega$.