Real Analysis Qualifying Exam

Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.

Question 1. Let $f : [0, 1] \to \mathbb{R}$ be a nonnegative Lebesgue measurable function such that $f > 0$ almost everywhere. Prove that for any $\delta > 0$, there exists $\epsilon > 0$ such that for any Lebesgue measurable subset $S \subset [0, 1]$ with $m(S) > \epsilon$, we have $\int_S f \, dm > \delta$. (Here, m denotes Lebesgue measure.)

Question 2. Let $f : (0, 1) \to \mathbb{R}$ be a Lebesgue integrable function. Define $g : (0, 1) \to \mathbb{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t} \, dm(t).$$

a. Show that for any $a \in (0, 1)$,

$$\int_a^1 g(x) \, dm(x) = \int_a^1 f(t) \, dm(t) - \int_a^1 \frac{a}{t} f(t) \, dm(t).$$

b. Show that g is Lebesgue integrable on $[0, 1]$ and

$$\int_0^1 g(x) \, dm(x) = \int_0^1 f(t) \, dm(t).$$

Question 3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a. State the Riesz Representation Theorem for the dual $(L^p)^*(\mu)$ of $L^p(\mu)$, $1 < p < \infty$.

b. Prove that if $F \in (L^p)^*(\mu)$, then there exists $g \in L^1(\mu)$ such that

$$F(\chi_A) = \int_A g \, d\mu$$

for all $A \in \mathcal{M}$. (Here, χ_A denotes the characteristic function of A.)

Question 4. Let $(X, ||\cdot||)$ be a normed ($\mathbb{R}$-)linear space and let $(X^*, ||\cdot||_{op})$ denote its dual Banach space of (real-valued) bounded linear functions (equipped with the operator norm). Prove that the linear map $\iota : X \to X^{**}$ given by

$$\iota(x)(f) = f(x)$$

is an isometry.

(You may use without proof the fact that for each $x \in X$ there exists $f \in X^*$ such that $||f||_{op} = 1$ and $||x|| = f(x)$.)