Real Analysis Qualifying Exam

Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.

**Question 1.** Let \( f : [0,1] \to \mathbb{R} \) be a nonnegative Lebesgue measurable function such that \( f > 0 \) almost everywhere. Prove that for any \( \delta > 0 \), there exists \( \epsilon > 0 \) such that for any Lebesgue measurable subset \( S \subset [0,1] \) with \( m(S) > \epsilon \), we have \( \int_S f \, dm > \delta \). (Here, \( m \) denotes Lebesgue measure.)

**Question 2.** Let \( f : (0,1) \to \mathbb{R} \) be a Lebesgue integrable function. Define \( g : (0,1) \to \mathbb{R} \) by
\[
g(x) = \int_x^1 \frac{f(t)}{t} \, dm(t).
\]
a. Show that for any \( a \in (0,1) \),
\[
\int_a^1 g(x) \, dm(x) = \int_a^1 f(t) \, dm(t) - \int_a^1 \frac{a}{t} f(t) \, dm(t).
\]
b. Show that \( g \) is Lebesgue integrable on \([0,1]\) and
\[
\int_0^1 g(x) \, dm(x) = \int_0^1 f(t) \, dm(t).
\]

**Question 3.** Let \((X, \mathcal{M}, \mu)\) be a finite measure space.

a. State the Riesz Representation Theorem for the dual \((L^p)^*\) of \(L^p\), \(1 < p < \infty\).

b. Prove that if \( F \in (L^p)^*(\mu) \), then there exists \( g \in L^1(\mu) \) such that
\[
F(\chi_A) = \int_A g \, d\mu
\]
for all \( A \in \mathcal{M} \). (Here, \( \chi_A \) denotes the characteristic function of \( A \).)
Question 4. Let \((X, \| \cdot \|)\) be a normed (\(\mathbb{R}\)-)linear space and let \((X^*, \| \cdot \|_{op})\) denote its dual Banach space of (real-valued) bounded linear functions (equipped with the operator norm). Prove that the linear map \(i : X \to X^{**}\) given by
\[
i(x)(f) = f(x)
\]
is an isometry.

(You may use without proof the fact that for each \(x \in X\) there exists \(f \in X^*\) such that \(\|f\|_{op} = 1\) and \(\|x\| = f(x)\).)