COMPLEX ANALYSIS QUALIFYING EXAM

Show all your work and explain all your reasoning. You may use any standard results, as long as you state clearly what results you are using. Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 *points.*

1. (10 points) Let f be a holomorphic function in the punctured unit disk $D\backslash\{0\}$. Suppose

$$|f(z)| \leq \frac{1}{\sqrt{|z|}}$$

for all $z \in D \setminus \{0\}$. Is the singularity of f at 0 removable, a pole, or essential? Justify your answer.

2. (10 points) Evaluate the integral

$$\int_0^\infty \frac{x^2 dx}{1+x^4}.$$

3. (10 points) Let U be an open, convex subset in $\mathbb{C}$. Suppose f is a holomorphic function in U such that $\mathrm{Re}(f'(z)) > 0$ for all $z \in U$. Prove that f is one-to-one.

4. (10 points) Let ω be a meromorphic one-form on the Riemann sphere $\mathbb{P}^1$. Namely, ω has local expressions $\{g(z)dz\}$ that are compatible with change of coordinates on $\mathbb{P}^1$, where the $g(z)$ are local meromorphic functions. Define the residue of ω at $p \in \mathbb{P}^1$ to be

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i}\int_\gamma \omega$$

for a small loop γ going around p counterclockwise such that inside γ the only possible singularity of ω is p. Prove that $\mathrm{Res}_p(\omega) = 0$ for all $p \in \mathbb{P}^1$ if and only if there exists a meromorphic function f on $\mathbb{P}^1$ such that $\omega = df$.

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