Show all your work and explain all your reasoning. You may use any standard results, as long as you state clearly what results you are using. Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 points.

1. (10 points) Let \( f \) be a holomorphic function in the punctured unit disk \( D \setminus \{0\} \). Suppose
\[
|f(z)| \leq \frac{1}{\sqrt{|z|}}
\]
for all \( z \in D \setminus \{0\} \). Is the singularity of \( f \) at 0 removable, a pole, or essential? Justify your answer.

2. (10 points) Evaluate the integral
\[
\int_0^\infty \frac{x^2}{1+x^4} dx.
\]

3. (10 points) Let \( U \) be an open, convex subset in \( \mathbb{C} \). Suppose \( f \) is a holomorphic function in \( U \) such that \( \text{Re}(f'(z)) > 0 \) for all \( z \in U \). Prove that \( f \) is one-to-one.

4. (10 points) Let \( \omega \) be a meromorphic one-form on the Riemann sphere \( \mathbb{P}^1 \). Namely, \( \omega \) has local expressions \( \{g(z)dz\} \) that are compatible with change of coordinates on \( \mathbb{P}^1 \), where the \( g(z) \) are local meromorphic functions. Define the residue of \( \omega \) at \( p \in \mathbb{P}^1 \) to be
\[
\text{Res}_p(\omega) = \frac{1}{2\pi i} \int_\gamma \omega
\]
for a small loop \( \gamma \) going around \( p \) counterclockwise such that inside \( \gamma \) the only possible singularity of \( \omega \) is \( p \). Prove that \( \text{Res}_p(\omega) = 0 \) for all \( p \in \mathbb{P}^1 \) if and only if there exists a meromorphic function \( f \) on \( \mathbb{P}^1 \) such that \( \omega = df \).

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