REAL ANALYSIS QUALIFYING EXAM

JUNE 2018

Answer all 4 questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Exercise 1. Let $f \in L^1(\mathbb{R}, m)$ be Lebesgue integrable and let $F(x) = \int_{-\infty}^{x} f(t)dt$.

(1) Show that F is continuous.
(2) For any $a > 0$, compute $\int_{-\infty}^{\infty} e^{-x^2} \cos(ax)dx$ and show that it is $\sqrt{\pi}e^{-\frac{a^2}{4}}$

Exercise 2.

(1) State the Radon-Nikodym theorem for finite positive measures.
(2) Let ρ, ν, and μ denote finite positive measures such that ν is absolutely continuous with respect to μ with Radon-Nikodym derivative f and ρ is absolutely continuous with respect to ν with Radon-Nikodym derivative g. Show that ρ is absolutely continuous with respect to μ with Radon-Nikodym derivative fg.

Exercise 3. Let X, Y be Banach spaces and let $\{T_n\}$ be a sequence in $L(X, Y)$ such that $\lim_{n\to\infty} T(x)$ exists for every $x \in X$. Show that $T : X \to Y$ defined by this limit is a bounded linear operator.

Exercise 4.

(1) Prove the uniqueness of a (left-invariant) Haar measure on a locally compact hausdorff topological group.
(2) Prove that Haar measure for a compact group or abelian group is both left and right invariant.

COMPLEX ANALYSIS QUALIFYING EXAM

Write your answers on the test pages. Show all your work and explain all your reasoning. You may use any standard result, as long as you state clearly what result you are using (including its hypotheses). Exception: you may not use a result which is the same as the problem you are being asked to do. Each problem has a noted value, in total 40 *points.*

Name: ____

1. (10 points) Prove that there is no function f such that f is analytic on the punctured unit disk $D \setminus \{0\}$ and that f' has a simple pole at 0.

2. (10 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^2}.$$

3. (10 points) Let f be an entire function such that $f(z+m+ni)=f(z)$ for all $z \in \mathbb{C}$ and all $m, n \in \mathbb{Z}$. Prove that f is constant.

4. (10 points) Let ω be a meromorphic one-form on the Riemann sphere, i.e. locally $\omega = f(z)dz$, where f is a meromorphic function of the local coordinate z and is compatible with change of coordinates.

(a) If ω has a unique pole, prove that the residue of ω at the pole is zero.

(b) If ω has a unique zero and two poles, prove that the residue of ω at each pole is nonzero.