ALGEBRA QUALIFYING EXAM SPRING 2018

Exercise 1. Let $q > 2$ be a prime and let $K/\mathbb{Q}$ denote the splitting field of the cyclotomic polynomial $\Phi_q(x) = x^{q-1} + x^{q-2} + \ldots + 1$ then $\mathcal{O}_K = \mathbb{Z}[\zeta_q]$ is its ring of integers (you do not need to prove this).

(1) Find all the primes $p \in \mathbb{Z}$ for which $p\mathcal{O}_K$ is ramified.
(2) For each of those primes compute the decomposition of $p\mathcal{O}_K$ into prime ideals (i.e., the number of prime ideals with multiplicities and the cardinality of the corresponding quotient rings).

[Hint: You may use without proof that the discriminant of Φ_q is $\Delta_q = (-1)^{\frac{q-1}{2}} q^{q-2}$.]

Exercise 2. Let R be an integral domain that is integrally closed in its field of fraction F.

(1) Show that an algebraic α is integral over R if and only if its minimal polynomial over F is a monic polynomial in $R[x]$.
(2) Show that for any monic $f(x) \in R[x]$, for any decomposition $f(x) = f_1(x)f_2(x)$ into monic polynomials in $F[x]$, the factors f_1, f_2 have coefficients in R.

Exercise 3. Let k be an algebraically closed field. Consider the affine variety $V = k^2$ (with coordinates x, y), and the affine variety $W = k^2$ (with coordinates s, t). Suppose $\varphi : V \to W$ is a morphism, and denote by $R \subseteq k[x, y]$ the image of the induced ring homomorphism $\tilde{\varphi} : k[s, t] \to k[x, y]$. For each of the following statements, give a proof or a counterexample.

(1) If φ has Zariski dense image, then φ is surjective.
(2) If $k[x, y]/R$ is an integral extension of rings, then φ is surjective.

Exercise 4. Consider the following situation: E/F is a Galois extension of degree 4 and K/F is a degree 5 extension that is **not** Galois such that the compositum KE/F is Galois. Either prove that such a situation is impossible, or give an example of such F, E, K and prove that your example works.

Exercise 5. Let $R = \mathbb{R}[x, y]$ and $M = \mathbb{R}[s, t]$ be an R-module via $\mathbb{R}$-algebra homomorphism $\phi\colon R \to M$ given by $\phi(x) = s$ and $\phi(y) = st$. Compute $\operatorname{Tor}_i^R(M, R/(x, y))$ for all i. Is M a flat R-module?

Exercise 6. Let R be a ring with some maximal ideal I satisfying $I^n = (0)$ for some integer $n \geq 1$. Suppose M is a finitely generated flat R-module. Prove that R is free.

Exercise 7. Suppose k be a field and $R = k[x, y, z]$ a polynomial ring. Compute

$$\operatorname{Ext}^i_R\big(R/(xz), R/(xy, xz)\big)$$

for all $i \geq 0$.

Exercise 8. Suppose p is a prime of the form $4k + 3$. Find the conjugacy class of every element of order 4 in $\mathrm{GL}_2(\mathbb{F}_p)$.