Exercise 1. Suppose $p$ is a prime. Show that the Galois group of $x^5 - 1 \in \mathbb{F}_p[x]$ depends only on $p \pmod{5}$, and compute it for each congruence class of $p \pmod{5}$.

Exercise 2. Let $R$ be a Dedekind domain with field of fractions $K$. Show that for any two proper fractional ideals $I, J$ there are $\alpha, \beta \in K$ with $\alpha I, \beta J \subseteq R$ integral and $\alpha I + \beta J = R$.

Exercise 3. Suppose that $R$ is a Noetherian ring and $p \subseteq R$ is a prime ideal such that $R_p$ is an integral domain. Show that there is an $f \in R \setminus p$ such that $R_f$ is an integral domain where $R_f = S^{-1}R$ with $S = \{1, f, f^2, f^3, \ldots\}$.

Exercise 4. Let $k$ be an algebraically closed field. Consider the affine variety $V = k^2$ (with coordinates $x, y$), and the affine variety $W = k^2$ (with coordinates $s, t$). Suppose $\varphi : V \to W$ is a morphism, and denote by $R \subseteq k[x, y]$ the image of the induced ring homomorphism $\hat{\varphi} : k[s, t] \to k[x, y]$. For each of the following statements, give a proof or a counterexample.

1. If $\varphi$ has Zariski dense image, then $\varphi$ is surjective.
2. If $k[x, y]/R$ is an integral extension of rings, then $\varphi$ is surjective.

Exercise 5. For every integer $n \geq 2$, do the following. Find all the primes $p$ such that $\text{GL}_n(\mathbb{Q})$ contains an element of order $p$; and describe the rational canonical form of every element of order $p$ in $\text{GL}_n(\mathbb{Q})$.

Exercise 6. Let $R$ be a commutative ring. Suppose $M$ is a projective $R$-module. Prove that $M$ is flat.

Exercise 7. Let $R = \mathbb{Q}[x, y]$ be a polynomial ring and $M = \mathbb{Q}[s, t]$ be an $R$-module via $\mathbb{Q}$-algebra homomorphism $\phi : R \to M$ given by $\phi(x) = s$ and $\phi(y) = st$. Compute $\text{Tor}_i^R(M, \mathbb{R}/(x, y))$ and $\text{Ext}_i^R(M, \mathbb{R}/(x, y))$ for all integers $i \geq 0$.

Exercise 8. Let $R$ be a commutative ring with an ideal $I$ satisfying $I^n = (0)$ for some integer $n \geq 1$. Let $f : M \to N$ be an $R$-module homomorphism such that the induced homomorphism

$$\overline{f} : M/IM \to N/IN$$

is surjective. Prove that $f$ is surjective.