ALGEBRA QUALIFYING EXAM FALL 2018

Exercise 1. Suppose p is a prime. Show that the Galois group of $x^5 - 1 \in \mathbb{F}_p[x]$ depends only on $p \pmod 5$, and compute it for each congruence class of $p \pmod 5$.

Exercise 2. Let R be a Dedekind domain with field of fractions K Show that for any two proper fractional ideals I, J there are $\alpha, \beta \in K$ with $\alpha I, \beta J \subseteq R$ integral and $\alpha I + \beta J = R$.

Exercise 3. Suppose that R is a Noetherian ring and $\mathfrak{p} \subseteq R$ is a prime ideal such that $R_{\mathfrak{p}}$ is an integral domain. Show that there is an $f \in R \setminus \mathfrak{p}$ such that R_f is an integral domain where $R_f = S^{-1}R$ with $S = \{1, f, f^2, f^3, \ldots\}$.

Exercise 4. Let k be an algebraically closed field. Consider the affine variety $V = k^2$ (with coordinates x, y), and the affine variety $W = k^2$ (with coordinates s, t). Suppose $\varphi : V \to W$ is a morphism, and denote by $R \subseteq k[x, y]$ the image of the induced ring homomorphism $\tilde{\varphi} : k[s, t] \to k[x, y]$. For each of the following statements, give a proof or a counterexample.

(1) If φ has Zariski dense image, then φ is surjective.
(2) If $k[x, y]/R$ is an integral extension of rings, then φ is surjective.

Exercise 5. For every integer $n \geq 2$, do the following. Find all the primes p such that $\mathrm{GL}_n(\mathbb{Q})$ contains an element of order p; and describe the rational canonical form of every element of order p in $\mathrm{GL}_n(\mathbb{Q})$.

Exercise 6. Let R be a commutative ring. Suppose M is a projective R-module. Prove that M is flat.

Exercise 7. Let $R = \mathbb{Q}[x, y]$ be a polynomial ring and $M = \mathbb{Q}[s, t]$ be an R-module via $\mathbb{Q}$-algebra homomorphism $\phi \colon R \to M$ given by $\phi(x) = s$ and $\phi(y) = st$. Compute $\mathrm{Tor}_i^R(M, R/(x, y))$ and $\mathrm{Ext}_R^i(M, R/(x, y))$ for all integers $i \geq 0$.

Exercise 8. Let R be a commutative ring with an ideal I satisfying $I^n = (0)$ for some integer $n \geq 1$. Let $f \colon M \to N$ be an R-module homomorphism such that the induced homomorphism

$$\overline{f} \colon M/IM \to N/IN$$

is surjective. Prove that f is surjective.