Problem 1. Prove that an Artinian ring has finitely many maximal ideals.

Problem 2. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Consider the subgroup

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}^\times, \ b \in \mathbb{F} \right\} < \text{GL}_2(\mathbb{F}).$$

Show that for any prime $p$ dividing $q - 1$, the number of Sylow $p$-subgroups of $G$ is $q$.

Problem 3. Let $R$ be a UFD and $a, b$ be coprime elements in $R$. For all $i \geq 0$, compute $\text{Tor}^{R/(ab)}_i(R/(a), R/(b))$.

Problem 4. Let $F$ be a field, and $D$ be an integral domain containing $F$. Suppose $D$ is finite dimensional as a vector space over $F$. For each $x \in D$, define the $F$-linear transformation $T_x : D \to D$ by $T_x(y) = xy$.

(a) Prove that $D$ is a field.

(b) Suppose $p = \text{char}(F) > 0$ and $\alpha \in D$ is purely inseparable over $F$. This means that the minimal polynomial of $\alpha$ over $F$ is $T^p e - r$ for some $r \in F$ and $e \geq 1$. Describe the Jordan canonical form of $T_\alpha$ over the algebraic closure of $F$.

Problem 5. Let $K$ be a field of characteristic $p > 0$ and $F = K(t)$ where $t$ is a variable. Let $f(x) = x^{2p} - tx^p + t \in F[x]$.

(a) Show that $f(x)$ is irreducible in $F[x]$.

(b) Let $E = F[s]$ where $s$ is a root of the polynomial $(x^p - t) \in F[x]$. If $L$ is the splitting field of $f(x)$ over $E$, show that $[L : E] \leq 2$.

(c) Show that $L = F[\alpha]$, where $\alpha$ is a root of $f(x)$.

Problem 6. Prove that a flat finitely-generated module over a Noetherian local ring is free.

Problem 7. Let $p$ be a prime integer, and $q$ be a power of $p$. Let $\mathbb{F}_q$ be the finite field with $q$ elements, and $\mathbb{F}_{q^n}$ be the degree $n$ extension of $\mathbb{F}_q$. Consider the map $N : \mathbb{F}_{q^n} \to \mathbb{F}_q$ defined by $N(x) = x^{1+q+\cdots+q^{n-1}}$.

(a) Prove that $N$ is surjective. (Hint: Recall that $\mathbb{F}_{q^n}^*$ is a cyclic group of order $q^n - 1$.)

(b) Prove that $N^{-1}(1)$ spans $\mathbb{F}_{q^n}$ as an $\mathbb{F}_q$-vector space.

Problem 8. Suppose $k$ is a field. Let $R = k[s^4, s^3t, st^3, t^4] \subset k[s, t]$.

(1) Compute the Krull dimension of $R$.

(2) Prove that $R$ is not Cohen-Macaulay. (Hint: Consider $R/s^4R$.)