1. Let $G$ be a group of order 108. Show that $G$ has a normal subgroup of order 9 or 27.

2. Let $R$ be a ring, and let $D$ be the set of all $x \in R$ such that $x$ is a zero divisor or $x = 0$. Show that $D$ is a union of prime ideals. (Hint: consider the set $\Sigma$ of all ideals contained in $D$. Show that $\Sigma$ contains maximal elements and every maximal element of $\Sigma$ is prime.)

3. a) Suppose that $V$ is a finite dimensional vector space over a field $F$ and $T \in \text{End}_F(V)$. Show that the characteristic polynomial of $T$ is irreducible over $F$ if and only if $V$ has no nontrivial proper $T$-invariant subspaces.

b) Let $V$ be a 3-dimensional vector space over $\mathbb{F}_5$, the field with 5 elements. Give an example of a linear transformation of $V$ that does not have a proper $T$-invariant subspace.

4. Let $p$ be a prime number and let $F$ be a field of characteristic 0. Suppose that every finite extension of $F$ has degree divisible by $p$. Show that in fact every finite extension of $F$ has degree a power of $p$.

5. Let $R$ be a local noetherian ring with maximal ideal $M$, let $A$ be a finitely generated nonzero $R$-module, and set $k = R/M$. Prove: $0 < \dim_k(k \otimes_R A) < \infty$.

6. a) Show $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

b) Is $\mathbb{Q}/\mathbb{Z}$ a projective $\mathbb{Z}$-module? Prove your answer.

7. Let $k$ be a field, $R = k[x, y, z]$ a polynomial ring.

Set $P_1 = (x, y), P_2 = (x, z), M = (x, y, z), I = P_1P_2$.

a) Prove that $M^2$ is a primary ideal in $R$.

b) Prove that $I = P_1 \cap P_2 \cap M^2$ is a minimal primary decomposition of $I$.

8. Let $k$ be a field, and $B$ a finitely generated $k$-algebra. Suppose $B$ is a field. Prove that $\dim_k(B)$ is finite.