1. Let $P$ be a finite $p$-group. Show that $P$ is not cyclic if and only if $P$ has a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

2. Let $R$ be a commutative ring with unity.
   (a) Let $S$ be a non-empty saturated multiplicative set in $R$, i.e. if $a, b \in R$, then $ab \in S$ if and only if $a, b \in S$. Show that $R - S$ (the complement of $S$ in $R$) is a union of prime ideals.
   (b) Suppose that $R$ is a domain such that every nonzero prime ideal in $R$ contains a prime element. Show that $R$ is a UFD. Hint: Use part (a). (Remark: the converse is also true, but not part of this problem.)

3. (a) Show that every $A$ in the group $GL_N(\mathbb{C})$ that is of finite order is conjugate to a diagonal matrix.
   (b) If $F$ is an algebraically closed field and $A \in GL_N(F)$ is of finite order, is $A$ always conjugate to a diagonal matrix? Why or why not?

4. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 4$ and let $K$ be a splitting field for $f(x)$ over $\mathbb{Q}$. Suppose that the Galois group $\text{Gal}(K/\mathbb{Q})$ is $S_n$. If $\alpha \in K$ is a root of $f(x)$, show that $\alpha^n \notin \mathbb{Q}$.

5. Let $k$ be a field and $k[x, y]$ the polynomial ring in two variables. Let $I$ be the principle ideal generated by $x^2 - y^2(1 + y)$.
   (a) Show that $R = k[x, y]/I$ is an integral domain.
   (b) Describe the integral closure $A$ of $R$ in its field of fractions $F$ explicitly by giving one or more elements of $F$ that generate $A$ over $R$, and prove your answer.

6. Let $A, B$ be two finitely generated $\mathbb{Z}$-modules. Prove that $\text{Tor}_n^\mathbb{Z}(A, B) = 0$ if $n > 1$.

7. Let $V$ be a vector space over a field $k$ and let $v_1, \ldots, v_n$ be linearly independent vectors in $V$. Let $p \geq 2$, and $w \in \wedge^p(V)$. Prove that $v_1 \wedge \cdots \wedge v_n \wedge w = 0$ if and only if there exist $y_1, \ldots, y_n \in \wedge^{p-1}V$ such that $w = \sum_{i=1}^n v_i \wedge y_i$.

8. Let $R$ be an integral domain, let $m$ be a maximal ideal in $R$ and set $k = R/m$. Let $P$ and $Q$ be finitely generated $R$-modules and $\phi : P \to Q$ a map of $R$-modules. Suppose that the induced map $P/mP \to Q/mQ$ is a surjection of $k$-vector spaces. Prove that there is an element $f \in R$, whose image in $k$ is nonzero, such that the map $\phi_f : P_f \to Q_f$ is a surjection of $R_f$-modules. (The subscript $f$ denotes localization at the multiplicative set $\{1, f, f^2, f^3, \ldots \}$.)