AFFIRMATIVE ACTION WITH OVERLAPPING RESERVES

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ABSTRACT. In a wide variety of real-life resource allocation problems such as school choice or assignment of public positions, implementation of affirmative action policies rely on choice rules that balance meritocracy with equity. We study choice rules where meritocracy is attained through reliance on a priority list, and equity is attained through reserved positions for target groups of disadvantaged individuals. Focusing on overlapping reserves, the case where an individual can belong to multiple types of reserved positions, we characterize choice rules that satisfy maximal compliance with reservations, elimination of justified envy, and non-wastefulness. When an individual accommodates only one of the reserved positions, the horizontal envelope choice rule is the only rule to satisfy these three axioms. When an individual accommodates each of the reserved positions she qualifies for, there are complementarities between individuals. Under this alternative convention, and assuming there are only two target groups, such as women and minorities, we show that paired-admission choice rules are the only ones to satisfy the three axioms. Building on these results, we provide improved allocation mechanisms for school choice in Chile, public position allocation in India, and college admissions in Brazil.

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1. Introduction

While affirmative action remains to be a highly contested topic in the U.S., it has been gaining widespread acceptance in much of the world, as public awareness has been increasing on inequalities faced by various disadvantaged groups. As a result, affirmative action systems have been adopted worldwide in a wide variety of resource allocation problems, such as allocation of public school seats, government positions, and legislative seats. Perhaps the most widely used class of affirmative action systems relies on reserving a certain fraction of seats for the members of the target beneficiary groups. In some applications, all items are identical, such as the allocation of legislative seats or decentralized admissions to a single university. In these applications, the outcome is determined through an (often rigorously defined) choice rule, which in general depends on the number of reserved seats for each target group, as well as other criteria such as a merit ranking of candidates (in applications such as allocation of school seats or assignment of government positions) or the number of votes received by the candidates (in applications such as allocation of legislative seats). These choice rules essentially capture those policies which define the “property rights” over these homogenous indivisible goods. Hence, the design of these choice rules can be viewed as an engineering exercise for the policies that govern allocation of these resources. In this paper, our focus is both the design and the analysis of these choice rules, as well as some of the direct policy implications of our findings on three large-scale real-life applications from Chile, India, and Brazil.

While we consider both allocation of homogenous goods as well as the more general case of heterogenous goods, the starting point of our analysis is the former of the two. We consider allocation problems where the two main elements of an affirmative action system are, (1) an exogenous priority ranking of individuals, which would have solely dictated the choice of selected individuals in the absence of affirmative action, and (2) a vector of reserves (or lower quotas) for each target group of individuals. Importantly, and deviating from much of the prior literature with a few exceptions, we allow for the general case where an individual can belong to multiple reserve-eligible target groups. That is, we consider the case of overlapping reserves, which has been becoming increasingly more relevant in real-life applications. For example, in Jordan and Pakistan there are reserves both for women and also for minorities for the seat allocation at their national assemblies \cite{Htun2004}. All of the three applications we present in Section 6 from Chile, India, and Brazil also have this feature. In those applications, an individual who belongs to two target beneficiary groups,

\footnote{In Jordanian House of representatives, of the 110 seats 6 are reserved for women and 12 are reserved for Christians and Chechens/Circassians. In Pakistani National Assembly, of the 342 seats 60 are reserved for women and 10 are reserved for minorities \cite{Htun2004}.}
for example a minority woman, can benefit both from the women reserve and also from the minority reserve.

Our approach is axiomatic, and throughout the paper we focus on choice rules that satisfy three basic properties (or axioms), each of which we believe is highly plausible in this framework. These three properties, which play a central role in our theoretical analysis, are as follows:

1. **Maximal compliance with reservations**: As many of the reserved seats as possible are to be allocated to qualified candidates from the target groups.
2. **Elimination of justified envy**: A lower-priority candidate is not to be assigned a seat at the expense of a higher-priority candidate, unless doing so enables channelling strictly higher number of the reserved seats to qualified candidates.
3. **Non-wastefulness**: All seats are to be allocated, to the extent there are sufficiently many candidates.

One distinction between various real-life implementations of overlapping reserves becomes critical for our modeling and analysis. Consider an individual who is a member of multiple reserve-eligible beneficiary groups. A key policy consideration is, whether an admission granted to this individual should count towards accommodating each one of the reserves she qualifies for, or only one of them. For example, suppose there is one seat reserved for women candidates and one seat reserved for minority candidates. When a minority woman candidate is admitted, should her admission count towards accommodating both the women reserve as well as the minority reserve, or only one of them? Essentially, the distinction here is about the accounting conventions for enforcing reserves, and both conventions are observed in real-life applications. We refer to these two conventions as **one-to-all reserve matching** and **one-to-one reserve matching** respectively.

While the choice of convention in enforcing reserves often depends on the specific application and it can be viewed as a policy decision, one of our main contributions is showing that it has important implications on the design. And while either accounting convention brings its own challenges into the design, adoption of the one-to-one reserve matching convention results in a much cleaner solution: We show in Theorem 1 that, under this convention the **horizontal envelope choice rule** introduced in Section 3 is the unique choice rule that satisfies non-wastefulness, maximal compliance with reservations, and elimination of justified envy. Before elaborating on the intuition and details of this choice rule, it will be helpful to outline the challenges either convention brings into design.

Consider one-to-all reserve matching, under which an individual upon admission counts towards each of the reserves she qualifies for. For example, admission of a minority female would count towards accommodating both the women reserves and the minority reserves under this convention. The resulting challenge is well-known in the literature: Adoption of
this convention introduces *complementarities* between various groups of individuals, which in general means admission of one of the individuals may hinge upon admission of another individual (see, for example, [Abdulkadiroğlu (2005)](https://doi.org/10.1257/aer.95.1.55)). For example there are complementarities between majority males and minority females when there are reserved seats both for women candidates and for minority candidates, and thus the admission of a majority male candidate may hinge upon the admission of a minority female candidate. Presence of complementarities is one of the main challenges in market design, and it virtually rules out the design of a well-behaved choice rule. Despite this challenge, in Section 4, we provide a full characterization of choice rules that satisfy our three axioms when the complementarities are in its most basic form restricted to only two types of individuals. Analysis of this special case is still of interest, because in many practical applications there are only two reserve-eligible groups (such as women and minorities). In other applications, there may be complementarities between only two groups of individuals, even if there are more than two reserve-eligible groups. Our analysis in Section 6.3 of the 2012 *Law of Social Quotas* from Brazil is about one of these applications.

Next consider one-to-one reserve matching, under which an individual upon admission counts towards accommodating only one of her eligible reserves. The resulting challenge under this alternative convention is technical, as it introduces an additional dimension to the allocation problem. Under this convention, a solution is to determine not only who receives the reserved seats, but also which type of a reserved seat an individual is to receive if she qualifies for multiple types. In other words, under this convention of enforcing reserves, there is a matching problem of individuals to different types of reserves, within the matching problem of individuals to positions. Fortunately, there is a clean solution to this more elaborate matching problem. We next provide an intuition for this solution.

Consider a small gathering where one of the two guests, Violet is a vegetarian, whereas the second guest Frank is flexible in his diet. Suppose there is one vegetable sandwich and one chicken sandwich prepared for the guests. If Frank is served his lunch prior to Violet, it would be a bad idea giving him the only vegetable sandwich, since that would mean Violet will have to skip the lunch. Frank is flexible in his diet, whereas Violet is not, and serving the vegetable sandwich to Frank results in giving up this valuable flexibility. A more careful server would have planned ahead taking into Violet’s dietary restriction into consideration, and thus would have offered Frank the chicken sandwich utilizing his flexible diet. Now consider allocation of public school seats to students. Suppose there is a female applicant Frida, a disabled female candidate Diana, one reserved seat for female applicants, and a second reserved seat for disabled applicants. Just as it is not plausible to

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2Interestingly, Hughes (2011) provides empirical evidence from some countries where majority male candidates are supportive of minority female candidates, since it helps increasing their representation.

3Notable exceptions are Ostrovsky (2008) and Alva (2019).
serve the vegetable sandwich to Frank in the above scenario, it is not plausible to assign Diana the reserved seat for females and deny Frida a seat since she is not qualified for the reserved seat for disabled candidates. Both candidates can receive reserved seats if Diana is instead assigned the reserved seat for disabled candidates. Ironically, many real-life applications do not utilize the flexibility generated by candidates who qualify for multiple types of reserves, and thus suffer from the very shortcoming we illustrate in these scenarios. For example, as we present in Section 6.1, the brand new school choice system in Chile suffers from this exact shortcoming, precisely for the reason we illustrate here. A better design would not give up the flexibility generated by the applicants who qualify for multiple reserves, and instead capitalize on it. This is the basic idea under our proposed horizontal envelope choice rule. We argue that this choice rule is the only plausible choice rule under the one-to-one reserve matching convention, for it is the unique choice rule that satisfies fairly basic desiderata in this framework.

When the allocation problem involves heterogenous goods, the design needs to be more involved, because not only the system has to determine the set of recipients of the set of available items, but it also has to determine which item each recipient receives. Over the last decade and a half, a mechanism that relies on the celebrated agent-proposing deferred acceptance algorithm \(^4\) (Gale and Shapley, 1962) has gained widespread acceptance worldwide as the mechanism of choice for this purpose in a wide variety of applications, including those that include affirmative action. Hence it is natural to rely on the same approach in this framework as well. Analysis of the more general version of the problem with heterogeneous goods reveals another advantage of the accounting convention one-to-one reserve matching, compared to one-to-all reserve matching. The agent-proposing deferred acceptance algorithm is well-defined and well-behaved only if the choice rule used by each institution satisfies a technical condition known as the substitutes condition. While the horizontal envelope choice rule satisfies the substitutes condition (Proposition 4), there is no choice rule that satisfies our three properties along with the substitutes condition under the one-to-all reserve matching convention (Proposition 5).

The following summary of our main theoretical results is helpful to compare and contrast the two alternative conventions for enforcing reserves.

1) Under one-to-one reserve matching:

(a) There is a unique choice rule, namely the horizontal envelope choice rule, that satisfies maximal compliance with reservations, elimination of justified envy, and non-wastefulness.

\(^4\) This approach started with the theoretical modeling and analysis of school choice mechanisms in Abdulkadiro˘glu and Sönmez (2003), and it was first put into practice with the subsequent school choice reforms in New York City (Abdulkadiro˘glu et al., 2005b) and Boston (Abdulkadiro˘glu et al., 2005a).
(b) The horizontal envelope choice rule satisfies the substitutes condition, and it can be jointly implemented with the agent-proposing deferred acceptance algorithm when items are heterogeneous.

(2) Under one-to-all reserve matching, and assuming there are only two target beneficiary groups:

(a) A choice rule satisfies maximal compliance with reservations, elimination of justified envy, and non-wastefulness if and only if it is a selection from the paired-admission choice correspondence.

(b) Presence of complementarities is reflected in the structure of the selections from the paired-admission choice correspondence: This class of choice rules has an optimal selection for individuals who are either members of both target groups or a member of neither target group. This selection is also pessimal for individuals who are members of only one of the target groups. Similarly, there is an optimal selection for individuals who are members of only one of the target groups, which is also the pessimal selection for individuals who are either members of both target groups or a member of neither target group.

(c) There is no selection from the paired-admission choice correspondence that satisfies the substitutes condition. Therefore, of the choice rules that satisfy non-wastefulness, maximal compliance with reservations, and elimination of justified envy, none of them can be jointly implemented with the agent-proposing deferred acceptance algorithm when items are heterogeneous.

Our results have direct policy implications for several real-life applications of affirmative action. Some of these policy implications, presented in Section 6, include:

1. Bringing into light two shortcomings of the recently designed school choice mechanism for Chile, as well as the design an improved mechanism that complies with the 2015 School Inclusion Law,

2. bringing into light two shortcomings of a Supreme Court-mandated affirmative action procedure from India, as well as it’s unique improvement that satisfies our three axioms, and

3. the provision of a class of college admissions mechanisms for Brazil that complies with The Law of Social Quotas, without imposing any additional artificial restrictions beyond those mandated by the law.

1.1. Related Literature. There is a large literature on market design under various classes of distributional constraints such as minimum guarantee reserves (or lower quotas), upper
What differentiates our paper from others is the following four features:

(1) Our focus is the analysis and design of affirmative action systems under overlapping reserves, which is a version of the problem that has not been well-understood or analyzed.

(2) In previous studies which adopted the one-to-one reserve matching convention, individuals are either assumed to have strict preferences between reserved seats of different types (as in Aygün and Turhan (2016) and Kurata et al. (2017)) or their indifferences are broken through fixed tie-breaking rules (as in Baswana et al. (2018) and Correa et al. (2019)). To the best of our knowledge, our proposed horizontal envelope choice rule is the first choice rule that utilizes the flexibility in reserve matching for an improved design.

(3) Relying on three intuitive axioms, we provide full axiomatic characterizations of choice rules for both of the reserve matching conventions considered in our paper. To the best of our knowledge, these are the first axiomatic characterizations under overlapping reserves.

(4) Our results have immediate policy implications for large scale applications from Chile, India, and Brazil.

The one paper that is closest to our study in its objective is Kurata et al. (2017). As in our paper, Kurata et al. (2017) also consider affirmative action with overlapping reserves, and due to possible non-existence of stable matchings under one-to-all reserve matching convention, they propose the one-to-one matching convention. This is where our approach diverges with Kurata et al. (2017). In order to invoke the analysis of Hatfield and Milgrom (2005) matching with contracts model, they assume that individuals have strict preferences between reserved seats of different types. This assumption together with the one-to-one matching convention assures that their model becomes a special case of the matching with contracts model, and all positive results follow directly. There is one footnote in Kurata et al. (2017) which states

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6 Earlier characterizations when each individual is eligible for only one reserve include Echenique and Yenmez (2015), Chambers and Yenmez (2018), and Doğan et al. (2019).

7 Their notion of stability is equivalent to elimination of justified envy together with non-wastefulness under the one-to-all reserve matching convention.
If a student is indifferent between different seats of the same school, we can break the ties arbitrarily to form a strict preference and apply our mechanism, which is still strategy-proof and obtains a stable matching. However, we can no longer guarantee student-optimality.

Indeed their suggestion of breaking ties arbitrarily to form a strict preference and applying the resulting choice rule in conjunction with agent-proposing deferred acceptance was followed by the research group that have recently designed the Chilean school choice mechanism. What is misleading in their footnote is that, their stability notion does not allow for schools to utilize their flexibility to move qualified individuals between different types of reserves to accommodate either higher priority students or to fill higher number of reserved seats. So while their stability notion is equivalent to the combination of elimination of justified envy and non-wastefulness for the one-to-all reserve matching convention, it is a weaker notion under the one-to-one reserve matching convention. In particular, when students are indifferent between different seats of the same school, their proposed choice rule fails not only maximal compliance with reservations (see Example 4 in Section 6.1), but also elimination of justified envy (see Example 3 in Section 6.1).

2. Model

There exist a set of individuals $I$ and a set of traits $T$. Each individual $i \in I$ has a set of traits $\tau(i) \subseteq T$. A trait can specify gender, race, or socioeconomic status of an individual. There are $q$ identical positions to allocate, which may correspond to positions at a job, seats at a school or seats for legislative organs. Individuals are strictly ranked according to a priority order $\pi$. Therefore, $i \pi j$ means that individual $i \in I$ has a strictly higher priority than individual $j \in I$. Without loss of generality, we assume that all individuals are eligible for this job.

For every trait $t$, $r_t \in \mathbb{N}$ number of positions are reserved for individuals with trait $t$. We assume that the number of reserved positions is less than the number of all positions, i.e., $\sum_{t \in T} r_t \leq q$. For any set of individuals $I \subseteq I$ and trait $t \in T$, let $I_t$ denote the set of individuals in $I$ who have trait $t$, i.e., $I_t = \{i \in I | t \in \tau(i)\}$.

A reservation market is a tuple $\langle I, T, \tau, \pi, q, (r_t)_{t \in T} \rangle$.

A choice rule $C$ is a function that takes a set of individuals $I \subseteq I$ and produces a subset $C(I) \subseteq I$ for a given reservation market. A choice rule may depend on the number of reserved positions for each trait and also the number of positions. For notational brevity, we omit this dependence when it is clear from the context.

We consider the following two conventions for implementing the reserves:

(1) **One-to-one reserve matching:** Upon admission, an individual with multiple traits accommodates only one of the reserves she qualifies for.
(2) **One-to-all reserve matching**: Upon admission, an individual with multiple traits accommodates each of the reserves she qualifies for.

Both conventions are prominent in real-life applications.

### 3. One-to-One Reserve Matching and Horizontal Envelope Choice Rule

In this section we adopt the one-to-one reserve matching convention. That is, throughout this section an individual is assumed to accommodate reserves for only one of her traits.

Consider a set of individuals $I \subseteq I$ who are applying for a job. Construct the following two-sided reservation graph. On one side of the graph, there are individuals in $I$. On the other side, there are reserved positions, i.e., there is a position for each reserved position. Therefore, for each trait $t$, there are $r_t$ positions and, there are $\sum_{t \in T} r_t$ positions in total. An individual and a position are connected if the individual has the trait of the position and thus they can be matched with each other. A one-to-one matching of individuals with positions has maximum cardinality in reserve matching if there exists no other one-to-one matching that assigns strictly higher number of reserved positions to individuals. Let $\rho(I)$ denote this maximum number of reserved positions that can be assigned to individuals. Say that an individual $i$ increases reserve utilization of $I$, if $\rho(I \cup \{i\}) = \rho(I) + 1$.

**Definition 1.** A set of individuals $I' \subseteq I$ maximally complies with the reservations for $I$, if, there exists a one-to-one matching of individuals in $I'$ to the reserved positions with maximum cardinality $\rho(I)$ in reserve matching.

A choice rule $C$ maximally complies with the reservations if, for every set $I \subseteq I$, $C(I)$ maximally complies with the reservations for $I$.

Consider the following choice rule that we call horizontal envelope:

**Choice Rule $C^\text{53}$**

**Step 1.1:** Choose the highest priority individual with a trait that has a reserved position. Let $i_1$ be this individual and $I_1$ be the set including only this individual. If there exists no such individual, go to Step 2.

**Step 1.k** ($k \in [2, \sum_{t \in T} r_t]$): Starting from the individual who has the next highest priority after $i_{k-1}$, check one by one if the next individual increases reserve utilization of $I_{k-1}$. If she does, choose this individual and denote her by $i_k$. In this case, let $I_k = I_{k-1} \cup \{i_k\}$ be the set of individuals chosen so far. Otherwise, if no such individual exists, go to Step 2.

**Step 2:** For unfilled positions, choose remaining individuals with the highest priority until all positions are filled or there are no unchosen individuals remaining.

When the number of individuals is less than $q$, this procedure selects all of them. Otherwise, if there are more than $q$ individuals, then it chooses a set with $q$ individuals.
We illustrate the envelope choice rule with the following example.

**Example 1.** Consider the following reservation market:

- \( \mathcal{I} = \{ i_1, i_2, i_3, i_4, i_5, i_6, i_7 \} \),
- \( \mathcal{T} = \{ t_1, t_2, t_3 \} \),
- \( i_1 \pi i_2 \pi i_3 \pi i_4 \pi i_5 \pi i_6 \pi i_7 \),
- \( \tau(i_1) = \emptyset, \tau(i_2) = \{ t_1, t_2, t_3 \}, \tau(i_4) = \{ t_1, t_2 \}, \tau(i_5) = \{ t_1 \}, \tau(i_6) = \{ t_3 \}, \tau(i_7) = \emptyset, \)
- \( r_{t_1} = r_{t_2} = r_{t_3} = 1 \), and \( q = 5 \).

The reservation graph for this market has one position for each trait and three positions in total. An individual is connected with a position if the individual has the corresponding trait. The reservation graph can be depicted as follows (see Figure 1).

![Figure 1](image-url)

**Figure 1.** The reservation graph of the reservation market in Example 1. Node \( s_n \) represents the reserved position for trait \( t_n \) for \( n \in \{1, 2, 3\} \).

The horizontal envelope choice rule works as follows. Only individuals \( i_2, i_4, i_5, i_6, \) and \( i_7 \) are qualified to receive a position at the first step because they have at least one trait. At Step 1.1. individual \( i_2 \) is selected because she is the highest priority individual who qualifies for a reserved position. At Step 1.2 individual \( i_4 \) is selected because she is the highest priority individual who can be matched along with \( i_2 \) to two reserved positions. For example, \( i_2 \) can be matched with \( s_1 \) and \( i_4 \) can be matched with \( s_2 \) (see the red matching in Figure 2).

At Step 1.3. individual \( i_5 \) is selected because she is the highest ranked individual with the property that she can be matched to a reserved position together with \( i_2 \) and \( i_4 \). However, implementation of such a matching requires that \( i_2 \) and \( i_4 \) are matched with different positions than the matching shown in red in Figure 2. To be more precise, \( i_2 \) can be matched with \( s_3 \), \( i_4 \) can be matched with \( s_2 \), and \( i_5 \) can be matched with \( s_1 \) (see the blue matching in Figure 3).
No remaining individuals can be matched with a reserved position together with \(i_2, i_4,\) and \(i_5,\) so we go to Step 2. At Step 2, individuals \(i_1\) and \(i_3\) are chosen because there are two vacant positions and they have the highest priority among the remaining individuals. Therefore, \(C^\epsilon(I) = \{i_1, i_2, i_3, i_4, i_5\}.\)

Our first result shows that the horizontal choice rule \(C^\epsilon\) selects higher priority individuals than any other choice rule that maximally complies with the reservations.

**Proposition 1.** Let \(C\) be a choice rule that maximally complies with the reservations. Then, for every set of individuals \(I,\)

\[(1) \ |C(I)| \leq |C^\epsilon(I)|\ and\]
for every $k \leq |\mathcal{C}(I)|$, if $i$ is the individual with the $k$-th highest priority in $\mathcal{C}(I)$ and $j$ is the individual with the $k$-th highest priority in $\mathcal{C}(I)$, then

$$i \pi j.$$ 

Now we introduce a fairness notion.

**Definition 2.** A choice rule $\mathcal{C}$ **eliminates justified envy** if for every $I \subseteq \mathcal{T}$, $i \in \mathcal{C}(I)$, and $j \in I \setminus \mathcal{C}(I)$,

$$j \pi i \implies \rho ((\mathcal{C}(I) \setminus \{i\}) \cup \{j\}) < \rho(\mathcal{C}(I)).$$

In words, if a high-priority individual $j$ is rejected even though a low-priority individual $i$ is chosen, then it must be the case that replacing $i$ with $j$ in the chosen set decreases the number of reserved positions that can be filled. When this condition is violated, we say that there is justified envy, which means that there exist a set of individuals $I$ and two individuals $i, j \in I$ such that

1. $i \in \mathcal{C}(I)$,
2. $j \notin \mathcal{C}(I)$,
3. $\rho ((\mathcal{C}(I) \setminus \{i\}) \cup \{j\}) \geq \rho(\mathcal{C}(I))$, and
4. $j \pi i$.

Therefore, when there is justified envy, a low-priority individual $i$ can be replaced with a high-priority individual $j$ without decreasing the number of reserved positions that can be filled.

Next we provide a standard property of choice rules.

**Definition 3.** A choice rule $\mathcal{C}$ is **non-wasteful** if for every $I \subseteq \mathcal{T}$,

$$|\mathcal{C}(I)| = \min\{|I|, q\}.$$ 

Equivalently, non-wastefulness requires that an individual is rejected only when all positions are allocated.

We now provide a characterization of the horizontal envelope choice rule.

**Theorem 1.** A choice rule maximally complies with the reservations, eliminates justified envy, and is non-wasteful if, and only if, it is the horizontal envelope choice rule.

### 4. One-to-All Reserve Matching with Two Traits

In this section we adopt the one-to-all reserve matching convention. That is, throughout this section an individual is assumed to accommodate each of the reserves for her traits.

Let $\mathcal{T} = \{t_1, t_2\}$ be the set of traits. Suppose that $r_{t_1} \leq q$ and $r_{t_2} \leq q$.

A few definitions have to be modified under the one-to-all reserve matching convention. Since an individual who has both traits counts against reservations for both traits, an
outcome under this convention is a many-to-one matching of individuals to reserved (or open) positions, where an individual who has both traits can be matched with a reserved position for each trait. A many-to-one matching of at most \( q \) individuals with positions has maximum cardinality in reserve matching if there exists no other many-to-one matching that assigns at most \( q \) individuals to strictly higher number of reserved positions. Let \( \rho(I) \) denote this maximum number of reserved positions that can be filled.

With this adjustment, maximal compliance with reservations can be defined as before:

**Definition 4.** A set of individuals \( I' \subseteq I \) maximally complies with the reservations for \( I \), if, there exists a many-to-one matching of individuals in \( I' \) to the reserved positions with maximum cardinality \( \rho(I) \) in reserve matching.

A choice rule \( C \) maximally complies with the reservations if, for every set \( I \subseteq \mathcal{I} \), \( C(I) \) maximally complies with the reservations for \( I \).

We need the following notation to define a class of choice rules. At Step \( k \) of the procedure, for any trait \( t \), let \( r_t(k) \) denote the number of reserved positions for \( t \) that are not filled and \( q(k) \) the number of remaining positions that are not filled before this step. Furthermore, let \( \Delta(k) = r_{t_1}(k) + r_{t_2}(k) - q(k) \) denote the number of remaining reservations in excess of the capacity. At Step 1, \( r_t(1) = r_t \), for every \( t \in \mathcal{T} \), and \( q(1) = q \).

For a given set of individuals \( I \) and \( i \notin I \), let \( \delta(i|I) \) denote the contribution of individual \( i \) to the remaining reservations in excess of the capacity when \( I \) is already chosen. More explicitly,

\[
\delta(i|I) = \begin{cases} 
1, & \text{if } \rho(I \cup \{i\}) = \rho(I) + 2 \\
0, & \text{if } \rho(I \cup \{i\}) = \rho(I) + 1 \\
-1, & \text{otherwise.}
\end{cases}
\]

Therefore, when \( I \) is the set of individuals chosen before Step \( k \) and \( i \) is chosen at Step \( k \), we have

\[
\Delta(k + 1) = \Delta(k) + \delta(i|I).
\]

Let \( I \) be the set of applicants and \( I(k) \) denote the set of individuals chosen before Step \( k \). Set \( I(1) = \emptyset \). The following choice procedure has two rounds. At the main round, one individual is chosen at every step. At the waitlist round, different sets of individuals are identified for all of the remaining positions. Each identified set determines a group of individuals that can be chosen. Therefore, if the procedure goes to the waitlist round, it will terminate and identify a choice correspondence that can potentially have different sets of individuals that can be chosen. However, if it terminates at the main round, it will produce a unique set of individuals that are chosen.

**Paired-Admission Choice Correspondence**
At every Step $k$ where $k \in [1, q]$, choose individuals depending on the case below. Stop if there are no remaining individuals.

**Main Round**

**Case 0:** $r_{t_1}(k) = 0$ or $r_{t_2}(k) = 0$. Choose the highest priority individual $i$ with the highest value of $\delta(i|I(k - 1))$.

**Case 1:** $r_{t_1}(k), r_{t_2}(k) > 0$ and $\Delta(k) > 0$. Choose the highest priority individual $i$ with the highest value of $\delta(i|I(k - 1))$.

**Case 2:** $r_{t_1}(k), r_{t_2}(k) > 0$ and $\Delta(k) < 0$. Choose the highest priority remaining individual.

**Case 3(i):** $r_{t_1}(k), r_{t_2}(k) > 0$ and $\Delta(k) = 0$. If $|I_t_1 \setminus I_t_1(k)| \leq r_{t_1}(k)$, then choose the highest priority remaining individuals who have trait $t_1$. Likewise, if $|I_t_2 \setminus I_t_2(k)| \leq r_{t_2}(k)$, then choose the highest priority remaining individual who has trait $t_2$.

**Case 3(ii):** $|I_t_1 \setminus I_t_1(k)| > r_{t_1}(k) > 0$, $|I_t_1 \setminus I_t_2(k)| > r_{t_2}(k) > 0$, and $\Delta(k) = 0$. If the remaining highest priority individual has at least one trait, then choose this individual.

**Case 3(iii):** $|I_t_1 \setminus I_t_1(k)| > r_{t_1}(k) > 0$, $|I_t_1 \setminus I_t_2(k)| > r_{t_2}(k) > 0$, and $\Delta(k) = 0$. If the remaining highest priority individual does not have a trait and there does not exist a remaining individual with both traits, then choose the highest priority individual who has the highest value of $\delta(\cdot|I_{k-1})$.

**Case 3(iv):** $|I_t_1 \setminus I_t_1(k)| > r_{t_1}(k) > 0$, $|I_t_1 \setminus I_t_2(k)| > r_{t_2}(k) > 0$, and $\Delta(k) = 0$. If $r_{t_1}(k) > r_{t_2}(k)$, then choose the highest priority remaining individual who has trait $t_1$. Otherwise, if $r_{t_2}(k) > r_{t_1}(k)$, then choose the highest priority remaining individual who has trait $t_2$.

**Case 3(v):** $|I_t_1 \setminus I_t_1(k)|, |I_t_1 \setminus I_t_2(k)| > r_{t_1}(k) = r_{t_2}(k) > 0$ and $\Delta(k) = 0$. Consider the remaining $r_{t_1}(k)$ highest priority individuals who have trait $t_1$ and the remaining $r_{t_2}(k)$ highest priority individuals who have trait $t_2$. If any of them has both traits, then choose the highest priority individual who has both traits.

**Waitlist Round**

**Case 3(vi):** $|I_t_1 \setminus I_t_1(k)|, |I_t_1 \setminus I_t_2(k)| > r_{t_1}(k) = r_{t_2}(k) > 0$ and $\Delta(k) = 0$. Let $r(t_1)(k) = r(t_2)(k) = r$. In this case the highest priority individual does not have any traits and no individual among the $r$ highest priority individuals who have trait $t_1$ and $r$ highest priority individuals who have trait $t_2$ has both traits. For $n \in \{1, \ldots, r\}$, let $p_n = \{i_n, j_n\}$ be the pair where $i_n$ is the $n$-th highest priority remaining individual who has trait $t_1$ and $j_n$ be the $n$-th highest priority remaining individual who has trait $t_2$. Likewise, form pair $q_n = \{i'_n, j'_n\}$ for $n \leq r$ where $i'_n$ is the $n$-th highest priority remaining individual with no traits and $j'_n$ is the $n$-th highest priority remaining individual with both traits. Let $m^* \leq r$ be the maximum
number such that \( q_{m^*} \) is well-defined and the individual with no traits in \( q_{m^*} \) has a higher priority than both individuals in \( p_{r-m^*+1} \). Then for every \( m \in \{0, 1, \ldots, m^*\} \), \( q_1 \cup \ldots \cup q_m \cup p_1 \cup \ldots \cup p_{r-m} \) can be chosen.

If this procedure terminates at Case 3(vi), then there are \( m^* + 1 \) different sets of individuals that can be chosen. However, if it ends at another case then there is only one set of individuals that is chosen. We call any selection from the paired-admission choice correspondence a **paired-admission choice rule**. The following example illustrates the paired-admission choice correspondence.

**Example 2.** Consider the following reservation market:

- \( \mathcal{I} = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\} \),
- \( i_1 \triangleright i_2 \triangleright i_3 \triangleright i_4 \triangleright i_5 \triangleright i_6 \triangleright i_7 \triangleright i_8 \),
- \( \tau(i_1) = \tau(i_8) = \{t_1, t_2\}, \tau(i_6) = \{t_1\}, \tau(i_7) = \{t_2\}, \tau(i_2) = \tau(i_3) = \tau(i_4) = \tau(i_5) = \emptyset \),
- \( r_{t_1} = r_{t_2} = 2 \), and \( q = 6 \).

The reservation graph for this market has two positions for each trait and four in total. An individual is connected with a reserved position if the individual has the corresponding trait. The reservation graph can be depicted as follows (see Figure 4).

The paired-admission choice rule works as follows. At Step 1, \( r_{t_1}(1) > 0, r_{t_2}(1) > 0 \) and \( r_{t_1}(1) + r_{t_2}(1) - q(k) = -2 < 0 \). Therefore, Case 2 holds and the highest priority individual \( i_1 \) is chosen. Since \( i_1 \) has both traits, \( r_{t_1}(2) = 1 \) and \( r_{t_1}(2) = 1 \). Therefore, at Step 2, \( r_{t_1}(2) + r_{t_2}(2) - q(k) = -3 < 0 \). Case 2 still holds and the next individual with the highest priority, who is \( i_2 \), is chosen. At Steps 3 and 4, Case 2 continues to hold and the highest priority remaining individuals, \( i_3 \) and \( i_4 \), are chosen.
At Step 5, \( r_{t_1}(5) = r_{t_2}(5) > 0 \) and \( r_{t_1}(5) + r_{t_2}(5) - q(5) = 0 \). Furthermore,

1. \(|I_1 \setminus I_{t_1}(5)| > r_{t_1}(5)\) and \(|I_2 \setminus I_{t_2}(5)| > r_{t_1}(5)\),
2. the remaining highest priority individual, \( i_5 \), does not have a trait,
3. there is a remaining individual, \( i_8 \), with both traits, and
4. neither the remaining highest priority individual with \( t_1 \), \( i_6 \), nor the remaining highest priority individual with \( t_2 \) has both traits.

Therefore, Case 3(vi) holds. Since \( r_{t_1}(5) = r_{t_2}(5) = 1 \), only one pair can be chosen: either the pair including an individual with no trait and an individual with both traits, \( (i_5, i_8) \), or the pair including an individual with only \( t_1 \) and an individual with only \( t_2 \), \( (i_6, i_7) \). Hence, there are two paired-admission choice rules, which are the two extremal ones \( C_{\text{min max}} \) and \( C_{\text{max min}} \), and furthermore

\[
C_{\text{min max}}(I) = \{i_1, i_2, i_3, i_4, i_5, i_8\}
\]
\[
C_{\text{max min}}(I) = \{i_1, i_2, i_3, i_4, i_6, i_7\}.
\]

Our main result of this section is as follows.

**Theorem 2.** A choice rule maximally complies with the reservations, eliminates justified envy, and is non-wasteful if, and only if, it is a paired-admission choice rule.

The paired-admission choice correspondence has two extremal choice rules that favor individuals with certain traits. The first one, \( C_{\text{min max}} \), favors individuals who do not have trait and individuals who have both traits. To be more specific, at the waitlist round of the procedure the maximum number of pairs that include an individual with no trait and an individual with both traits are chosen. Therefore, \( q_1 \cup \ldots \cup q_{m^*} \cup p_1 \cup \ldots \cup p_{r-m^*} \) is the set of individuals chosen at the last step. Likewise, there is a second choice rule, \( C_{\text{max min}} \) that favors individuals who have exactly one trait. At the waitlist round, this choice rule selects all pairs that have an individual with trait \( t_1 \) and an individual with trait \( t_2 \). Therefore, at this last step, the set of individuals chosen is \( p_1 \cup \ldots \cup p_r \).

In the next proposition, we study properties of \( C_{\text{min max}} \) and \( C_{\text{max min}} \).

**Proposition 2.** \( C_{\text{min max}} \) and \( C_{\text{max min}} \) have the following properties.

1. Let \( C \) be any paired-admission choice rule.
   (a) For any \( I \subseteq \mathcal{I} \) and \( i \in I \) such that \( \tau(i) = \emptyset \) or \( \tau(i) = \{t_1, t_2\} \),
   \[
i \in C(I) \implies i \in C_{\text{min max}}(I).
   \]
   (b) For any \( I \subseteq \mathcal{I} \) and \( i \in I \) such that \( \tau(i) = \{t_1\} \) or \( \tau(i) = \{t_2\} \),
   \[
i \in C_{\text{min max}}(I) \implies i \in C(I).
   \]
(c) For any $I \subseteq \mathcal{I}$, the highest priority individual in $I \setminus C_{\text{min max}}(I)$ has a weakly lower priority than the highest priority individual in $I \setminus C(I)$.

(2) Let $C$ be any paired-admission choice rule.

(a) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \{t_1\}$ or $\tau(i) = \{t_2\}$,

\[ i \in C(I) \implies i \in C_{\text{max min}}(I). \]

(b) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \emptyset$ or $\tau(i) = \{t_1, t_2\}$,

\[ i \in C_{\text{max min}}(I) \implies i \in C(I). \]

(c) For any $I \subseteq \mathcal{I}$, the lowest priority individual in $C_{\text{max min}}(I)$ has a weakly higher priority than the lowest priority individual in $C(I)$.

The next result shows that certain pairs of individuals are always selected together, thus motivating the term “paired-admissions choice rule.”

**Proposition 3.** Let $C$ and $C'$ be two paired-admission choice rules and $I$ be a set of individuals. If either

1. $i$ is the lowest priority individual with no traits in $C(I)$ and $j$ is the lowest priority individual with both traits in $C(I)$ or
2. $i$ is the lowest priority individual with only trait $t_1$ in $C(I)$ and $j$ is the lowest priority individual with only trait $t_2$ in $C(I)$, then

\[ i \in C'(I) \iff j \in C'(I). \]

5. Heterogeneous Jobs

In this section, we consider the case when there is more than one type of job. Let $\mathcal{A}$ be the set of jobs. For each job $a \in \mathcal{A}$, there exists a priority order $\pi_a$ over individuals.

Extension of our analysis to the case when individuals have strict preferences over jobs is straightforward for the case when the choice rules satisfy the following two conditions.

**Definition 5.** A choice rule $C$ satisfies the **substitutes** condition, if, for every $I \subseteq \mathcal{I}$,

\[ i \in C(I) \text{ and } j \neq i \implies i \in C(I \setminus \{j\}). \]

[Kelso and Crawford (1982)] introduced the substitutes condition in a market-design context.

**Definition 6.** A choice rule $C$ satisfies the **irrelevance of rejected individuals** condition, if, for every $I \subseteq \mathcal{I}$,

\[ i \in I \setminus C(I) \implies C(I \setminus \{i\}) = C(I). \]
See Aygün and Sönmez (2013) for a study of the irrelevance of rejected individuals condition.\footnote{The conjunction of substitutes condition and irrelevance of rejected individuals condition is equivalent to a property called path independence. See Chambers and Yenmez (2017) for a study of path-independent choice rules in a matching context.}

Over the last fifteen years, the following algorithm Gale and Shapley (1962) has become one of the mechanisms of choice for priority-based allocation with heterogenous goods, where the policies of the institutions are captured through the choice rules that are used in conjunction with this algorithm.

**Deferred Acceptance Algorithm (DA)**

**Step 1:** Each individual applies to her most preferred acceptable job if such a job exists. Suppose that $I_{a}^1$ is the set of individuals who apply to job $a$. Job $a$ tentatively accepts candidates in $C_{a}^{\top}(I_{a}^1)$ and permanently rejects the rest. If there are no rejections, then stop.

**Step $k$:** Each individual who was rejected in Step $k-1$ applies to her next preferred acceptable job, if such a job exists. Suppose that $I_{a}^k$ is the union of the set of individuals who were tentatively accepted by job $a$ in Step $k-1$, and the set of individuals who just proposed to job $a$. Job $a$ tentatively accepts individuals in $C_{a}^{\top}(I_{a}^k)$ and permanently rejects the rest. If there are no rejections, then stop.

As the following two results imply, while a joint implementation of the agent-proposing deferred acceptance algorithm with the horizontal envelope choice rule is straightforward under the one-to-one reserve matching convention, this is not possible for any paired-admission choice rule under the one-to-all reserve matching convention.

**Proposition 4.** $C^{\top}$ satisfies the substitutes condition and the irrelevance of rejected individuals condition.

**Proposition 5.** There exists no paired-admission choice rule that satisfies the substitutes condition.

Since $C^{\top}$ is also non-wasteful individuals cannot benefit from misreporting their preferences over jobs when DA is implemented by a clearinghouse. In other words, DA is strategy-proof.\footnote{When choice rules of jobs satisfy the substitutes condition and the irrelevance of rejected individuals condition, then DA produces a stable matching. Furthermore, if they also satisfy the law of aggregate demand, DA is strategy-proof for individuals. See Hatfield and Milgrom (2005). Non-wastefulness alone implies the law of aggregate demand.}

### 6. Applications

In this section, we present two large-scale practical applications of our model from Chile and India, and present how our proposed mechanism improves upon the mechanisms of choice in these applications.


6.1. School Inclusion Law in Chile. With the promulgation of the School Inclusion Law in Chile in 2015, a centralized school choice system has been in the process of being adopted in Chile, following a similar series of reforms throughout the world (Correa et al., 2019). The system is the product of an ongoing collaboration between the Chilean Ministry of Education and a team of researchers from operations research and economics, and it covers all grades prior to higher education (i.e. Pre-K to grade 12). The system was first implemented in 2016 as a pilot program in the smallest of the sixteen regions of Chile, and it has been adopted in all regions but the Metropolitan Area of Santiago by 2019, where over 274,000 students applied to more than 6400 schools. The system is expected to be adopted throughout Chile in 2020.

As many of its predecessors, the Chilean school choice system is based on the celebrated deferred acceptance algorithm, and the following three features in its design makes it a perfect application of our model:

(1) In order to promote diversity, the School Inclusion Law includes affirmative action policies for financially disadvantaged students and children with special needs. Under the new system, these policies are implemented through reserved seats at each school. In addition, a number of schools are allowed to reserve seats for high-achieving students. Hence, using our terminology there are three traits Financially disadvantaged, Special needs, High-achieving, where a student potentially can have any subset of these traits, possibly including none of them.

(2) While a student with multiple traits (say a financially disadvantaged student who is also high-achieving) is eligible for reserved seats for each of her traits, she “consumes” only one of the reserved seats upon receiving a seat. This feature in Chilean design eliminates potential complementarities between the regular students and students with multiple traits.

(3) Reserved seats at each school are implemented in the form of a soft lower bound (i.e. as a minimum guarantee).

A subtle implication of the second design feature is, it allows the model to be interpreted as an application of the matching with contracts model by Hatfield and Milgrom (2005), where the contractual term between a school and a student specifies which of the four types of seats (i.e., open seats, reserved seats for financially disadvantaged students, reserved seats for special needs students, and reserved seats for high-achieving students) the student receives (Kurata et al., 2017). However, the theory of matching with contracts is developed under the assumption that students have strict preferences over all their contracts, which in this context corresponds to them having strict preferences on the specific type of seats they receive at each school. Since students have preferences over school only

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10 Students with none of the three traits are referred as Regular.
in the Chilean context, a tie-breaking rule is used to construct student preferences over specific type of seats at each school. In [Correa et al. (2019)], the designers emphasize that the choice of a tie-breaking rule is not straightforward, and it has distributional consequences. In order to implement the reserves in the form of a minimum guarantee, they break ties in a way each student is assumed to prefer reserved seats for any of their traits to open seats. When each student has at most one trait, this construction assures that the reserves are implemented as a minimum guarantee (Hafalir et al. (2013), Sönmez and Yenmez (2019a)). However, as we present in the following two examples, interpreting this problem as an application of matching with contracts and relying on tie-breaking between reserved seats results in undesired outcomes.

Example 3. Suppose there is only one school $s$ with three seats. There are four students $i, j, k, \ell$ who are priority ranked as follows:

$$ i \pi_s j \pi_s k \pi_s \ell $$

One of the seats is reserved for students with trait financially disadvantaged $t_d$, and one of the seats is reserved for students with trait high-performing $t_h$. Student $i$ is a regular student with neither of the traits, whereas student $j$ has both traits, student $k$ has the high-performing trait $t_h$ only, and student $\ell$ has the financially disadvantaged trait $t_d$ only.

Let $s_o$ denote the open seat, $s_d$ denote the reserve seat for financially disadvantaged students, and $s_h$ denote the reserve seat for high-achieving students. Hence student $j$ receives preferential treatment for both seat $s_d$ and seat $s_h$, whereas student $k$ receives preferential treatment for seat $s_h$ only, and student $\ell$ receives preferential treatment for seat $s_\ell$ only. Hence, student priorities for each seat is given as follows:

$$ i \pi_{s_o} j \pi_{s_o} k \pi_{s_o} \ell $$
$$ j \pi_{s_h} k \pi_{s_h} i \pi_{s_h} \ell $$
$$ j \pi_{s_d} \ell \pi_{s_d} i \pi_{s_d} k $$

As for the tie-breaking, let us assume reserve seat $s_h$ is preferred to reserve seat $s_d$ for any student who either receives preferential treatment for both reserve seats or for neither of the reserve seats. Since by the Chilean design each student is also assumed to prefer seats at one of her traits to open seats, this results in the following construction of preferences

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11See Dur et al. (2018) for an unintended consequence of an erroneous choice of a tie-breaking rule by the Boston Public Schools system: The choice of the tie-breaking rule undermined the stated policy objectives of the school district, and resulted in a de facto elimination of walk-zone priority by implementing the walk-zone reserves on a minimum guarantee basis rather than over-and-above basis.
for the students:

\[ s_o \succ_i s_h \succ_i s_d \]
\[ s_h \succ_j s_d \succ_j s_o \]
\[ s_h \succ_k s_o \succ_k s_d \]
\[ s_d \succ_\ell s_o \succ_\ell s_h \]

So under the student-proposing deferred acceptance algorithm, at Step 1 student \( i \) applies to the open seat \( s_o \), whereas students \( j \) and \( k \) both apply for the reserved seat \( s_h \), and student \( \ell \) applies to the reserved seat \( s_d \). Reserved seat \( s_h \) holds student \( j \) and rejects student \( k \), whereas the open seat \( s_o \) and the reserved seat \( s_d \) each hold their only applicant. At Step 2, student \( k \) applies to the open seat \( s_o \), only to be rejected again since student \( i \) who is on hold for the open seat has higher priority for the open seat. Finally at Step 3, student \( k \) applies to the reserve seat \( s_d \), and gets rejected for a third time since student \( \ell \) who is on hold for the reserved seat \( s_d \) has higher priority for seat \( s_d \) having the financially disadvantaged trait. This results in the following matching

\[
\begin{pmatrix}
  s_o & s_h & s_d \\
  i & j & \ell
\end{pmatrix}
\]

of students to seats, and hence the set of students admitted to school \( s \) are \( \{i, j, \ell\} \). This outcome is undesired for the following reason: Observe that student \( j \) has the highest priority at not only seat \( s_h \) but also \( s_d \). Therefore, had she not been artificially assumed to prefer seat \( s_h \) to \( s_d \), she could have been instead assigned the seat \( s_d \), which in turn would allow student \( k \) to receive the seat \( s_h \) resulting in the matching,

\[
\begin{pmatrix}
  s_o & s_h & s_d \\
  i & k & j
\end{pmatrix}
\]

This alternative outcome is preferred to the outcome of the Chilean system, because it replaces the third priority student \( k \) with the fourth priority student \( \ell \), while still satisfying both reserves. This outcome is indeed the outcome of the horizontal envelope choice rule, which is agnostic about which type of reserved seat an agent receives when she has multiple traits.

The Chilean system induces a choice rule \( C^{Chile} \) for any given tie-breaking rule, and in Example 3 the choice rule \( C^{Chile} \) uses reservations fully. Hence, by Proposition 1 in Section 4, the choice rule \( C^{\boxtimes} \) admits higher priority students than the choice rule \( C^{Chile} \) in all examples where the latter choice rule uses reservations fully. But our next example shows that, in general the choice rule \( C^{Chile} \) may fail to use reservations fully.
Example 4. Suppose there is only one school $s$ with three seats. There are four students $i, j, k, \ell$ who are priority ranked as follows:

$$i \pi_s \ell \pi_s j \pi_s k$$

One of the seats is reserved for students with trait financially disadvantaged $t_d$, and one of the seats is reserved for students with trait high-performing $t_h$. Students $i, \ell$ are both regular students with neither of the traits, whereas student $j$ has both traits, and student $k$ has the financially disadvantaged trait $t_d$ only.

Let $s_o$ denote the open seat, $s_d$ denote the reserve seat for financially disadvantaged students, and $s_h$ denote the reserve seat for high-achieving students. Hence student $j$ receives preferential treatment for both seats $s_d$ and $s_h$, whereas student $k$ receives preferential treatment for seat $s_d$ only. Hence, student priorities for each seat is given as follows:

$$i \pi_{s_o} \ell \pi_{s_o} j \pi_{s_o} k$$
$$j \pi_{s_h} i \pi_{s_h} \ell \pi_{s_h} k$$
$$j \pi_{s_d} k \pi_{s_d} i \pi_{s_d} \ell$$

As for the tie-breaking, let us assume reserve seat $s_d$ is preferred to reserve seat $s_h$ for any student who either receives preferential treatment for both reserve seats or for neither of the reserve seats. Since by the Chilean design each student is also assumed to prefer seats at one of her traits to open seats, this results in the following construction of preferences for the students:

$$s_o \succ_i s_d \succ_i s_h$$
$$s_d \succ_j s_h \succ_j s_o$$
$$s_d \succ_k s_o \succ_k s_h$$
$$s_o \succ_\ell s_d \succ_\ell s_h$$

So under the student-proposing deferred acceptance algorithm, at Step 1 students $i$ and $\ell$ both apply to the open seat $s_o$, whereas students $j$ and $k$ both apply for the reserved seat $s_d$. Open seat $s_o$ holds student $i$ and rejects student $\ell$, whereas the reserved seat $s_d$ holds student $j$ and rejects student $k$. At Step 2, student $\ell$ applies to the reserved seat $s_d$ and student $k$ applies to the open seat $s_o$, and both students are rejected since these seats are each holding higher-priority students. Finally at Step 3, both students $k$ and $\ell$ apply to the reserved seat $s_h$, which holds student $\ell$ and rejects student $k$. Since student $k$ is rejected from all seats, the algorithm terminates at the end of Step 3 finalizing all assignments. This
results in the following matching
\[
\mu = \begin{pmatrix}
  s_o & s_h & s_d \\
  i & \ell & j
\end{pmatrix}
\]
of students to seats, and hence the set of students admitted to school \( s \) are \( \{i, \ell, j\} \). This outcome is undesired for the following reason: While it is possible to assign both reserve seats to students who have priority for these seats through an alternative matching of
\[
v = \begin{pmatrix}
  s_o & s_h & s_d \\
  i & j & k
\end{pmatrix},
\]
matching \( \mu \) instead only utilizes the reserve for financially disabled trait, and thus it de facto “converts” the reserve for high-achieving trait to an open position. Hence, the tie-breaking rule results in a situation where student \( j \) who has the flexibility to receive either of the two reserved seats to give up this flexibility forcing her to rigidly receive the reserve seat \( s_d \), which in turn means no one else can benefit from the high-achieving reserve. The end result is, only one of the reserve seats is utilized for affirmative action under the choice rule \( C_{Chile} \) even though both could have been utilized for this purpose. In contrast, both reserved seats are utilized under the horizontal envelope choice rule, which is agnostic about which type of reserved seat a student receives when she has multiple traits.

Example 3 shows that the choice rule induced by the Chilean system allows for justified envy, whereas Example 4 shows that it fails to use the reservations fully. In contrast, not only the horizontal envelope choice rule \( C_{\oplus} \) satisfies these properties, it is also the unique choice rule that satisfies these properties along with the modest efficiency requirement of non-wastefulness.

6.2. Supreme Court of India Horizontal Adjustment Subroutine. Built into the country’s constitution, one of the world’s most comprehensive affirmative action programs is implemented in India. Allocation of government positions and seats at publicly funded educational institutions have to comply with rules outlined by the landmark Supreme Court judgement \( Indra Sawhney and others v. Union of India (1992) \)¹² widely known as the Mandal Commission Case. Under these rules, an allocation system that relies on an objective merit list of candidates is integrated with two types of reservations referred as vertical reservations and horizontal reservations. Of the two types of reservations, vertical reservations are intended as the higher-level provisions, targeted for classes that faced historical discrimination, and implemented on a “set aside” (or equivalently “over-and-above”) basis. Horizontal reservations, on the other hand, are intended as lower-level provisions, targeted for other disadvantaged groups (such as women or disabled citizens), and implemented on a

¹²The case is available at [https://indiankanoon.org/doc/1363234/](https://indiankanoon.org/doc/1363234/) (last accessed on 11/29/2019),
“minimum guarantee” basis. The distinction between the higher-level provisions role of vertical reservations and the lower-level provisions role of horizontal reservation is justified through the mechanisms of their implementation, i.e. on a “set aside” (or equivalently “over-and-above”) basis for vertical reservations and on a “minimum guarantee” basis for horizontal reservation (Sönmez and Yenmez, 2019a). In the absence of horizontal reservations, implementation of vertical reservations is straightforward: First open (i.e. unreserved) positions are allocated based on merit scores only, and next for each of the (mutually exclusive) groups eligible for vertical reservation, reserved positions are allocated to the remaining members of the group based on their merit scores.

What is less clear is how to implement horizontal reservations to eligible groups (that could potentially overlap) in the form of minimum guarantees. The subroutine for implementing horizontal reservations is first introduced by the Supreme Court judgement Anil Kumar Gupta v. State of U.P. (1995) and further elaborated in the Supreme Court judgement Rajesh Kumar Daria v. Rajasthan Public Service Commission and others (2007) as follows:

If 19 posts are reserved for SCs (of which the quota for women is four), 19 SC candidates shall have to be first listed in accordance with merit, from out of the successful eligible candidates. If such list of 19 candidates contains four SC women candidates, then there is no need to disturb the list by including any further SC women candidate. On the other hand, if the list of 19 SC candidates contains only two woman candidates, then the next two SC woman candidates in accordance with merit, will have to be included in the list and corresponding number of candidates from the bottom of such list shall have to be deleted, so as to ensure that the final 19 selected SC candidates contain four women SC candidates. (But if the list of 19 SC candidates contains more than four women candidates, selected on own merit, all of them will continue in the list and there is no question of deleting the excess women candidate on the ground that ‘SC-women’ have been selected in excess of the prescribed internal quota of four.)

This procedure, which we refer as Supreme Court of India horizontal adjustment subroutine (or in short SCI horizontal subroutine) is mandated throughout India for each horizontal trait, and urged to be implemented independently for each of the vertical categories; i.e for the open category, to be followed by each of the vertical categories. This is referred as compartmentalized horizontal reservation.

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In India the only federally mandated horizontal reservation is for disabled people, and it is known as persons with disability (PwD) reservation.\footnote{Horizontal PwD reservation is federally mandated by the Supreme Court judgement \textit{Union Of India & Anr vs National Federation Of The Blind & ... on 8 October, 2013}, available at \url{https://indiankanoon.org/doc/178530295/} (last accessed on 11/29/2019).} In applications with only one horizontal trait, integrating SCI horizontal subroutine with vertical reservation is straightforward: The subroutine is to be directly applied to each vertical category.\footnote{Even with only one horizontal trait, however, there is one subtle aspect of horizontal reservation (Sönmez and Yenmez, 2019a). Under Indra Sawhney (1992), while all candidates are eligible for open positions (without any horizontal adjustments), only members of General Category (i.e., candidates who do not qualify for vertical reservation) are eligible for horizontal adjustment for open positions. Horizontal adjustments for other candidates is to be carried out in their respective vertical categories. While this restriction does not make application of the SCI horizontal subroutine any more complicated, it introduces a number of shortcomings in the resulting choice rule. See Sönmez and Yenmez (2019a) for numerous lawsuits due to these shortcomings, as well as a recommended resolution.}

However, in many applications there are multiple horizontal traits. For example, horizontal reservation for women is mandated in several states including in Bihar with 35\%, Andhra Pradesh with 33\frac{1}{3}\%, and Madhya Pradesh, Uttarakhand, Chhattisgarh, Rajasthan, and Sikkim with 30\% each. In many applications there are other horizontal traits as well, such as ex-servicemen, sportsmen, etc. In those applications, the standard procedure is implementing the SCI horizontal subroutine for each of the horizontal traits. Processing multiple horizontal reservations is also straightforward when each individual is either qualified for at most one horizontal trait, or she is forced to declare at most one horizontal trait. In that case, it is immaterial in what order the adjustments are made via SCI horizontal subroutine. Sönmez and Yenmez (2019a) analyze the Supreme Court’s allocation procedure for this case where it is well-defined.

What is analytically more challenging, however, is the case individuals qualify for multiple horizontal reservations. Before analyzing the impact of sequential implementation of the SCI horizontal subroutine for each trait, it is important to clarify one important aspect of this version of the problem: Suppose there are two horizontal traits, one for women and another for persons with disability, and consider a female candidate with a disability who is admitted. A key policy question is, whether this candidate counts for both of this horizontal reservations or only one of the horizontal reservations. To the best of our knowledge, this critical question is not addressed by any of the court cases. Hence as far as we can tell, both interpretations are consistent with Indra Sawhney (1992) and the subsequent Supreme Court rulings. However, we believe the more widespread interpretation is the case where each candidate counts for only one of the horizontal traits. That is because, in most applications in India, the number of positions are explicitly given for each vertical category-horizontal trait pair, and as a consequence individuals are assigned to positions attached to category-trait pairs. For example, when positions are announced for the
classifications Open Category-Women and Open Category-PwD, a disabled woman can be assigned a position from either classification (or simply the Open Category without using either one of her horizontal traits). Hence, most of the applications are consistent with individuals “using up” a position from only one of the horizontal reservations. Adhering to this interpretation of horizontal reservation, we next present two shortcomings of the practice of sequentially implementing the SCI horizontal adjustment subroutine \[17\]

**Example 5.** Suppose there is a job with three (open category) positions, where one position is horizontally reserved for women and one position is horizontally reserved for persons with disability. Let us denote the horizontal trait of being female as (W) and the horizontal trait of being disabled as (D).

There are three male candidates \(i_1, i_2, i_3\), and two female candidates \(i_4, i_5\). Only one of the candidates, the female candidate \(i_4\) is disabled. Candidates are merit ranked as follows:

\[
i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5
\]

Prior to horizontal adjustments, the three highest merit ranking candidates \(i_1, i_2, i_3\), are tentatively assigned to available positions. Since neither the women reservation nor the disability reservation is satisfied, adjustments are needed for both horizontal traits.

If, the SCI horizontal adjustment subroutine is used in the sequence D-W, then

1. first the highest priority unmatched disabled candidate \(i_4\) receives the horizontally reserved position for disabled at the expense of the lowest merit ranking non-disabled candidate \(i_3\) who was on hold, and
2. subsequently the highest priority unmatched female candidate \(i_5\) receives the horizontally reserved position for women at the expense of the lowest merit ranking male candidate \(i_2\) who was on hold,

resulting in the admission of the set of individuals \(\{i_1, i_4, i_5\}\).

If, on the other hand, the SCI horizontal adjustment subroutine is used in the sequence W-D, then

1. first the highest priority unmatched female candidate \(i_4\) receives the horizontally reserved position for women at the expense of the lowest merit ranking male candidate \(i_3\) who was on hold, and
2. next due to the lack of disabled candidates among unmatched candidates, no additional adjustment can be made

resulting in the admission of the set of individuals \(\{i_1, i_2, i_4\}\).

\[17\] See Example 1 in Sönmez and Yenmez (2019a) for a shortcoming of the SCI horizontal adjustment subroutine under the alternative interpretation where an individual uses up a position from each of the horizontal traits she has upon admission.
Example 3 highlights two shortcomings of sequential implementation of SCI horizontal adjustment subroutine: First, the outcome of this process depends on the order the horizontal traits. This means, unless the process order is explicitly stated, the resulting mechanism is not mathematically well-defined. Perhaps the more important shortcoming is the second one. When the traits are processed in the sequence W-D, only one of the horizontally reserved positions are filled with candidates who are eligible for these positions, even though it is feasible to assign both of these positions who are eligible for horizontal reservation. Processing of trait W prior to trait D results in a “mismatch” of the horizontally reserved W position to the disabled female candidate $i_4$, who could have as well received the horizontally reserved D position. Therefore the resulting choice rule fails to use the reservations fully.

Observe that adjustment for horizontal traits is a direct application of our model, and both shortcomings can be avoided by replacing sequential implementation of the SCI horizontal adjustment subroutine with the horizontal envelope choice rule $C^\Box$ whenever there are overlapping horizontal traits.

6.3. The Law of Social Quotas in Brazil. On August 29, 2012, President Dilma Rousseff enacted the Law of Social Quotas in Brazil, which requires public colleges to reserve at least half of their seats for graduates of public high schools. The law also requires that, within seats reserved for graduates of public high schools, half should be reserved for low-income families. Finally the law also requires that, again within seats reserved for graduates of public high schools, a percentage equal to their share in the State population should be reserved for those who declare themselves as black, mixed or indigenous. So for our application in Brazil, there are three traits $p, \ell, m$, where

- trait $p$ indicates graduating from a public school,
- trait $\ell$ indicates being low income, and
- trait $m$ indicates being of black, mixed or indigenous descent.

By the Law of Social Quotas, there are reserved seats for

1. students who have trait $p$,
2. students who have traits $p$ and $\ell$, and
3. students who have traits $p$ and $m$.

Let $r_p$ be the number seats reserved for graduates of public schools, $r_\ell$ be the number of seats reserved for graduates of public schools who are low income, and $r_m$ be the number of seats reserved for graduates of public schools who are of black, mixed or indigenous descent. Since the latter two reserves are within the reserves for public school graduates, we have $r_\ell \leq r_p$ and $r_m \leq r_p$. 
Observe that we have overlapping traits in this application. Moreover, the traits $\ell$ and $m$ are useful only if an applicant also has the trait $p$. So even though there are three traits in this application, there are complementarities between only two groups of individuals, namely individuals with trait $p$ only, and individuals with all three traits $p, \ell, m$. So while our results in Section 4 do not immediately apply to the application for Brazil, we can build on these results to design a class of choice rules since complementarities are restricted to two groups for this application as well.

While the Law of Social Quotas requires reserved seats for students with sets of trait \{p\}, \{p, \ell\}, and \{p, m\}, it does not provide any choice rule to implement these reserves. Due to the challenges of implementing overlapping reserves, several universities have adopted choice rules that rely on a dividing the reserves given by the law into finer reserves for each combination of traits that qualify for reserves. Under these rules, there are (non-overlapping) reserves for

1. public school graduates who are low income and of black, mixed or indigenous descent,
2. public school graduates who are low income but not of black, mixed or indigenous descent,
3. public school graduates who are of black, mixed or indigenous descent but not low income, and
4. public school graduates who are neither low income, nor of black, mixed or indigenous descent.

Aygün and Bö (2016) analyze several of these choice rules, presenting their shortcomings and correcting their shortcomings. However their analysis also rely on the finer division of the reserves that introduces some artificial restrictions in addition to the restrictions that are imposed by the law. In contrast, in our design we maintain the overlapping reserves structure given in the Law of Social Quotas, and directly implement the provisions in the law without imposing any additional restrictions.

The following choice rule for admissions of students in Brazil can be used in conjunction with any paired-admission choice rule $C$ where $C$ is used to allocate public school seats first and the two traits are minority and low income. Let $I$ be a set of applicants.

**Horizontal Choice Rule $C_{\text{hor}} \circ C$**

**Step 1:** Choose $C(I)$ for the seats reserved for students graduating from public schools.

**Step 2:** Choose remaining individuals with the highest priorities for the remaining seats.

The set of individuals chosen is denoted by $(C_{\text{hor}} \circ C)(I)$. In particular, the two extremal paired-admission choice rules can be used in this construction.
Proposition 6. $\text{C}_{\text{hor}} \circ \text{C}_{\text{min max}}$ and $\text{C}_{\text{hor}} \circ \text{C}_{\text{max min}}$ have the following properties.

(1) Let $C$ be any paired-admission choice rule.
(a) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \{p, \ell, m\}$,

\[ i \in (\text{C}_{\text{hor}} \circ C)(I) \implies i \in (\text{C}_{\text{hor}} \circ \text{C}_{\text{min max}})(I). \]

(b) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \{p, \ell\}$ or $\tau(i) = \{p, m\}$,

\[ i \in (\text{C}_{\text{hor}} \circ \text{C}_{\text{min max}})(I) \implies i \in (\text{C}_{\text{hor}} \circ C)(I). \]

(2) Let $C$ be any paired-admission choice rule.
(a) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \{p, \ell\}$ or $\tau(i) = \{p, m\}$,

\[ i \in (\text{C}_{\text{hor}} \circ C)(I) \implies i \in (\text{C}_{\text{hor}} \circ \text{C}_{\text{max min}})(I). \]

(b) For any $I \subseteq \mathcal{I}$ and $i \in I$ such that $\tau(i) = \{p, \ell, m\}$,

\[ i \in (\text{C}_{\text{hor}} \circ \text{C}_{\text{max min}})(I) \implies i \in (\text{C}_{\text{hor}} \circ C)(I). \]

(c) For any $I \subseteq \mathcal{I}$, the lowest priority individual in $(\text{C}_{\text{hor}} \circ \text{C}_{\text{max min}})(I)$ has a weakly higher priority than the lowest priority individual in $(\text{C}_{\text{hor}} \circ C)(I)$.

Remark 1. For an individual $i$ such that $\tau(i) = \emptyset$ and $\tau(i) = \{p\}$, there is no preference comparison between $\text{C}_{\text{hor}} \circ \text{C}_{\text{min max}}$ and $\text{C}_{\text{hor}} \circ \text{C}_{\text{max min}}$. For a set of individuals, the highest priority rejected individual by $\text{C}_{\text{hor}} \circ \text{C}_{\text{min max}}$ can have a strictly lower or strictly higher priority than the highest priority rejected individual by $\text{C}_{\text{hor}} \circ \text{C}_{\text{max min}}$.

7. Conclusion

We have presented a theory of overlapping reserves both for the case of one-to-one reserve matching and also for the case of one-to-all reserve matching when there are complementarities between two groups only. There is a natural choice rule that emerges under the first convention, the horizontal envelope choice rule, and hence if there is any flexibility to select one of the conventions we believe the case for the one-to-one reserve matching is much stronger.
Appendix A. Mathematical Preliminaries

Let $G$ be the reservation graph of a reservation market. The vertices of $G$ are the individuals in $I$ and reserved positions. There exists an edge between an individual $i$ and a position reserved for trait $t$ if $i$ has trait $t$. A matching is a set of edges without common vertices. A matching covers a vertex if it has an edge adjacent to that vertex.

**Lemma 1** (Mendelson-Dulmage Theorem). Consider a reservation graph. Suppose that there exist a matching that covers individuals in $I$ and a matching that covers reserved positions in $S$. Then there exists a matching that covers both $I$ and $S$.

See Theorem 4.1 in Lawler (2001, Page 191) for a proof of this lemma.

An alternating path between matching $M_1$ and matching $M_2$ is a path of connected edges that starts at a vertex covered by $M_1$ but not by $M_2$ and ends at a vertex covered by $M_2$ but not by $M_1$ such that edges of the path belong alternately to $M_1$ and $M_2$.

**Lemma 2** (Alternating Path). Let $M_1$ and $M_2$ be two distinct matchings that cover the same set of positions in a reservation graph. Suppose that there exists a vertex $i$ covered by $M_1$ but not by $M_2$. Then there exists an alternating path between matching $M_1$ and matching $M_2$ that starts at $i$.

**Proof.** Let $i_1 \equiv i$ and $(i_1, s_1)$ be the edge that covers $i_1$ in $M_1$. Since $M_1$ and $M_2$ cover the same set of positions, there exists an edge $(i_2, s_1)$ in $M_2$. If $i_2$ is not covered by $M_1$, then we are done. Otherwise, $i_2$ is covered by both $M_1$ and $M_2$. Let $(i_2, s_2)$ be the edge in $M_1$ that covers $i_2$. Since $M_1$ and $M_2$ cover the same set of positions, there exists an edge $(i_3, s_2)$ in $M_2$. If $i_3$ is not covered by $M_1$, then we are done. Otherwise, $i_3$ is covered by both $M_1$ and $M_2$. Continue this construction. Since there exists a finite number of vertices, this construction ends in finite time at a vertex $i_k$ covered by $M_2$ but not by $M_1$. This finishes the construction of an alternating path starting at $i$. See Figure 5 for an illustration of the alternating path that is constructed. □

![Figure 5](image.png)

**Figure 5.** The alternating path between $M_1$ and $M_2$ constructed in the proof of Lemma 2. The edges in $M_1$ are solid and the edges in $M_2$ are dashed.
Appendix B. Proofs

Proof of Proposition 1. Let $I \subseteq \mathcal{I}$ be a set of individuals. To show part (1), note that $|C^\otimes(I)| = \min\{q, |I|\}$. Furthermore, for choice rule $C$, $C(I) \subseteq I$ and $|C(I)| \leq q$. Therefore, $|C(I)| \leq \min\{q, |I|\} = |C^\otimes(I)|$.

We show part (2) by mathematical induction on parameters $(q, (r_t)_{t \in \tau})$. We show the claim that for an ordering of agents in $C^\otimes(I) \setminus C(I)$ and $C(I) \setminus C^\otimes(I)$ that the $k$-th agent in $C^\otimes(I) \setminus C(I)$ has a higher priority than the $k$-th agent in $C(I) \setminus C^\otimes(I)$, which implies part 2.

For the base case when there are no reserved positions, statement (2) holds because $C^\otimes$ chooses individuals with the highest priority at Step 2. Now suppose that the claim holds for all parameters bounded above by $(q, (r_t)_{t \in \tau})$. Consider parameters $(q, (r_t)_{t \in \tau})$. If all individuals in $C^\otimes(I) \setminus C(I)$ are chosen at Step 2, then the claim holds as in the base case because individuals in $C(I) \setminus C^\otimes(I)$ are available at Step 2 in the construction of $C^\otimes(I)$.

Consider the situation when there exists at least one individual in $C^\otimes(I) \setminus C(I)$ chosen at Step 1. Let $i$ be the individual with the highest priority in $C^\otimes(I) \setminus C(I)$ chosen at Step 1 and $t$ be the trait of the position that she is matched with. By Lemma 3, $C^\otimes$ maximally complies with the reservations, so in the reservation graph, there exists a matching $M_1$ that matches $C^\otimes(I)$ to a set of reserved positions $S$ with maximum cardinality $\rho(I)$. Since $C$ also maximally complies with the reservations, by Mendelson-Dulmage Theorem (see Lemma 1) there exists another matching $M_2$ that matches $C(I)$ to $S$ both of which have cardinality $\rho(I)$. By Lemma 2, there exists an alternating path that starts at $i$ and ends at an individual $j \in C(I) \setminus C^\otimes(I)$. Therefore, individual $i$ can be replaced with individual $j$ in $C^\otimes(I)$ without changing the set of positions covered in the reservation graph for $I$. Hence, by construction of $C^\otimes(I)$, $i \not\sim j$ because $j$ is available when $i$ is chosen at Step 1.

Now consider the reduced market when capacity $q$ and reservation for trait $t$ are both reduced by one and the set of individuals is $I \setminus \{i, j\}$. In this reduced market, $C^\otimes(I \setminus \{i, j\})$ is equal to $C^\otimes(I) \setminus \{i\}$ for the original market because $j \notin C^\otimes(I)$ and the construction of $C^\otimes(I \setminus \{i, j\})$ chooses individuals in the same order as they are chosen in $C^\otimes(I)$. In particular, the set of individuals chosen before $i$ at $C^\otimes(I)$ are chosen in the same order in $C^\otimes(I \setminus \{i, j\})$. Furthermore, after $i$ is chosen the set of updated parameters are exactly the same. Therefore, the same set of individuals are chosen in the same order after $i$ is chosen in $C^\otimes(I)$. In addition, $C(I \setminus \{j\}$ maximally complies with the reservations and $i \notin C(I \setminus \{j\})$. By the induction hypothesis, the individuals in $C^\otimes(I \setminus \{i, j\})$ and $C(I \setminus \{j\}$ can be ordered with the required property, which implies the hypothesis. Therefore, the hypothesis holds for every set of parameters $(q, (r_t)_{t \in \tau})$. □
Proof of Theorem 1. We first show that \( C \) satisfies the stated properties in several lemmas and then show that the unique choice rule satisfying these properties is \( C \).

Lemma 3. \( C \) maximally complies with the reservations.

Proof. Suppose, for contradiction, that \( C \) does not maximally comply with the reservations. Hence, there exists \( I \subseteq \mathcal{I} \) such that \( C(I) \) does not maximally comply with the reservations for \( I \). Therefore, in the reservation graph for \( C(I) \), the maximum cardinality that can be attained by a matching is strictly less than \( \rho(I) \). Let \( \bar{I} \subseteq C(I) \) be the set of individuals who are matched to reserved positions in a maximum matching for the reservation graph for \( C \). By construction, \( |\bar{I}| < \rho(I) \). Now consider a maximum matching for the reservation graph for \( I \). Let \( S \) be the set of positions matched, so \( |S| = \rho(I) \). By Mendelson-Dulmage Theorem (see Lemma 1), there exists a matching that assigns every individual in \( \bar{I} \) and every reserved position in \( S \) in the reservation graph of \( I \). But this is a contradiction to the construction of \( C \) because this choice rule finds a maximal matching in the reservation graph of \( I \), which implies that there cannot exist another matching that assigns a set of individuals that is a proper superset of \( C(I) \) to positions in the reservation graph of \( I \).

Lemma 4. \( C \) eliminates justified envy.

Proof. Suppose, for contradiction, that \( C \) does not eliminate justified envy. Therefore, there exist a set of individuals \( I \subseteq \mathcal{I} \), individuals \( i \in C(I) \), \( j \in I \setminus C(I) \) with \( \rho \left( (C(I) \setminus \{i\}) \cup \{j\} \right) \geq \rho(C(I)) \), and \( j \pi i \). Consider choice rule \( C \) such that

\[
C(I') = \begin{cases} 
C(I'), & \text{if } I' \neq I \\
(C(I) \setminus \{i\}) \cup \{j\}, & \text{if } I' = I.
\end{cases}
\]

Since \( C \) maximally complies with the reservations, \( C(I') \) maximally complies with the reservations for \( I' \) whenever \( I' \neq I \) because \( C(I') = C(I') \). Furthermore,

\[
\rho(C(I)) = \rho \left( (C(I) \setminus \{i\}) \cup \{j\} \right) \geq \rho(C(I))
\]

and the fact that \( C \) maximally complies with the reservations by Lemma 3 (i.e., \( \rho(C(I)) = \rho(I) \)) imply that \( \rho(C(I)) = \rho(I) \) because of the fact that \( \rho(I) \) is the maximum cardinality. Hence, \( C(I) \) maximally complies with the reservations for \( I \). By Proposition 1, for every \( k \leq |C(I)| \), the individual with the \( k \)-th highest priority in \( C(I) \) has a weakly higher priority than the individual with the \( k \)-th highest priority in \( C(I) \). This is a contradiction to the construction of \( C \) because

\[
C(I) = (C(I) \setminus \{i\}) \cup \{j\}
\]

and \( j \pi i \).
Lemma 5. \(C^{\box right}\) is non-wasteful.

Proof. \(C^{\box right}\) is non-wasteful because at the second step all the remaining positions are filled with the remaining individuals until all positions are filled or all individuals are assigned to positions.

Lemma 6. If a choice rule maximally complies with the reservations, eliminates justified envy, and is non-wasteful, then it has to be the horizontal envelope choice rule.

Proof. Let \(C\) be a choice rule that maximally complies with the reservations, eliminates justified envy, and is non-wasteful. Suppose, for contradiction, that \(C \neq C^{\box right}\). Therefore, there exists \(I \subseteq \mathcal{I}\) such that \(C(I) \neq C^{\box right}(I)\). Since both choice rules are non-wasteful

\[|C(I)| = |C^{\box right}(I)|.\]

Since \(C(I) \neq C^{\box right}(I)\), this equation implies that

\[\left|C^{\box right}(I) \setminus C(I)\right| = \left|C(I) \setminus C^{\box right}(I)\right| > 0.\]

We consider two cases depending on the value of \(\rho(I)\).

Case 1: If \(\rho(I) = 0\), then no individual in \(I\) has a trait that has a positive reservation. Therefore, \(C^{\box right}(I)\) consists of \(\min\{|I|, q\}\) individuals with the highest priority in \(I\). This is a contradiction to the assumption that \(C(I)\) eliminates justified envy because any individual \(i \in C^{\box right}(I) \setminus C(I) = \emptyset\) has a higher priority than any individual \(j \in C(I) \setminus C^{\box right}(I) \neq \emptyset\) and \(\rho((C(I) \setminus \{j\}) \cup \{i\}) \geq \rho(C(I)) = 0\). Hence, there is an instance of justified envy for \(C(I)\) involving \(i \in I \setminus C(I)\) and \(j \in C(I)\), which is a contradiction.

Case 2: Let \(\rho(I) = k > 0\). Therefore, there are \(k\) individuals chosen at Step 1 of \(C^{\box right}(I)\). For \(1 \leq n \leq k\), let \(i_n\) be the \(n\)-th individual chosen at Step 1 of \(C^{\box right}(I)\). Consider a maximum matching \(M_1\) in the reservation graph for \(C^{\box right}(I)\) that matches \(I_1 \equiv \{i_1, \ldots, i_k\}\). We show that \(C(I) \supseteq I_1\). Let \(S\) be the set of positions that are matched in \(M_1\). Since \(C\) maximally complies with the reservations, there exists a maximum matching in the reservation graph for \(C(I)\) that has cardinality \(k\). Furthermore, by Mendelson-Dulmage Theorem (see Lemma 1), there exists a matching of a subset of \(C(I)\) to positions in \(S\). Let \(I_2 \subseteq C(I)\) be the set of these individuals and \(M_2\) be this matching. Suppose, for contradiction, that \(I_1 \setminus C(I) \neq \emptyset\). Let \(i_n\) be the individual with the lowest index in \(I_1 \setminus C(I)\). By Lemma 2, there exists an alternating path between \(M_1\) and \(M_2\) that starts at \(i_n\) and ends at a vertex \(j\) covered by \(M_2\) but not by \(M_1\). Therefore, \(i_n\) and \(j\) can be replaced with each other in both \(M_1\) and \(M_2\) without decreasing the maximum cardinality. By construction of \(C^{\box right}\), \(i_n \succ_{\rho} j\) because \(j\) is available when \(i_n\) is chosen. But this is a contradiction to the assumption that \(C(I)\) eliminates justified envy because \(j \in C(I)\), \(i_n \in I \setminus C(I)\), \(i_n \succ_{\rho} j\), and \(\rho(C(I)) = \rho((C(I) \setminus \{j\}) \cup \{i_n\})\). Therefore, \(I_1 \subseteq C(I)\).
By construction of $C^{\Omega}(I)$, every individual in $C^{\Omega}(I) \setminus I_1$ is chosen at Step 2. Therefore, these individuals have a higher priority ranking than any individual in $I \setminus C^{\Omega}(I)$. Let $j \in C(I) \setminus C^{\Omega}(I)$, which is non-empty by assumption. Therefore, $j \in I \setminus C^{\Omega}(I)$, which means that any individual $i \in C^{\Omega}(I) \setminus C(I)$ has a strictly higher priority than $j$. This is a contradiction to the assumption that $C(I)$ eliminates justified envy because $j \in C(I)$, $i \in I \setminus C(I)$, $i \triangleright j$, and $\rho(C(I)) = k = \rho((C(I) \setminus \{j\}) \cup \{i\})$ where the last equation follows from $I_1 \subseteq (C(I) \setminus \{j\}) \cup \{i\}$ and the fact that $\rho(I_1) = k$. ■

This finishes the proofs of all the statements. □

**Proof of Proposition 4**. Let $I \subseteq \mathcal{I}$, $i \in C^{\Omega}(I)$, and $j \in I$ such that $j \neq i$. To prove the substitutes condition, we show that $i \in C^{\Omega}(I \setminus \{j\})$. If $j \notin C^{\Omega}(I)$ or $i \triangleright j$, then $i \in C^{\Omega}(I \setminus \{j\})$ is satisfied trivially by the construction of $C^{\Omega}$.

Now assume that $j \triangleright i$ and $j \in C^{\Omega}(I)$. If $i$ is chosen at Step 2 of $C^{\Omega}(I)$, then, when the set of individuals is $I \setminus \{j\}$, it will be chosen either at the first step or the second step. Consider the case when $i$ is chosen at Step 1 of $C^{\Omega}(I)$. Let $j$ be matched with a reserved position for trait $t$. The first steps of $C^{\Omega}(I)$ and $C^{\Omega}(I \setminus \{j\})$ are the same until the priority ranking of the individual considered is below that of $j$. If an individual with a priority higher than $i$ is chosen from $I \setminus C^{\Omega}(I)$ when $I \setminus \{j\}$ is the set of applicants, then the first such individual can be matched with a reserved position for trait $t$ that $j$ was matched with because this individual is not chosen by $C^{\Omega}(I)$. After this individual is chosen, the construction of $C^{\Omega}(I \setminus \{j\})$ proceeds as in the construction of $C^{\Omega}(I)$. Therefore, $i$ is chosen in this case. However, if no individual with a priority higher than $i$ is chosen from $I \setminus C^{\Omega}(I)$ in the construction of $C^{\Omega}(I \setminus \{j\})$, then $i$ will also be chosen.

It is well known that the substitutes condition and non-wastefulness imply the irrelevance of rejected individuals condition.

□

**Proof of Theorem 2**. We provide a series of lemmas to show each claim separately. Recall that for any Step $k$ of the procedure $\Delta(k) = r_{t_1}(k) + r_{t_2}(k) - q(k)$.

**Lemma 7.** A paired-admission choice rule is non-wasteful.

*Proof.* A paired-admission choice rule is non-wasteful because individuals are chosen until all positions are filled or all individuals are chosen. ■

**Lemma 8.** A paired-admission choice rule maximally complies with the reservations.

*Proof.* Let $C$ be a paired-admission choice rule. Suppose, for contradiction, that it does not maximally complies with the reservations. Then there exists a set of individuals $I$ such that $C(I)$ does not have the maximum cardinality of $\rho(I)$ in the reservation graph. Therefore,
at least one reservation for one of the traits, say $t_1$ without loss of generality, is not filled by $C(I)$ and there exists an individual in $I \setminus C(I)$ who has $t_1$. Therefore, the last Step $K$ of the choice rule must be in the main round and $\Delta(K) > 0$ because $q(K) = 1$ and $r_{t_1}(K) + r_{t_2}(K) \geq 2$ since $r_{t_1}(K) \geq 1$ and $r_{t_1}(K) + r_{t_2}(K) = 1$ implies that $r_{t_1}(K) = 1$ and $r_{t_2}(K) = 0$. But if this were the case then at Case 0 an individual with trait $t_1$ would have been selected at Step $K$ since an individual with trait $t_1$ is available. Furthermore, whenever $\Delta(k) > 0$, the choice rule selects the highest priority remaining individual who has the maximum number of traits for which reservations are not filled among the remaining individuals ($\Delta(k) > 0$ can only happen for Case 0 and Case 1). If $\Delta(k) > 0$ for every Step $k$, we get a contradiction to the assumption that $C(I)$ does not maximally comply with the reservations for $I$. Therefore, there exists a last Step $k'$ at which $\Delta(k') = 0$ because, for every Step $k$ of the main round,

$$\Delta(k + 1) - \Delta(k) = \delta(i|I(k)) \in \{1, 0, -1\},$$

where $i$ is the individual chosen at Step $k$. Hence, the difference can change at most by one at a step. Since this difference goes up by one after this step, the individual who was chosen did not have a trait for which there were positive reservations. However, $r_{t_1}(k') > 0$ since $r_{t_1}(K) > 0$ and there exists at least one unchosen individual with trait $t_1$. At Step $k'$, we choose an individual either by Case 0 or Cases 3(i)-3(v) since $\Delta(k') = 0$ and we are in the main round. Therefore, an individual with a trait for which there are positive remaining reservations must be chosen at Step $k'$. We get a contradiction.

**Lemma 9.** A paired-admission choice rule eliminates justified envy.

*Proof.* Let $C$ be a paired-admission choice rule. Suppose, for contradiction, that it does not eliminate justified envy. Then, there exist $I \subseteq \mathcal{I}$, $i \in C(I)$, and $j \in I \setminus C(I)$ such that $j \succ i$ and $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$. Since $i$ is chosen and $j$ is rejected even though $j$ has a higher priority than $i$, it must be the case that $i$ has a trait that $j$ does not because of the construction of $C(I)$. Without loss of generality, let $t_1$ be this trait. We consider two cases depending on whether $i$ has trait $t_2$ or not.

**Case 1:** Suppose that $i$ has trait $t_2$, so $\tau(i) = \{t_1, t_2\}$. Since $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$, there must be at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait $t_1$. Therefore, the paired-admission choice correspondence must have terminated at the main round. If there are also at least $r_{t_2} + 1$ individuals in $C(I)$, then at the last Step $K$, $\Delta(K) = -1$. Furthermore, $i$ cannot be chosen at Case 0 for any step, since this would imply that another individual should have been chosen if both remaining reservations are zero or that if only one remaining reservation is positive, then the last individual chosen with this trait has lower priority than $j$. Likewise, $i$ cannot be chosen at Case 2, since $j$ is available and has higher priority than $i$ at this step. Therefore, when $i$ is chosen at a step, say $k'$, $\Delta(k') \geq 0$. Note that $k' \neq K$ because $\Delta(K) = -1$. Hence, there is a last step, say $k^*$ such that $\Delta(k^*) = 0$. 

**Case 2:** Suppose that $i$ does not have trait $t_2$, so $\tau(i) = \{t_1\}$. Let $\Delta(I(i)) = 1$, where $\Delta(I(i))$ is the number of traits for which reservations are not filled among the remaining individuals. Since $i$ is chosen at a step, say $k$, $\Delta(k) = 1$. Suppose, for contradiction, that $i$ is rejected, then $\Delta(k+1) = 0$. Therefore, there exists a last Step $K$, say $K(1)$, for which $\Delta(k') = 0$. 

**Proof.** Let $C$ be a paired-admission choice rule. Suppose, for contradiction, that it does not eliminate justified envy. Then, there exist $I \subseteq \mathcal{I}$, $i \in C(I)$, and $j \in I \setminus C(I)$ such that $j \succ i$ and $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$. Since $i$ is chosen and $j$ is rejected even though $j$ has a higher priority than $i$, it must be the case that $i$ has a trait that $j$ does not because of the construction of $C(I)$. Without loss of generality, let $t_1$ be this trait. We consider two cases depending on whether $i$ has trait $t_2$ or not.

**Case 1:** Suppose that $i$ has trait $t_2$, so $\tau(i) = \{t_1, t_2\}$. Since $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$, there must be at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait $t_1$. Therefore, the paired-admission choice correspondence must have terminated at the main round. If there are also at least $r_{t_2} + 1$ individuals in $C(I)$, then at the last Step $K$, $\Delta(K) = -1$. Furthermore, $i$ cannot be chosen at Case 0 for any step, since this would imply that another individual should have been chosen if both remaining reservations are zero or that if only one remaining reservation is positive, then the last individual chosen with this trait has lower priority than $j$. Likewise, $i$ cannot be chosen at Case 2, since $j$ is available and has higher priority than $i$ at this step. Therefore, when $i$ is chosen at a step, say $k'$, $\Delta(k') \geq 0$. Note that $k' \neq K$ because $\Delta(K) = -1$. Hence, there is a last step, say $k^*$ such that $\Delta(k^*) = 0$.
and $\Delta(k^* + 1) = -1$ (which can happen at Cases 3(i), 3(ii), 3(iv), and 3(v)). At Step $k^*$, $r_1(k^*), r_2(k^*) > 0$ and an individual with both traits must be chosen so that $\Delta(k^* + 1) = -1$. By construction, $\Delta(k) < 0$ for every $k > k^*$. Let $\tilde{i}$ be the last individual with both traits chosen. By construction, this individual is chosen at Step $k^*$ or after. Therefore, by construction, $\tilde{i}$'s priority is higher than an individual with at least one trait that is chosen when the remaining reservations are zero and that individual must have a strictly higher priority than $j$. Furthermore, by construction, $\tilde{i}$ must have a weakly higher priority than $i$ since $\tilde{i}$ is the last individual with both traits who is chosen. We get a contradiction that $i$ must have a strictly higher priority than $j$.

Otherwise, if there are at most $r_t$ individuals with trait $t_2$ in $C(I)$, then $j$ must also have $t_2$ because $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$. Furthermore, there must be at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait $t_1$. Therefore, all individuals must be chosen at the main round. In the last Step $K$, if $r_{t_2}(K) = 1$, then the last individual to be chosen, say $i'$, has trait $t_2$. Furthermore, since there are at least $r_{t_1} + 1$ individuals in $C(I)$ with trait $t_2$, $r_{t_1}(K) = 0$. This implies that $i'$ is chosen by Case 0 and, therefore, this individual has a higher priority than $j$. If $i'$ also has trait $t_1$, then we get a contradiction because $i$ has a weakly higher priority than $i'$. Therefore, $i'$ does not have trait $t_1$. Consider the last individual $i^*$ with trait $t_1$ to be chosen. At the step $i^*$ is chosen, the remaining reservations for $t_1$ is zero. Furthermore, there is a positive number of reservations for $t_2$ and there is at least one remaining individual with $t_2$. Therefore, $i^*$ also has $t_2$. But this implies that $i$ has a weakly higher priority than $i^*$, who in turn has a strictly higher priority than $j$ since $j$ is not chosen. This is a contradiction. If $r_{t_2}(K) \neq 1$, then $r_{t_2}(K) = 0$ and $\Delta(K) = -1$. If $\Delta(k) < 0$ for every Step $k$, then we get a contradiction because $i$ must be chosen by Case 0 for one of the traits which cannot be $t_2$ because $j$ also has trait $t_2$ and $j$ has a higher priority than $i$. If $i$ is chosen by Step 0 because the remaining reservations for $t_1$ is positive then an individual with trait $t_1$ is chosen later when there are no remaining reservations for $t_1$. Hence this individual has a higher priority than $j$, which is a contradiction. Therefore, it cannot be that $\Delta(k) < 0$ for every Step $k$. Let $k^*$ be the last step such that $\Delta(k^*) = 0$ and $\Delta(k^* + 1) = -1$. At Step $k^*$, $r_{t_1}(k^*) > 0$, $r_{t_2}(k^*) > 0$, and an individual with both traits must have been chosen. Let $i'$ be the last individual with both traits who is chosen. Therefore, $i'$ is chosen at a Step $k$ where $k \geq k^*$ such that $\Delta(k) \leq 0$. If $i'$ is chosen by Case 0 because of trait $t_1$ then the last individual chosen with trait $t_1$ has a higher priority than $j$, who in turn has a weakly lower priority than $i$, which is a contradiction. Likewise, if $i'$ is chosen by Case 0 because of trait $t_2$ or when no reservations remain, we get a contradiction. However, if $i'$ is chosen by Case 2 because he has the highest priority, we get a contradiction because $i'$ has a weakly lower priority than $i$. Finally, if $i'$ is chosen by one of Case 3(i), Case 3(ii), Case 3(iv), or Case 3(v), then we get a similar contradiction as in Case 0.
Case 2: Suppose that $i$ does not have trait $t_2$, so $T(i) = \{t_1\}$. Since $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$, if there are at least $r_{t_2} + 1$ individuals in $C(I)$ who have trait $t_2$, then there are also at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait $t_1$. Furthermore, when $i$ is chosen, there must be positive reservations for $t_1$ because $j$ is available and not chosen at this step. If the last chosen individual with trait $t_1$, say $i'$, does not have trait $t_2$, then we get a contradiction as this individual must have strictly lower priority than $j$ and there are no remaining reservations for $t_1$. If $i'$ has trait $t_2$ but there are no remaining reservations for $t_2$, we get a similar contradiction. Then the last individual with trait $t_2$, say $i^*$, does not have $t_1$ and there are no remaining reservations for $t_2$. We get a contradiction because $i^*$ has a weakly lower priority than $i'$ and $i'$ has a weakly lower priority than $i$. But $i^*$ is chosen while $j$ is rejected when there are no remaining reservations, so $i^*$ must have a strictly higher priority than $j$.

Now consider the case when there are at most $r_{t_2}$ individuals in $C(I)$ with trait $t_2$. If there are at most $r_{t_1}$ individuals in $C(I)$ who have trait $t_1$, then $j$ must have trait $t_2$ and the number of individuals in $C(I)$ with trait $t_2$ must be strictly less than $r_{t_2}$ because $\rho((C(I) \setminus \{i\}) \cup \{j\}) \geq \rho(C(I))$. But at the step when $i$ is chosen, $j$ must also have been considered because each one has a trait for which there is a positive remaining reservation, which is a contradiction. However, if there are at least $r_{t_1} + 1$ individuals in $C(I)$ who have trait $t_1$, then when the lowest priority individual in $C(I)$ who has trait $t_1$ is chosen if $j$ has trait $t_2$, then $j$ should have been chosen because $j$ has a strictly higher priority than $i$, which is a contradiction. But if $j$ does not have any trait, then the individual in $C(I)$ with the $r_{t_1} + 1$-th highest priority who has trait $t_1$ must also have trait $t_2$. Call this individual $i'$. But when $i$ is chosen at an earlier Step $k'$, we must have $r_{t_1}(k'), r_{t_2}(k') > 0$. If $\Delta(k') > 0$, then because of Case 1, the highest priority remaining individual who has both traits should have been chosen, which is a contradiction since $i'$ is available at that step. If $\Delta(k') < 0$, then because of Case 2, the highest priority remaining individual should have been chosen, which is also a contradiction since $j$ is available. If $\Delta(k') = 0$, then $i$ must have been chosen by Case 3(iv), which implies that $i$ has a strictly higher priority than $i'$. Furthermore, at this step there are more reservations for $t_1$ than $t_2$. From this step on, a combination of Case 2, Case 3(iv), and Case 3(v) must lead to a step where we have Case 3(vi). This is a contradiction because at Case 3(vi) reservations are filled without any excess for any of the traits.

Lemma 10. Every choice rule that maximally complies with the reservations, eliminates justified envy, and is non-wasteful has to be a selection from the paired-admission choice correspondence.

Proof. Let $C$ be a choice rule that maximally complies with the reservations, eliminates justified envy, and is non-wasteful. Let $I$ be a set of individuals. If $|I| \leq q$, then any non-wasteful choice rule selects $I$. Therefore, we assume that $|I| \geq q$ and $|C(I)| = q$. 

We prove the claim by mathematical induction on $q$. For the base case, when $q = 1$, we consider three possibilities:

1. If $r_{t_1}, r_{t_2} > 0$, then $\Delta(1) > 0$. Therefore, individual $i$ chosen by the paired-admission choice rule has the highest priority among individuals who maximize $\delta(\cdot | \emptyset)$. If $C(I)$ maximally complies with the reservations, then the individual in $C(I)$ and $i$ have the same number of traits. Furthermore, if $C(I) \neq \{i\}$, then $i$ has a strictly higher priority than the individual in $C(I)$, so there is an instance of justified envy. Therefore, $C(I) = \{i\}$.

2. If only reservations for one of the traits, say $t_1$ without loss of generality, is positive, then individual $i$ chosen by the paired-admission choice rule Case 0 has the highest priority among individuals who maximize $\delta(\cdot | \emptyset)$. If $C(I)$ maximally complies with the reservations, then either the individual in $C(I)$ and $i$ both have $t_1$ or they both do not have $t_1$. Furthermore, if $C(I) \neq \{i\}$, then $i$ has a strictly higher priority than the individual in $C(I)$, so there is an instance of justified envy. Therefore, $C(I) = \{i\}$.

3. If $r_{t_1}, r_{t_2} = 0$, then the individual with the highest priority $i$ must be chosen so that there is no instance of justified envy. Therefore, $C(I) = \{i\}$.

Suppose that the hypothesis is true up to capacity $q$. Now we show it for $q$. We consider two possibilities depending on the case at the first step of the paired-admission choice correspondence.

1. Suppose that at the first step of the paired-admission choice correspondence only one individual $i$ is chosen in the main round. Then $i \in C(I)$ otherwise either $C(I)$ does not maximally comply with the reservations for $I$ or there is an instance of justified envy. Likewise $i$ is in any paired-admission choice rule. Now consider the reduced market when $I \setminus \{i\}$ is the set of individuals, $q - 1$ is the capacity, and positive reservations for a trait is reduced by one if $i$ has that trait. In this reduced reservation market, $C(I) \setminus \{i\}$ is non-wasteful, maximally complies with the reservations for $I \setminus \{i\}$, and eliminates justified envy. Therefore, by mathematical induction hypothesis, $C(I) \setminus \{i\}$ is equal to the outcome of a paired-admission choice rule for the reduced reservation market. Hence, $C(I)$ is equal to the outcome of a paired-admission choice rule for the original reservation market.

2. Suppose that at the first step of the paired-admission choice correspondence, all individuals are chosen at the waitlist round. Thus, $r_{t_1} = r_{t_2} = r > 0$, $|I_{t_1}|, |I_{t_2}| > r$, and $q = 2r$. Furthermore, the highest priority individual does not have any traits and there exists an individual with both traits. Finally, no individual among the $r$ highest priority individuals who have trait $t_1$ and $r$ highest priority individuals who have trait $t_2$ has both traits. Let $i$ be the highest priority individual with no
traits and \( j \) be the highest priority individual with no traits. If \( C(I) \) has an individual with no traits, then \( i \in C(I) \) to eliminate justified envy. To maximally comply with the reservations, an individual with both traits must also be in \( C(I) \). Therefore, \( j \in C(I) \) to eliminate justified envy. Hence, if \( C(I) \) has an individual with no traits, then \( \{i, j\} \subseteq C(I) \). Now, consider the reduced reservation problem with the set of individuals \( I \setminus \{i, j\} \), capacity \( q - 2 \), reservations for both traits \( r - 1 \). Then in the reduced problem, \( C(I) \setminus \{i, j\} \) maximally complies with the reservations, eliminates justified envy, and is non-wasteful. By the mathematical induction hypothesis, \( C(I) \setminus \{i, j\} \) is the outcome of a paired-admission choice rule. Since \( \{i, j\} \) is selected by one of the paired-admission choice rules in the original reservation problem, we conclude that \( C(I) \) is a paired-admission choice rule. The second possibility is that \( C(I) \) does not have any individuals with no traits. In this case, \( C(I) \) also cannot have any individual with both traits because there are \( r \) individuals with trait \( t_1 \) only and \( r \) individuals with trait \( t_2 \) only who have a higher priority than the highest priority individual with both traits. Therefore, \( C(I) \) must have all these individuals, which is the outcome of a paired-admission choice rule.

This completes the proofs of all claims.

\[\Box\]

**Proof of Proposition 2.** Let \( I \subseteq \mathcal{I} \) be a set of applicants and \( C \) a paired-admission choice rule. We prove each statement separately.

**Proof of (1a):** Let \( i \in C(I) \) with \( \tau(i) = \emptyset \) or \( \tau(i) = \{t_1, t_2\} \). If \( i \) is chosen in the main round, \( i \in C_{\min\max}(I) \) because the same set of individuals is chosen for every paired-admission choice rule in the main round. However, if \( i \) is chosen in the waitlist round, then \( i \in C_{\min\max}(I) \) because \( C_{\min\max} \) chooses the maximum number of pairs including an individual with no traits and an individual with both traits.

**Proof of (1b):** Let \( i \in C_{\min\max}(I) \) with \( \tau(i) = \{t_1\} \) or \( \tau(i) = \{t_2\} \). If \( i \) is chosen in the main round, \( i \in C(I) \) because the same set of individuals is chosen for every paired-admission choice rule in the main round. However, if \( i \) is chosen in the waitlist round, then \( i \in C(I) \) because \( C \) chooses the minimum number of pairs including an individual with only trait \( t_1 \) and an individual with only trait \( t_2 \).

**Proof of (1c):** If the paired-admission choice correspondence does not have a waitlist round, then the claim is trivial. Likewise if \( |I| \leq q \), the claim is trivial. Suppose that the correspondence ends at the waitlist round and \( |I| > q \). Then the highest priority individual in \( I \setminus C_{\min\max}(I) \) is either \( i_{m^*+1} \) or \( j_{m^*+1} \). If \( C(I) \neq C_{\min\max}(I) \), the highest priority individual in \( I \setminus C_{\min\max}(I) \) has a weakly higher priority than \( i_{m^*}' \). By construction of the correspondence, \( i_{m^*}' \) has a strictly higher priority than both \( i_{m^*+1} \) and \( j_{m^*+1} \).
Proof of (2a): Let \( i \in C(I) \) with \( \tau(i) = \{ t_1 \} \) or \( \tau(i) = \{ t_2 \} \). If \( i \) is chosen in the main round, \( i \in C_{\text{max min}}(I) \) because the same set of individuals is chosen for every paired-admission choice rule in the main round. However, if \( i \) is chosen in the waitlist round, then \( i \in C_{\text{max min}}(I) \) because \( C_{\text{max min}} \) chooses the maximum number of pairs including an individual with only trait \( t_1 \) and an individual with only trait \( t_2 \).

Proof of (2b): Let \( i \in C_{\text{max min}}(I) \) with \( \tau(i) = \emptyset \) or \( \tau(i) = \{ t_1, t_2 \} \). If \( i \) is chosen in the main round, \( i \in C(I) \) because the same set of individuals is chosen for every paired-admission choice rule in the main round. However, if \( i \) is chosen in the waitlist round, then \( i \in C(I) \) because \( C_{\text{max min}} \) chooses the minimum number of pairs including an individual with no traits and an individual with both traits.

Proof of (2c): If the paired-admission choice correspondence does not have a waitlist round, then the claim is trivial. Likewise if \( |I| \leq q \), the claim is trivial. Suppose that the correspondence ends at the waitlist round and \( |I| > q \). Then the lowest priority individual in \( C_{\text{max min}}(I) \) is either \( i_r \) or \( j_r \). If \( C(I) \neq C_{\text{min max}}(I) \), then the lowest priority individual in \( C(I) \) has a weakly lower priority than \( j'_1 \). By construction of the correspondence, both \( i_r \) and \( j_r \) have a strictly higher priority than \( j'_1 \). \( \square \)

Proof of Proposition 3: If the paired-admission choice correspondence terminates at the main round, then the claim is satisfied trivially. Otherwise, if the paired-admission choice correspondence terminates at the waitlist round, then \( i \) and \( j \) belong to the same pair \( q \) or \( p \) described in the construction of the paired-admission choice correspondence. By construction, for any paired-admission choice rule, the intersection of the outcome of this choice rule when \( I \) is the set of applicants and the pair is either the empty set or the pair. The conclusion follows. \( \square \)

Proof of Proposition 5: Consider the following reservation market:

- \( I = \{ i_1, i_2, i_3, i_4 \} \),
- \( T = \{ t_1, t_2 \} \),
- \( i_1 \pi i_2 \pi i_3 \pi i_4 \),
- \( \tau(i_1) = \emptyset, \tau(i_2) = \{ t_1 \}, \tau(i_3) = \{ t_1, t_2 \}, \tau(i_4) = \{ t_2 \} \), and
- \( r_{t_1} = r_{t_2} = 1 \) and \( q = 2 \).

When the set of applicants is \( I \), the correspondence terminates at the main round and chooses \( \{ i_1, i_3 \} \). Therefore, all paired-admission choice rules have to select this set. Likewise, when the set of applicants is \( \{ i_1, i_2, i_4 \} \), the correspondence still terminates at the main round and chooses \( \{ i_2, i_4 \} \), which has to be the outcome for all paired-admission choice rules. We conclude that no paired-admission choice rule \( C \) can satisfy the substitutes condition because

\[ i_1 \in C(I) \text{ and } i_1 \notin C(I \setminus \{ i_3 \}) \].
Proof of Proposition 6. Let $I \subseteq \mathcal{I}$ be a set of applicants and $C$ a paired-admission choice rule. We prove each statement separately.

Proof of (1a): Let $i \in (C_{\text{hor}} \circ C)(I)$ with $\tau(i) = \{p, \ell, m\}$. If $i \in C(I)$, then by Proposition 2, $i \in C^{\text{min max}}(I)$, which implies $i \in (C_{\text{hor}} \circ C^{\text{min max}})(I)$. Otherwise, if $i \notin C(I)$, then $i$ must have been chosen at the second step of $C_{\text{hor}} \circ C$. If $i \in C^{\text{min max}}(I)$, we are done. Suppose that $i \notin C^{\text{min max}}(I)$. This implies that every individual in $C^{\text{min max}}(I)$ has a higher priority than $i$ because $\tau(i) = \{t_1, t_2\}$. Since $i$ must have been chosen at the second step of $C_{\text{hor}} \circ C$, every individual with a higher priority ranking than $i$ must also be in $(C_{\text{hor}} \circ C)(I)$. Therefore, $C^{\text{min max}}(I) \subseteq (C_{\text{hor}} \circ C)(I)$. Since individuals with the highest priority ranking is chosen at the second step of both $C_{\text{hor}} \circ C$ and $C_{\text{hor}} \circ C^{\text{min max}}$, this implies $(C_{\text{hor}} \circ C)(I) = (C_{\text{hor}} \circ C^{\text{min max}})(I)$, so $i \in (C_{\text{hor}} \circ C^{\text{min max}})(I)$.

Proof of (1b): Let $i \in (C_{\text{hor}} \circ C^{\text{min max}})(I)$ with $\tau(i) = \{p, \ell\}$ or $\tau(i) = \{p, m\}$. Without loss of generality assume that $\tau(i) = \{p, \ell\}$. If $i \in C^{\text{min max}}(I)$, then $i \in C(I)$ by Proposition 2, which implies that $(C_{\text{hor}} \circ C)(I)$. Suppose that $i \notin C^{\text{min max}}(I)$, then $i$ must have been chosen at the second step of $C_{\text{hor}} \circ C^{\text{min max}}$ and all individuals with a higher priority than $i$ must also be in $(C_{\text{hor}} \circ C^{\text{min max}})(I)$. If, for every $j \in C(I)$, $j \pi i$, then $i \in (C_{\text{hor}} \circ C)(I)$ because there are no individuals in $C(I)$ who are ranked lower than $i$ whereas there is at least one individual with the set of traits $\{p, \ell, m\}$ in $C^{\text{min max}}(I)$ who is ranked lower than $i$. If there is an individual $j \in C(I)$ with $\tau(j) = \tau(i)$ such that $i \pi j$, then $i \in C(I)$, which implies $(C_{\text{hor}} \circ C)(I)$. Suppose that for every $j \in C(I)$ with $\tau(j) = \tau(i)$, $j \pi i$. Therefore, there exists a number of individuals in $C(I)$ with the set of traits $\{p, m\}$ who have lower priority than $i$. Let $n$ be this number. By construction, there are at least $n$ pairs in $C^{\text{min max}}(I)$ which have an individual with the set of traits $\{p, \ell, m\}$. These individuals have a lower priority than every individual in $C(I)$. Therefore, there are at least $n$ individuals in $C^{\text{min max}}(I)$ who have a lower priority than $i$. Hence, $i \in (C_{\text{hor}} \circ C^{\text{min max}})(I)$ implies that $i \in (C_{\text{hor}} \circ C)(I)$ because the number of positions that are filled in the second step are the same for both choice rules and the number of individuals with a priority lower than $i$ that are admitted at the first step are weakly higher in $(C_{\text{hor}} \circ C^{\text{min max}})(I)$ than that in $(C_{\text{hor}} \circ C)(I)$.

Proof of (2a): Let $i \in (C_{\text{hor}} \circ C)(I)$ with $\tau(i) = \{p, \ell\}$ or $\tau(i) = \{p, m\}$. If $i \in C(I)$, then $i \in C^{\text{max min}}(I)$ by Proposition 2, which implies that $(C_{\text{hor}} \circ C^{\text{max min}})(I)$. Likewise if $i \in C^{\text{max min}}(I)$, then we get $(C_{\text{hor}} \circ C^{\text{max min}})(I)$. Suppose that $i \notin C^{\text{max min}}(I)$ and $i \notin C(I)$. Then $i$ must have been chosen at the second step of $C_{\text{hor}} \circ C$ and all individuals with a higher priority than $i$ must also be in $(C_{\text{hor}} \circ C)(I)$. Since $i \notin C^{\text{max min}}(I)$, this implies $C^{\text{max min}}(I) \subseteq (C_{\text{hor}} \circ C)(I)$. Therefore, $i \in (C_{\text{hor}} \circ C^{\text{max min}})(I)$ because there must be enough capacity at the second step of $C_{\text{hor}} \circ C^{\text{max min}}$ because $C_{\text{hor}} \circ C^{\text{max min}}(I)$ does not
have any pairs including low-priority individuals with the set of traits \(\{p, \ell, m\}\) whereas \(C(I)\) may have some.

**Proof of (2b):** Let \(i \in (\text{Chor} \circ \text{Cmax min})(I)\) with \(\tau(i) = \{p, \ell, m\}\). If \(i \in \text{Cmax min}(I)\), then \(i \in C(I)\) by Proposition 2, which implies \(i \in (\text{Chor} \circ C)(I)\). Suppose that \(i \notin \text{Cmax min}(I)\), then \(i\) must have been chosen at the second step of \((\text{Chor} \circ \text{Cmax min})(I)\), and all individuals with a higher priority than \(i\) must also have been chosen. Since \(C(I)\) may have some individuals with the set of traits \(\tau(i) = \{p, \ell, m\}\) who have a lower priority than \(i\), \(i \in (\text{Chor} \circ C)(I)\) because the number of individuals in \((\text{Chor} \circ C)(I)\) and \((\text{Chor} \circ \text{Cmax min})(I)\) are the same.

**Proof of (2c):** Let \(i\) be the lowest priority individual in \((\text{Chor} \circ \text{Cmax min})(I)\). If \(i \in \text{Cmax min}(I)\), then \(i\) has a weakly higher priority than the lowest priority individual in \(C(I)\) by Proposition 2, which implies that \(i\) has a weakly higher priority than the lowest priority individual in \((\text{Chor} \circ C)(I)\). If \(i \notin \text{Cmax min}(I)\) and \(C(I) \subseteq (\text{Chor} \circ \text{Cmax min})(I)\), then \((\text{Chor} \circ \text{Cmax min})(I) = (\text{Chor} \circ C)(I)\) and the claim follows. If \(i \notin \text{Cmax min}(I)\) and \(C(I) \notin (\text{Chor} \circ \text{Cmax min})(I)\), then there is an individual with the set of traits \(\{p, \ell, m\}\) in \(C(I)\) who has a lower priority than \(i\), which implies that \(i\) has a higher priority than the lowest priority individual in \((\text{Chor} \circ C)(I)\).

\(\square\)

**References**


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