Equilibria in Bottleneck Games*

Ryo Kawasaki Hideo Konishi Junki Yukawa

January 20, 2018

Abstract

This paper introduces a bottleneck game with finite sets of commuters and departing time slots as an extension of congestion games by Milchtaich (1996). After characterizing Nash equilibrium of the game, we provide sufficient conditions for which the equivalence between Nash and strong equilibria holds. Somewhat surprisingly, unlike in congestion games, a Nash equilibrium in pure strategies may often fail to exist, even when players are homogeneous. In contrast, when there is a continuum of atomless players, the existence of a Nash equilibrium and the equivalence between the set of Nash and strong equilibria hold as in congestion games (Konishi, Le Breton, and Weber, 1997a).

1 Introduction

A bottleneck model is used in analyzing traffic congestion during rush hours, where commuters depart from their origins (e.g. their houses) to their destinations (e.g. their workplaces). The simplest model was independently analyzed by Vickrey (1969) and Hendrickson and Kocur (1981), where a continuum of commuters depart from a single origin to a single destination connected by a single road with continuous time horizon. Along the road, there is a bottleneck in which a queue forms if the number of commuters exceeds the capacity of the road at a given time, where the capacity is defined as the maximum number of commuters that can pass through it in each slot. In these papers, commuters decide on the departure time based on the trade-offs between congestion and their optimal arrival time. Players are assumed to have the same preferred time of arrival and a specific form of the trip cost function. Subsequent papers,

*The authors thank conference participants at the 22nd Decentralization Conference in Japan, UECE Lisbon Meetings in Game Theory and Applications 2016, the 11th International Conference on Game Theory and Management, the 13th European Meeting on Game Theory and the East Asian Game Theory Conference 2017. This research was carried out while Yukawa was a visiting scholar at Boston College. Their hospitality and research supports are appreciated. Kawasaki acknowledges financial support from the Japan Society for the Promotion of Science (JSPS) through the Grants-in-Aid for Young Scientists (B) (No. 26780115). Yukawa acknowledges financial support from Program for Leading Graduate School "Academy for Co-creative Education of Environment and Energy Science", MEXT, Japan.

such as Smith (1983), Daganzo (1985) and Arnott et al. (1990), introduce some heterogeneity of commuters.

In this paper, we define a bottleneck game with a finite set of departure time slots. Each commuter has preferences on two arguments: her departure time and the length of the queue in which she has to wait to pass through the bottleneck. Our game is an anonymous game with congestion generated by a queue structure without imposing a specific form of trip costs function. In this sense, our model can be regarded as an abstract generalization of bottleneck models in the aforementioned papers. Moreover, this abstract setup allows us to interpret our model in a different context other than traffic congestion. For example, consider a location choice problem along a river, in which residents pollute the river while the river has an ability to abate pollution up to some level (capacity) at each location of the river. We can allow residents' arbitrary preferences over locations (such as scenic and/or convenient locations) on the river, resulting in emergence of congested locations causing pollutions for downstream locations.

Mathematically, our model is also an extension of the congestion game by Milchtaich (1996), which has following three properties:[1] Anonymity (A), Congestion (C) and Independence of Irrelevant Choices (IIC). First, A requires that the payoff of each player depends on the number of players who choose each action and not on the players' names. Second, C states that the payoff of each player increases if another player who had chosen the same strategy chooses a different strategy. Finally, IIC states that the payoff of a player is not affected even if another player that chooses a different strategy from hers switches to another strategy that is also a different strategy from hers. In this game, Milchtaich (1996) shows that a congestion game always has a Nash equilibrium in pure strategies. Konishi et al. (1997a) shows that in the same model, any strictly improving coalitional deviation from a Nash equilibrium results in another Nash equilibrium, thus implying a congestion game also admits a strong equilibrium that is immune to any strictly improving coalitional deviation. They also show that if there is a continuum of atomless players, then the sets of Nash and strong equilibria coincide with each other.

Our bottleneck game does not satisfy IIC, whereas the other two conditions hold (though C applies in a strict sense only after a queue forms by exceeding the capacity). Specifically, IIC would be violated in the case where a player who had departed later then switched to an earlier departure time and thereby possibly creating a longer queue for some of those players which she leaps over. With this

[1] The name "congestion game" is sometimes attributed to a class of games introduced by Rosenthal (1973), who considers a situation in which players choose a combination of primary factors out of a certain number of alternatives. Each player's payoff is determined by the sum of the costs of each primary factor she chooses, while the cost of each primary factor depends on the number of players who choose it, and not on the players' names. Rosenthal (1973) proved that there always exists at least one pure-strategy Nash equilibrium by constructing a potential function, which is later formalized by Monderer and Shapley (1996). However, these congestion games do not require payoffs being negatively affected by the population while requiring that payoff functions have the same form among the players who take same factors. We refer Milchtaich's game by "congestion games."

difference, we first show that the equivalence between Nash and strong equilibria under some conditions (Propositions 2, 3, and 4), show that a Nash equilibrium may not exist even when players are Homogeneous (H) and other stringent conditions such as Single-Peakedness (SP) and Order-Preservation (OP) on the payoff function are satisfied (Examples 4 and 5). With an even more stringent condition, we show the existence of Nash equilibrium (Proposition 5). These results are in stark contrast with the ones in Milchtaich's congestion games: Nash equilibrium always exists, and it is hard to ensure the equivalence between Nash and strong equilibrium due to coordination failures unless players are homogeneous. In contrast, when players are atomless, we can establish both the existence of Nash equilibrium and equivalence between Nash and strong equilibria exactly in the same way as in congestion games (Proposition 6).

The rest of the paper is organized as follows. In Section 2, we define our bottleneck game with a finite number of players. In Section 3, we provide three sufficient conditions under which Nash and strong equilibria are equivalent to each other. In Section 4, we show that our bottleneck game may not have a Nash equilibrium in pure strategies even when players are homogeneous. We also provide a positive result for the existence although the conditions are very stringent. Section 5 introduces a bottleneck game with atomless players, and we show that the existence of Nash and the equivalence between Nash and strong equilibria all hold in this idealized environment. Section 6 concludes.

2 The Model with a Finite Number of Players

We consider a commuting road with a finite number of departing time slots. Let $t = 1, ..., T$ be available departing time slots ($t = 1$ is the earliest). Each discrete time unit can represent every minute or every five minutes, for example. Let the set of departing time slots be $\mathcal{T} = \{1, ..., T\}$. Let q_{t-1} be the length of the resulting queue at departing time slot $t-1$. Then, the length of the queue at time slot t can be calculated as $q_t = \max\{0, q_{t-1} + m_t - c\}$, where m_t is the number of players who depart at time slot t, and $c \in \mathbb{Z}_+$ is the capacity of the bottleneck. We also introduce notation $\tilde{q}_t = q_{t-1} + m_t - c$ to describe possible slacks: $\tilde{q}_t < 0$ means that the road capacity is not binding at time slot t, and the queue at time slot t does not develop even if an additional car joins. Let $i = 1, ..., n$ be players who can be heterogeneous in their preferences. The set of players is denoted by $N = \{1, ..., n\}$. Player i's choice (strategy) of departing time is denoted $\tau_i \in \mathcal{T}$. A strategy profile is $\tau = (\tau_1, ..., \tau_n) \in \mathcal{T}^N$, and resulting queue lengths at all time slots are described by a vector $\tilde{q}(\tau) = (\tilde{q}_1(\tau), ..., \tilde{q}_T(\tau))$ and $q(\tau) = (q_1(\tau), ..., q_T(\tau))$, respectively. Denote player i's payoff from choosing time slot t with queue length q_t is denoted by $u^i(t, q_t)$. Note that by this specification, we are assuming **Anonymity (A)** implicitly – it does not matter who is in the queue.

Congestion (C) For all $i \in N$, $u^i(t, k) > u^i(t, k+1)$ holds for all $t \in \mathcal{T}$ and all $k \in \mathbb{Z}_+$.

Note that although $q_t = 0$ holds irrespective of $\tilde{q}_t = 0$ or $\tilde{q}_t < 0$, these two cases make some difference when an additional car arrives at time slot t. In the former case, a queue develops with an additional car while in the latter case it does not develop. A strategy profile τ is a **Nash equilibrium** if and only if for all $i \in N$ and all $t \in \mathcal{T}$, $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t, q_t(t, \tau_{-i}))$ holds. Before we present a characterization of Nash equilibrium, we introduce some new terms.

Definition 1.

1. A single slot t is said to be a **basin** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) < 0$ and $\tilde{q}_{t-1}(\tau) \leq 0$.
2. A single slot t is a **single terrace** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) = 0$ and $\tilde{q}_{t-1}(\tau) \leq 0$.
3. A consecutive slots $I = [t_1, t_2]$ with $1 \leq t_1 < t_2 \leq T$ is said to be a **connected terrace** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) > 0$ for all $t \in [t_1, t_2)$, $\tilde{q}_{t_1-1}(\tau) \leq 0$, and $\tilde{q}_{t_2}(\tau) \leq 0$.

The following is a characterization of Nash equilibria in this game (straightforward).

Proposition 1. A strategy profile τ is a Nash equilibrium if and only if for all $i \in N$,

1. for all $t' < \tau_i$, $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', \max\{\tilde{q}_{t'}(\tau) + 1, 0\})$
2. for all $t' > \tau_i$,
   (a) $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', \max\{\tilde{q}_{t'}(\tau), 0\})$ if $t' \in [t_1, t_2]$, where $[t_1, t_2]$ is a connected terrace at τ such that $\tau^i \in [t_1, t_2]$,
   (b) $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', \max\{\tilde{q}_{t'}(\tau) + 1, 0\})$, otherwise.

With IIC, Konishi et al. (1997a) shows that with strict preferences, every Nash equilibrium has the same structure (the same distribution of strategies—the game satisfies anonymity) in their domain. However, in our domain, there may be Nash equilibria with multiple distinct queue structures even under strict preferences.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{1, 2, 3, 4\}$ with capacity $c = 1$. Players 1 and 2 are attached to time slots 1 and 2, respectively (that is, preferences are such that they will not choose to move to any other time slot under any circumstance). Players 3 and 4 have the following preferences, respectively:

$$u^3(2,0) > u^3(1,0) > u^3(2,1) > u^3(3,0) > u^3(4,0) > u^3(1,1) > \mathit{others}$$
$$u^4(4,0) > u^4(1,1) > u^4(3,0) > u^4(2,1) > u^4(3,1) > \mathit{others}$$

There are two Nash equilibria: $\tau = (\tau_1, \tau_2, \tau_3, \tau_4) = (1, 2, 4, 1)$ and $\tau' = (1, 2, 2, 4)$. These two Nash equilibria have two distinct queue structures, and the latter is a strong equilibrium (see the next section).□

3 Equivalence between Nash and Strong Equilibria

A **coalitional deviation** from τ is a pair of $(C, \hat{\tau}^C)$ such that (i) $C \neq \emptyset$, and (ii) for all $i \in C$, $u^i(\hat{\tau}) > u^i(\tau)$, where $\hat{\tau} = (\hat{\tau}^C, \tau^{-C})$. A **strong equilibrium** is a strategy profile such that there is no coalitional deviation from τ. In a special case, we can show that Nash equilibrium is unique and is equivalent to strong equilibrium. This is a unique result in our domain, since in the domain of Konishi et al. (1997a), it is virtually impossible to exclude coordination failure: that is, it is not easy to show the equivalence between Nash and strong equilibria.

Proposition 2. Suppose that there is a Nash equilibrium τ with a unique connected terrace $[t_1, t_2]$, and $\tilde{q}_t(\tau) < 0$ for all $t \notin [t_1, t_2]$. Then, τ is a strong equilibrium.

We will prove this result with the following two claims.

Claim 1. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. Then, $q_t(\hat{\tau}) \leq q_t(\tau)$ for all $t \in \mathcal{T}$, where $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$.

Proof. Suppose not. Then, there exists at least one slot t such that

$$q_t(\hat{\tau}) > q_t(\tau). \tag{1}$$

If multiple slots are found, take the earliest such slot. Since the queue-length at slot t strictly increases, there must be at least one player who deviates to slot t at $\hat{\tau}$, i.e., $m_t(\hat{\tau}) > m_t(\tau)$. Then, we can find at least one member i of C who deviates to $\hat{\tau}_i = t$ from $\tau_i \neq t$ since only the members of C can change their strategies. Since the deviation is strictly improving, it must hold that

$$u^i(t, q_t(\hat{\tau})) > u^i(\tau_i, q_{\tau_i}(\tau)). \tag{2}$$

Note that (1) can be rewritten as

$$q_t(\hat{\tau}) \geq q_t(\tau) + 1 > q_t(\tau),$$

and by C we have

$$u^i(t, q_t(\hat{\tau})) \leq u^i(t, q_t(\tau) + 1) < u^i(t, q_t(\tau)).$$

Together with (2), we have

$$u^i(\tau_i, q_{\tau_i}(\tau)) < u^i(t, q_t(\tau) + 1).$$

This shows that under τ, player i could have switched to slot t and obtained a higher payoff. This contradicts that τ is a Nash equilibrium. $\square$

Claim 2. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. Then, no member of C deviates to slots t such that $\tilde{q}_t(\tau) < 0$.

Proof. Suppose not. Then, there exists at least one member $i \in S$ such that $\tilde{q}_t(\tau) < 0$ with $t = \hat{\tau}_i$. Letting $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$, we consider two cases:

$$\begin{aligned} &\text{(i)} \quad \tilde{q}_t(\tau) < \tilde{q}_t(\tau) + 1 = \tilde{q}_t(t, \tau_{-i}) \le \tilde{q}_t(\hat{\tau}) \le 0, \\ &\text{(ii)} \quad \tilde{q}_t(\tau) < \tilde{q}_t(\tau) + 1 = \tilde{q}_t(t, \tau_{-i}) \le 0 < \tilde{q}_t(\hat{\tau}). \end{aligned}$$

Since the deviation is strictly improving, it must follow that

$$u^i(t, q_t(\hat{\tau})) > u^i(\tau_i, q_{\tau_i}(\tau)).$$

In case (i), noting $q_t(\tau) = q_t(t, \tau_{-i}) = q_t(\hat{\tau}) = 0$, we obtain

$$u^i(t, q_t(t, \tau_{-i})) = u^i(t, q_t(\hat{\tau})) > u^i(\tau_i, q_{\tau_i}(\tau)).$$

This shows that under τ, player i could have switched to slot t and obtained higher payoff. This contradicts that τ is a Nash equilibrium.

In case (ii), we immediately obtain

$$0 = q_t(\tau) < q_t(\hat{\tau}),$$

contradicting Claim 1. $\square$

Now, we are ready to prove Proposition 2.

Proof of Proposition 2. Suppose that there is a coalitional deviation $(C, \hat{\tau}_C)$. By Claim 2, all members of C choose time slots in $[t_1, t_2]$ under τ, and no member of C will not go out of $[t_1, t_2]$ under $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$.

Denote by $\bar{t}$, $\hat{t}$ the last slots which coalition members choose at τ, $\hat{\tau}$, respectively, i.e., $\bar{t} = \max\{\tau_i : i \in C\}$ and $\hat{t} = \max\{\hat{\tau}_j : j \in C\}$. We consider two cases: (i) $\hat{t} < \bar{t}$ and (ii) $\hat{t} \ge \bar{t}$.

In case (i), noting that $\left|\{i \in C : \tau_i \in [t_1, \hat{t}]\}\right| \le |C|-1$ and $\left|\{i \in C : \hat{\tau}_i \in [t_1, \hat{t}]\}\right| = |C|$, we have $\Delta q_{\hat{t}} := q_{\hat{t}}(\hat{\tau}) - q_{\hat{t}}(\tau) \ge 1$, since all slots in $[t_1, \bar{t}]$ belong to connected terrace $[t_1, t_2]$. However, this contradicts Claim 1.

Then, we consider case (ii). First we have $\Delta q_{\hat{t}} = 0$, i.e., $\tilde{q}_{\hat{t}}(\tau) = \tilde{q}_{\hat{t}}(\hat{\tau})$, since $\left|\{i \in C : \tau_i \in [t_1, \hat{t}]\}\right| = \left|\{i \in C : \hat{\tau}_i \in [t_1, \hat{t}]\}\right| = |C|$ by the definitions of $\bar{t}$ and $\hat{t}$, and all slots in $[t_1, \hat{t}]$ belong to connected terrace $[t_1, t_2]$. Moreover, the deviation is strictly improving, and there must be member j of C such that $\hat{\tau}_j = \hat{t}$ and $\tau_j \ne \hat{t}$, which implies $\tau_j < \hat{\tau}_j = \hat{t}$. That is, player j delayed her departure time. These suggest that she could have done that under τ as well. This is a contradiction with τ's being a Nash equilibrium. $\square$

The above result relies both on the uniqueness of connected terrace and the absence of single terraces in equilibrium. The next example shows that the equivalence result may not hold if the conditions are not satisfied.

Example 2. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{T} = \{1, 2, 3, 4, 5\}$ with capacity $c = 1$. Players 1, 2, 3 and 4 are attached to time slots 1, 2, 4, and 5, respectively. Players 5 and 6 have the following preferences, respectively:

$$u^5(1,0) > u^5(2,0) > u^5(4,0) > u^5(5,0) > u^5(1,1) > u^5(2,1) > u^5(4,1) > u^5(5,1) > others$$

$$u^6(4,0) > u^6(5,0) > u^6(1,0) > u^6(2,0) > u^6(4,1) > u^6(5,1) > u^6(1,1) > u^6(2,1) > others$$

There are two Nash equilibria: $\tau = (1, 2, 4, 5, 1, 4)$ and $\tau' = (1, 2, 4, 5, 4, 1)$. In these cases $\tilde{q}_3 = 0$. Only τ is a strong equilibrium. □

An additional natural condition allows Proposition 2 to extend to the case with multiple connected terraces. We say that the time slot $t_i^* \in \mathcal{T}$ is an **optimal slot** for player $i \in N$ if $u^i(t_i^*, 0) > u^i(t, 0)$ for all $t \in \mathcal{T}, t \neq t_i^*$.

Single-Peakedness (SP). Let player i's optimal slot be $t_i^* \in \mathcal{T}$. Then, for all $i \in N$, and all $t' < t < t_i^*$ or $t_i^* < t < t'$, $u^i(t, 0) > u^i(t', 0)$ holds.

Proposition 3. Suppose that there is a Nash equilibrium τ in which (i) there is no single terrace, and (ii) any pair of connected terraces is separated by a basin. Then, τ is a strong equilibrium if we assume SP in addition to C and A.

Proof. First note that Claims 1 and 2 apply to this case. Under the assumption, there are K connected terraces $[t_1, t_2], ..., [t_{2k-1}, t_{2k}], ..., [t_{2K-1}, t_{2K}]$ such that $\tilde{q}_t < 0$ for all $t \in (t_{2k}, t_{2k+1})$ for each $k = 1, ..., K$. Focus on the kth connected terrace $[t_{2k-1}, t_{2k}]$. Since τ is a Nash equilibrium, any player i with $\tau_i \in [t_{2k-1}, t_{2k}]$ satisfies $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t_{2k-1} - 1, 0)$ and $u^i(\tau_i, t_{2k} + 1, 0)$. By SP, we have $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t, 0)$ for all $t \leq t_{2k-1} - 1$ and $t \geq t_{2k} + 1$. Thus, for any coalitional deviation from τ, $(C, \hat{\tau}_C)$, if $i \in C$ with $\tau_i \in [t_{2k-1}, t_{2k}]$ then $\hat{\tau}_i \in [t_{2k-1}, t_{2k}]$ must hold. As a result, Proposition 2 extends to this domain. □

If connected terraces are not separated by $\tilde{q}_t < 0$, the equivalence between Nash and strong equilibria need not hold.

Example 3. Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{T} = \{1, 2, 3, 4, 5\}$ with capacity $c = 2$. Players 1, 2, and 3 are attached to time slot 1, and players 4 and 5 are attached to time slot 3. Players 6 and 7 have the following preferences:

$$u^6(3,0) > u^6(3,1) > u^6(2,0) > u^6(1,0) > u^6(4,0) > u^6(3,2) > others$$

$$u^7(2,0) > u^7(3,0) > u^7(3,1) > u^7(4,0) > u^7(2,1) > u^7(1,0) > others$$

There is a Nash equilibrium $\tau = (1,1,1,3,3,2,3)$, but $(C, \hat{\tau}_C) = (\{6,7\},(3,2))$ is a coalitional deviation from τ. The destination profile $\hat{\tau} = (1,1,1,3,3,3,2)$ is a strong equilibrium. In this example, SP is satisfied, but $\tilde{q}_2(\tau) = \tilde{q}_2(\hat{\tau}) = 0$, and the two connected terraces $[1,2]$ and $[3,4]$ are not separated by $\tilde{q}_2 < 0$. □

The following variation of Example 3 also violates SP, but it shows that we cannot ensure the existence of strong equilibrium even if there exists a Nash equilibrium.

Example 3'. Let $N = \{1,2,3,4,5\}$ and $\mathcal{T} = \{1,2,3,4,5\}$ with capacity $c = 1$. Players 4 and 5 are attached to time slots 1 and 4, respectively. Players 1,2, and 3 have the following preferences, respectively:

$$\begin{aligned} u^1(1,1) &> u^1(4,1) > u^1(1,2), \\ u^2(2,0) &> u^2(1,1) > u^2(1,2) > u^2(2,1), \\ u^3(3,0) &> u^3(1,2) > u^3(4,1) > u^3(3,1), \end{aligned}$$

There is only one Nash equilibrium $\tau = (1,1,4,1,4)$. However, $(C, \hat{\tau}_C) = (\{1,2,3\},(4,2,3))$ is a coalitional deviation. Hence, τ is not a strong equilibrium. □

We can prove the equivalence between Nash and strong equilibria without imposing any condition on a resulting equilibrium queue structure when players have the same payoff functions, although the proof is surprisingly involved.[2]

Homogeneity (H). For all players $i, j \in N$, $u^i = u^j$.

Proposition 4. Under H, the set of Nash equilibrium coincides with the set of strong equilibrium.

We start by proving the following claim.

Claim 3. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. Then, for any connected terrace $[t_1, t_2]$ at τ, we have $|\{i \in C : \tau_i \in [t_1,t_2], \hat{\tau}_i \notin [t_1,t_2]\}| \geq |\{i \in C : \tau_i \notin [t_1,t_2], \hat{\tau}_i \in [t_1,t_2]\}|$.

Proof. Suppose not then there exists a connected terrace at τ, $[t_1,t_2]$ such that $|\{i \in C : \tau_i \in [t_1,t_2], \hat{\tau}_i \notin [t_1,t_2]\}| < |\{i \in C : \tau_i \notin [t_1,t_2], \hat{\tau}_i \in [t_1,t_2]\}|$. We consider two cases: (i) $\tilde{q}_{t_2}(\tau) = 0$, and (ii) $\tilde{q}_{t_2}(\tau) < 0$.

In case (i), we have

$$\tilde{q}_{t_2}(\hat{\tau}) \geq \sum_{t=t_1}^{t_2} m_t(\hat{\tau}) - c\,(t_2 - t_1 + 1) > \sum_{t=t_1}^{t_2} m_t(\tau) - c\,(t_2 - t_1 + 1) = \tilde{q}_{t_2}(\tau) = 0,$$

[2] In congestion games, it is trivial to show the equivalence between Nash and strong equilibria if players are homogeneous. It is because swapping strategies would not improve all coalition members.

contradicting Claim 1. In case (ii), by Claim 2, $|\{i \in C : \hat{\tau}_i = t_2\}| = 0$ holds. Thus, we have $|\{i \in C : \tau_i \in [t_1, t_2), \hat{\tau}_i \notin [t_1, t_2)\}| < |\{i \in C : \tau_i \notin [t_1, t_2), \hat{\tau}_i \in [t_1, t_2)\}|$, implying

$$\tilde{q}_{t_2-1}(\hat{\tau}) \geq \sum_{t=t_1}^{t_2-1} m_t(\hat{\tau}) - c\,(t_2 - t_1 + 1) > \sum_{t=t_1}^{t_2-1} m_t(\tau) - c\,(t_2 - t_1 + 1) = \tilde{q}_{t_2-1}(\tau).$$

This is a contradiction with Claim 1. The proof is complete. □

Claim 4. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. Then, there does not exist $i \in C$ such that $\tilde{q}_{\tau_i}(\tau) < 0$ and $\hat{\tau}_j \in [t_1, t_2]$, where $[t_1, t_2]$ is a connected terrace at τ. That is, for all $i \in C$, both τ_i and $\hat{\tau}_i$ must belong to either a single terrace or a connected terrace.

Proof. Suppose not. Then, there exists at least one $j \in C$ such that $\tilde{q}_{\tau_j}(\tau) < 0$ and $\hat{\tau}_j \in [t_1, t_2]$.

Note that no player $i \in C$ can deviate at $\hat{\tau}$ to the slots which is a basin at τ.

Due to the addition of this player j to $\hat{\tau}_j \in [t_1, t_2]$ and the finiteness of the number of connected terraces at any profile, we can find some connected terrace at τ, $[t'_1, t'_2]$ with $t_1 \leq t_2$ such that

$$\left|\, \{i \in C : \tau_i \in [t'_1, t'_2], \hat{\tau}_i \notin [t'_1, t'_2]\} \,\right| < \left|\, \{i \in C : \hat{\tau}_i \in [t'_1, t'_2], \tau_i \notin [t'_1, t'_2]\} \,\right|.$$

However, this contradicts to Claim 3. □

From the previous claim, the next claim follows.

Claim 5. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. Then, for any connected terrace at τ, $[t_1, t_2]$ with $t_1 \leq t_2$, we have

$$\left|\, \{i \in C : \tau_i \in [t_1, t_2], \hat{\tau}_i \notin [t_1, t_2]\} \,\right| = \left|\, \{i \in C : \hat{\tau}_i \in [t_1, t_2], \tau_i \notin [t_1, t_2]\} \,\right|.$$

Proof. Suppose not. That is, suppose that for some connected terrace at τ, $[t_1, t_2]$ with $t_1 \leq t_2$,

$$\left|\, \{i \in C : \tau_i \in [t_1, t_2], \hat{\tau}_i \notin [t_1, t_2]\} \,\right| > \left|\, \{i \in C : \hat{\tau}_i \in [t_1, t_2], \tau_i \notin [t_1, t_2]\} \,\right|.$$

Note that from Claim 2 no player involving an improving coalitional deviation takes the slots that is a basin at τ or at τ'.

Due to the finiteness of the number of players in C, $|C|$, we can find another connected terrace at τ, $[t'_1, t'_2]$ with $t'_1 \leq t'_2$ such that

$$\left|\, \{i \in C : \tau_i \in [t'_1, t'_2], \hat{\tau}_i \notin [t'_1, t'_2]\} \,\right| < \left|\, \{i \in C : \hat{\tau}_i \in [t'_1, t'_2], \tau_i \notin [t'_1, t'_2]\} \,\right|.$$

Again, this contradicts Claim 3. □

Proof of Proposition 4. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. We will derive a contradiction.

Step 1. Find $t \in \mathcal{T}$ such that $q_t(\hat{\tau}) < q_t(\tau)$. If there exist multiple such slots, take the earliest one. Denote by $[\underline{t}, \bar{t}]$ the connected terrace where t belongs. Note that some player $i \in C$ switches to $\hat{\tau}_i \notin [\underline{t}, \bar{t}]$ at $\hat{\tau}$.

Step 2. Find a player who deviates to slots in $[\underline{t}, \bar{t}]$ at $\hat{\tau}$.
By Claim 5, there must be at least one such player. Among these players, let the player who chooses the latest slot at $\hat{\tau}$ be player $j \in C$. Note that player j chooses τ_j at τ which does not belong to $[\underline{t}, \bar{t}]$, say $[\underline{t}', \bar{t}']$. That is, player j chooses $\tau_j \in [\underline{t}', \bar{t}']$ at τ and $\hat{\tau}_j \in [\underline{t}, \bar{t}]$ at $\hat{\tau}$.

Step 3. Find a player who deviates to slots in $[\underline{t}', \bar{t}']$ at $\hat{\tau}$, and name player k the one among such players who chooses the latest slot at $\hat{\tau}$.
Likewise in Step 2, such player must be found due to player j's deviation from $[\underline{t}', \bar{t}']$. Let player k choose $\tau_k \in [\underline{t}'', \bar{t}''] \neq [\underline{t}', \bar{t}']$. That is, player k chooses $\tau_k \in [\underline{t}'', \bar{t}'']$ at τ and $\hat{\tau}_k \in [\underline{t'}, \overline{t'}]$ at $\hat{\tau}$.

Step 4. In this sequence of terraces, by finiteness in the number of connected terraces at τ, there must be a cycle; that is, there exists a player who deviated from a connected terrace we have identified earlier.

Let $[\underline{t}^{(1)}, \bar{t}^{(1)}], [\underline{t}^{(2)}, \bar{t}^{(2)}], \ldots, [\underline{t}^{(k)}, \bar{t}^{(k)}]$ be a cycle of connected terraces, where $[\underline{t}^{(k+1)}, \bar{t}^{(k+1)}] \equiv [\underline{t}^{(1)}, \bar{t}^{(1)}]$. Moreover, denote, by $i(1), i(2), \ldots, i(k) \in C$ with $k \in \mathbb{N}$ and $i(k+1) \equiv i(1)$, the player who takes $\tau_{i(l)} \in [\underline{t}^{(l)}, \bar{t}^{(l)}]$ at τ and $\hat{\tau}_{i(l)} \in [\underline{t}^{(l+1)}, \bar{t}^{(l+1)}]$ at $\hat{\tau}$ for $l = 1, \ldots, k$.

Since the payoffs of players $i(1), i(2), \ldots, i(k)$ must improve under the deviation,

$$u^{i(l)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})) > u^{i(l)}(\tau_{i(l)}, q_{\tau_{i(l)}}(\tau)) \tag{3}$$

for all $l = 1, \ldots, k$.

From Propositon 1, for all $l = 1, \ldots, k$,

$$u^{i(l)}(\tau_{i(l)}, q_{\tau_{i(l)}}(\tau)) > u^{i(l)}(t, q_t(t, \tau_{-i(l)}))$$

for all $t \in [\underline{t}^{(l)}, \bar{t}^{(l)}] \setminus \{\tau_{i(l)}\}$. Specifically, for $i(l+1)$-th player and $\hat{\tau}_{i(l)} \in [\underline{t}^{(l+1)}, \bar{t}^{(l+1)}]$,

$$u^{i(l+1)}(\tau_{i(l+1)}, q_{\tau_{i(l+1)}}(\tau)) > u^{i(l+1)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)})). \tag{4}$$

We would see that $q_{\hat{\tau}_{i(l)}}(\tau) = q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)}) \leq q_{\hat{\tau}_{i(l)}}(\hat{\tau})$.
Thus,

$$u^{i(l+1)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)})) \geq u^{i(l+1)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})). \tag{5}$$

Note that by H,

$$u^{i(l+1)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})) = u^{i(l)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})). \tag{6}$$

Hence, from (3), (4), (5) and (6), we obtain

$$\begin{aligned}
u^{i(l+1)}(\tau_{i(l+1)}, q_{\tau_{i(l+1)}}(\tau)) &> u^{i(l+1)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)})) \\
&\geq u^{i(l+1)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})) \\
&= u^{i(l)}(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})) \\
&> u^{i(l)}(\tau_{i(l)}, q_{\tau_{i(l)}}(\tau)).
\end{aligned}$$

However, this yields a cycle on the preference:

$$\begin{aligned}
u^{i(1)}(\tau_{i(1)}, q_{\tau_{i(1)}}(\tau)) &< u^{i(1)}(\hat{\tau}_{i(1)}, q_{\hat{\tau}_{i(1)}}(\hat{\tau})) \\
&< u^{i(2)}(\tau_{i(2)}, q_{\tau_{i(2)}}(\tau)) \\
&< u^{i(2)}(\hat{\tau}_{i(2)}, q_{\hat{\tau}_{i(2)}}(\hat{\tau})) \\
&\;\;\vdots \\
&< u^{i(k)}(\tau_{i(k)}, q_{\tau_{i(k)}}(\tau)) \\
&< u^{i(k)}(\hat{\tau}_{i(k)}, q_{\hat{\tau}_{i(k)}}(\hat{\tau})) \\
&< u^{i(k+1)}(\tau_{i(k+1)}, q_{\tau_{i(k+1)}}(\tau)) \\
&= u^{i(1)}(\tau_{i(1)}, q_{\tau_{i(1)}}(\tau)),
\end{aligned}$$

which is a contradiction. □

4 (Non)existence of Nash Equilibrium

Unfortunately, even under homogeneity, the existence of Nash equilibrium is not guaranteed. In fact, the following simple example shows that there may not be a Nash equilibrium even under H together with SP and another stringent condition, Order Preservation (OP) introduced by Konishi et al. (1997b) that investigates positive externality games (see below).

Order Preservation (OP). For all $i \in N$, all $t, t' \in \mathcal{T}$ and all $k, k' \in \mathbb{Z}_+$,

$$u^i(t,k) \geq u^i(t',k') \Longleftrightarrow u^i(t,k+1) \geq u^i(t',k'+1).$$

The following Boundedness (B) condition together with OP enables us a tractable representation of payoff functions.

Boundedness (B). Suppose that C holds. For all $t, t' \in \mathcal{T}$ with $u^i(t,0) < u^i(t',0)$ there exists $k_{tt'} \in \mathbb{Z}_+$ such that $u^i(t,0) > u^i(t',k_{tt'})$.

The following result is a variation of the result in Konishi and Fishburn (1996).[3]

Fact. Under A, B, C, and OP, utility function u^i has a quasi-linear representation. There is a vector $v^i = (v^i(1), ..., v^i(T)) \in \mathbb{R}^{\mathcal{T}}$ such that for all $t, t' \in \mathcal{T}$, and all $k, k' \in \mathbb{Z}_+$,

$$u^i(t,k) \geq u^i(t',k') \Longleftrightarrow v^i(t) - k \geq v^i(t') - k'.$$

Example 4. Consider the following three-player, three-time-slot game with A, B, C, H, OP, and SP (capacity $c = 1$): $v(1) > v(2) > v(1) - 1 > v(3) > v(2) - 1 > v(1) - 2 > ...$ Then, there is no pure strategy equilibrium. To see this, first note at least one player chooses 1 in a Nash equilibrium. Let player 1 be such a player. Without loss of generality, player 2 weakly earlier departure time than player 3. There are five cases: (i) $(1,1,1)$ then a player moves to 3, (ii) $(1,1,2)$ then player 3 moves to 3, (iii) $(1,1,3)$ then player 1 or 2 moves to 2, (iv) $(1,2,2)$ then player 2 or 3 moves to 3, and (v) $(1,2,3)$ then player 3 moves to 1. Thus, there is no Nash equilibrium in pure strategy. $\square$

Therefore we seek a stronger concept, which we call symmetric single-peakedness (SSP). Symmetric single-peakedness reflects a player who values the trade-off between departing at her optimal slot and the queue-length at a one-to-one ratio. That is, departing k slots later (earlier) than the optimal slot is equivalent to facing an added queue-length of k at her optimal slot. Formally,

Symmetric single-peakedness (SSP). For all $i \in N$, let $t_i^* \in \mathcal{T}$ be an optimal slot. Player i's payoff function satisfies

$$u(t_i^*, k) = u(t_i^* \pm k, 0) \quad \text{for all } k \in \mathbb{N}.$$

Note that SSP implies B and SP. Finally, we show that SSP enables us to establish the existence of Nash equilibria.

Proposition 5. Under A, B, C, H, SSP, and OP, there exists a Nash equilibrium with pure strategies.

We show that the allocation generated by the following procedure is indeed an equilibrium allocation. Let n' denote the number of players yet to be allocated.

Procedure.

[3] Note that this Boundedness condition differs from the one in Konishi and Fishburn (1996). Their Boundedness goes: "For all $t, t' \in \mathcal{T}$, there exists $k_{tt'} \in \mathbb{Z}_+$ such that $u^i(t, k_{tt'}) > u^i(t', 0)$," and the resulting utility representation is $v_t^i + q_t$ (conformity instead of congestion).

Step 1 Set $n' = n$.

Step 2 At slot t^*, put $(c+1)$ players whenever possible, and proceed to Step 3. If $n' < c+1$, put all n' players at slot t^*, and terminate.

Step 3 Update n' with $n' - (c-1)$, i.e., $n' \to n' - (c-1)$.

Step 4 Set $\kappa = 1$.

Step 5 While $t^* - \kappa > 0$ and $n' > 0$:

- **Step 5-1** At slot $(t^* + \kappa)$, put $(c-1)$ players whenever possible, and proceed to Step 5-2. Otherwise, put all n' players at slot $(t^* + \kappa)$, and terminate.
- **Step 5-2** Update $n' \to n' - (c-1)$.
- **Step 5-3** At slot $t^* - \kappa$, put $(c+1)$ players whenever possible, and proceed to Step 5-4. Otherwise, put all n' players at slot $(t^* - \kappa)$, and terminate.
- **Step 5-4** Update $n' \to n' - (c+1)$ and $\kappa \to \kappa + 1$.

Step 6 While $n' > 0$:

- **Step 6-1** At slot $(t^* + \kappa)$, put $(c-1)$ players whenever possible, and proceed to Step 6-2. Otherwise, put all n' players at slot $(t^* + \kappa)$, and terminate.
- **Step 6-2** Update $n' \to n' - (c-1)$.
- **Step 6-3** At slot 1, put one player whenever possible, and proceed to Step 6-4. Otherwise, terminate.
- **Step 6-4** Update $n' \to n' - (c+1)$ and $\kappa \to \kappa + 1$.

When $c = 1$, steps 5-1, 5-2, 6-1 and 6-2 can be skipped since $c - 1 = 0$. Also, note that if there are enough number of players (so that Step 5 in the procedure is implemented), then the procedure allocates $(c+1)$ players to all slots $t \in [2, t^*]$, $(c-1)$ players to all slots $t \in [t^*+1, t']$ with some $t' > t^*$, and the remaining players to slot 1.

Proof. Let τ be a profile resulting from this procedure. If the total number of players $n \leq c+1$, then $\tau_i = t^*$ is trivially a Nash equilibrium. Thus, we consider the case in which $n > c+2$.

(A) Suppose $t^* \geq 2$. There exists one and only one connected terrace $[t_1, t_2]$ at τ. We consider further two cases: (i) $t_1 > 1$ and (ii) $t_1 = 1$.

(i) When $t_1 > 1$, we will have

$$m_t(\tau) = c+1 \quad t \in [t_1, t^*],$$
$$m_t(\tau) = c-1 \quad t \in [t^*+1, t_2-1].$$

At this profile the queue-length vector $q(\tau)$ becomes

$$\begin{aligned} q(\tau) &= (q_1, ..., q_{t_1-1}, q_{t_1}, q_{t_1+1}, ..., q_{t^*}, q_{t^*+1}, ..., q_{t_2-1=2t^*-t_1}, q_{t_2}, ...) \\ &= (0, ..., 0, 1, 2, ..., t^* - t_1 + 1, t^* - t_1, ..., 1, 0, ...). \end{aligned} \tag{7}$$

SSP and OP imply

$$\begin{aligned} u(t_1 - 1, 0) = u(t_1, 1) = \cdots = u(t^*, t^* - t_1 + 1) \\ = u(t^* + 1, t^* - t_1) = \cdots = u(t_2 - 1, 1) = u(t_2, 0). \end{aligned}$$

First, note that player i with $\tau_i \in [t_1, t_2]$ cannot improve by departing later in $[t_1, t_2]$, since the queue-length at switched slot, τ_i' is the same as in (7), so player i is indifferent between τ_i and τ_i'.

In addition, these players cannot improve by departing earlier in $[t_1, t_2]$, since the queue-length at switched slot, τ_i', compared to (7), increases by one, so they are worse off by switching to τ_i'.

Next we consider the case when they depart later or earlier out of the connected terrace. At τ_i', they face a queue of length zero or one if $\tau_i' = t_1 - 1$ or of length zero otherwise. If $\tau_i' = t_1 - 1$ and $q_{t_1-1}(\tau_i', \tau_{-i}) = 0$, player i is indifferent between $\tau_i' = t_1 - 1$ and τ_i. If $\tau_i' = t_1 - 1$ and $q_{t_1-1}(\tau_i', \tau_{-i}) = 1$, $\tau_i' = t_1 - 1$ is worse than τ_i. If $\tau_i' \neq t_1 - 1$, $u(\tau_i', 0) < u(t_1 - 1, 0) = u(t_2, 0)$, they making worse-off.

Player i in slot $t_1 - 1$, if any, does not depart earlier than slot $t_1 - 1$ or later than slot t_2 by the same logic in the above. Player i also does not switch to $\tau_i \in [t_1, t_2]$, since the queue-length at switched slot, τ_i', compared to (7), increases by one, so they are worse.

Thus, since no player has an incentive to switch their slots in the case (A)-(i), τ is a Nash equilibrium.

(ii) Case $t_1 = 1$. We will have

$$\begin{aligned} &m_1(\tau) \geq c + 1, \\ &m_t(\tau) = c + 1 \quad t \in [2, t^*], \\ &m_t(\tau) = c - 1 \quad t \in [t^* + 1, t_2 - 1]. \end{aligned}$$

Depending on the value of n, there may be some player(s) at slot t_2, say

$$m_{t_2}(\tau) = k \quad \text{for some } k \in [0, c - 1].$$

Let $m_1(\tau) \equiv q_1^*$. At this profile the queue-length vector $q(\tau)$ becomes

$$\begin{aligned} q(\tau) &= (q_1, q_2, ..., q_{t^*}, q_{t^*+1}, ..., q_{t_2-1}, q_{t_2}, ...) \\ &= (q_1^*, q_1^* + 1, ..., q_1^* + (t^* - 2 + 1), q_1^* + (t^* - 2 + 1) - 1, ..., ..., 1, 0, ...). \end{aligned}$$

In this case, using a similar argument as in case (A)-(i), no player has an incentive to switch their slots.

(B) Suppose $t^* = 1$. This is a variant of the case (A)-(ii), and it is shown that no player has an incentive to switch their slots. □

Any property imposed on Proposition 5 seems required for a Nash equilibrium to exist. Indeed, once OP is dropped, then the existence of Nash equilibria may not be guaranteed any more as the following example shows.

Example 4. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{1, 2, 3, 4\}$ with capacity $c = 1$. Players have the following preferences.

$$u(2,0) > u(1,0) = u(2,1) = u(3,0) > u(1,1) > u(3,1) > u(2,2) = u(4,0) > others.$$

In this example, H and SSP with optimal time slot $t^* = 2$ are satisfied, while OP is not, since $u(2, 0) > u(1, 0)$ but $u(2, 1) = u(1, 0) > u(1, 1)$. Then, this example does not admit any pure strategy Nash equilibrium. To see, first consider four cases: (i) $(1, 2, 2, 3)$ then player 4 moves to 1. (ii) $(1, 2, 2, 1)$ then player 3 moves to 3. (iii) $(1, 2, 3, 1)$ then player 4 moves to 2. (iv) $(1, 2, 3, 2)$ then player 3 moves to 1. Since the queue-structure when (i) and (iv) are the same and H holds, the cycle is started by player 3, and never stops. Moreover, even when stating from an arbitrary profile, the deviation process is finally absorbed to this cycle. □

5 Bottleneck Games with Atomless Players

When there are a finite number of players, the existence of Nash equilibrium and in the equivalence between Nash and strong equilibria held under special circumstances. The primary reason for the number of negative results may be the asymmetry of the effect of a player deviating to an earlier slot versus deviating to a later slot within a connected terrace (Proposition 1). This is coming from the fact that players are atoms in that deviating players taken into account the change in the queue distribution caused by their deviations. In this section we will consider an idealized game in which players are atomless as in Vickrey (1969).

The set of players is the interval $I = [0, 1]$ endowed with Lebesgue measure λ. There is a finite set of alternatives $\mathcal{T} = \{1, ..., T\}$. A strategy profile is a measurable function $\tau : I \to \mathcal{T}$. Each player $i \in I$ has a payoff function $u^i(\tau)$, but without confusion, we write it $u^i(\tau_i, \tau)$ to clarify which strategy player i is choosing.

We assume anonymity in the sense of Schmeidler (1973). For each strategy profile τ, let $\mu(\tau)$ be a T-dimensional vector $\mu(\tau) = (\mu_1(\tau), \mu_2(\tau), ..., \mu_T(\tau))$, where $\mu_t(\tau) = \lambda(\{i \in I : \tau(i) = t\})$ for each $t \in \mathcal{T}$. We say that a game satisfies **anonymity (A)** if the following condition is satisfied: for all $i \in I$, all $t \in \mathcal{T}$, and τ, τ', $u^i(t, \tau) = u^i(\tau')$ whenever $\mu(\tau) = \mu(\tau')$. A strategy profile τ is a **Nash equilibrium** if $u^i(\tau^i, \tau) \geq u^i(t, \tau)$ for almost all $i \in I$ and all

$t \in \mathcal{T}$. Note that we can define $\tilde{q}(\tau)$ and $q(\tau)$ exactly in the same way as before: $\tilde{q}_t(\tau) = q_{t-1}(\tau) + \mu_t(\tau) - c$, and $q_t(\tau) = \max\{\tilde{q}_t(\tau), 0\}$. By A, the payoff function $u^i(t,\tau)$ can also be written as $u^i(t,\tau) = v^i(t, q_t(\tau))$.

Under the atomless player assumption, we will assume Schmeidler's technical assumption.

Regularity (R) (Schmeidler, 1973). (i) For all $i \in I$, and all $t \in \mathcal{T}$, $u^i(t,\tau)$ is continuous. Thus, all utility functions are uniformly bounded and there exists a positive constant K such that $\left|u^i(t,\tau)\right| < K$ for all $i \in I$, $t \in \mathcal{T}$, and τ. (ii) For all τ and all $t, t' \in \mathcal{T}$, the set $\left\{i \in I : u^i(t,\tau) > u^i(t',\tau)\right\}$ is measurable.

Proposition (Schmeidler, 1973). Under A and R, there exists a Nash equilibrium in pure strategies.

A strategy profile is a **strong equilibrium** if there is no measurable subset $C \subset I$ with $\lambda(C) > 0$ and a strategy profile $\hat{\tau}$ of players in C such that $u^i(\hat{\tau}_i, \hat{\tau}) > u^i(\tau_i, \tau)$ almost everywhere on C, where $\hat{\tau} = ((\hat{\tau}_i)_{i \in C}, (\tau_i)_{i \notin C})$. We will impose the following congestion condition.

Congestion (C) $v^i(t, q_t)$ is strictly decreasing in q_t for all $t \in \mathcal{T}$ and all $q_t \in \mathbb{R}_+$.

The main result of this section is:

Proposition 6. Consider an atomless game. Under A, C, and R, the sets of Nash and strong equilibria coincide with each other.

Proof. Suppose that τ is a Nash equilibrium while it is not a strong equilibrium. Then, there exist a coalition C with $\lambda(C) > 0$ and a strategy profile $\hat{\tau}$ for C such that $u^i(\hat{\tau}_i, \hat{\tau}) > u^i(\tau_i, \tau)$, where $\hat{\tau} = ((\hat{\tau}_i)_{i \in C}, (\tau_i)_{i \notin C})$. Note that $\hat{\tau}_i \notin \{t' \in \mathcal{T} : q_{t'}(\tau) \leq 0\}$ holds for all $i \in C$. It is because player i would have moved under strategy profile τ, contradicting τ's being a Nash equilibrium, otherwise. Thus, $\{t' \in \mathcal{T} : q_{t'}(\tau) > 0\} \supseteq \{t' \in \mathcal{T} : q_{t'}(\hat{\tau}) > 0\}$.

Assume now that there is a time slot $t \in \{t' \in \mathcal{T} : q_{t'}(\hat{\tau}) > 0\}$ with $q_t(\hat{\tau}) > q_t(\tau)$. Take the earliest time slot of this kind t. Then, $C \cap \{i' \in N : \hat{\tau}_{i'} = t\} \neq \emptyset$. Let i be such a player. Since τ is a Nash equilibrium, $v^i(\tau_i, q_{\tau_i}(\tau)) \geq v^i(t, q_t(\hat{\tau}))$ must hold. This is a contradiction with C's being profitable deviation. Thus, for all $t \in \{t' \in \mathcal{T} : q_{t'}(\hat{\tau}) > 0\}$, $q_t(\hat{\tau}) \leq q_t(\tau)$ holds. Since $\{t' \in \mathcal{T} : \tilde{q}_{t'}(\tau) > 0\} \supseteq \{t' \in \mathcal{T} : \tilde{q}_{t'}(\hat{\tau}) > 0\}$, $q_t(\hat{\tau}) = q_t(\tau)$ holds for all $t \in \{t' \in \mathcal{T} : q_{t'}(\tau) > 0\} = \{t' \in \mathcal{T} : q_{t'}(\hat{\tau}) > 0\}$. Hence, a deviation C with $\hat{\tau}$ cannot improve on Nash equilibrium τ. This implies that a Nash equilibrium τ is a strong equilibrium.□

6 Concluding Remarks

We have investigated the bottleneck games with finite players and atomless players. Although the bottleneck game is a natural extension of congestion game by Milchtaich (1996) and Konishi et al. (1997a), the results of these two games differ from each other in the finite case. Somewhat surprisingly, the presence/absence of single-terraces (time slots that are chosen by the same number of players as the capacities) can alter the structure of the equilibria of the bottleneck game. This is because there is an asymmetry between an increase and a reduction in population at single-terraces: the former reduces payoffs while the latter has no effect on them. In contrast, in an atomless bottleneck game, we need essentially no condition for the result. There is no such asymmetry: players can simply choose the most preferable time slot given the queue structure without affecting the queues. This is why we can recover the nice equivalence result between Nash and strong equilibria as in Konishi et al. (1997a).

Thus, whether the traffic bottleneck model started by Vickrey (1969) would provide us useful insights or not depends on how we interpret the ”atomless” assumption of the model. If we accept this assumption as an reasonable approximation of the real world, we can enjoy nice properties and rich results of the model. However, if we question the legitimacy of atomless players, then we need to suffer from the ill-behaved model coming from finite problems.

References

Arnott, R., A. De Palma, and R. Lindsey (1990). Economics of a bottleneck. *Journal of Urban Economics 27*(1), 111–130.

Daganzo, C. F. (1985). The uniqueness of a time-dependent equilibrium distribution of arrivals at a single bottleneck. *Transportation Science 19*(1), 29–37.

Hendrickson, C. and G. Kocur (1981). Schedule delay and departure time decisions in a deterministic model. *Transportation Science 15*(1), 62–77.

Konishi, H. and P. Fishburn (1996). Quasi-linear utility in a discrete choice model. *Economics Letters 51*(2), 197 – 200.

Konishi, H., M. Le Breton, and S. Weber (1997a). Equilibria in a model with partial rivalry. *Journal of Economic Theory 72*(1), 225–237.

Konishi, H., M. Le Breton, and S. Weber (1997b). Pure strategy nash equilibrium in a group formation game with positive externalities. *Games and Economic Behavior 21*(1), 161–182.

Milchtaich, I. (1996). Congestion games with player-specific payoff functions. *Games and Economic Behavior 13*(1), 111–124.

Monderer, D. and L. S. Shapley (1996). Potential games. *Games and Economic Behavior 14*(1), 124 – 143.

Rosenthal, R. W. (1973). A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory 2*(1), 65–67.

Schmeidler, D. (1973). Equilibrium points of nonatomic games. *Journal of Statistical Physics 7*(4), 295–300.

Smith, M. J. (1983). The existence and calculation of traffic equilibria. *Transportation Research Part B: Methodological 17*(4), 291–303.

Vickrey, W. S. (1969). Congestion theory and transport investment. *American Economic Review 59*(2), 251–260.