Allocation Mechanisms with Mixture-Averse Preferences*

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Abstract

Consider an economy with equal amounts of N types of goods, to be allocated to agents with strict quasi-convex preferences over lotteries. We show that ex-ante, all feasible and Pareto efficient allocations give almost all agents binary lotteries. Therefore, even if all preferences are the same, some identical agents necessarily receive different lotteries. Our results imply that many of the popular allocation mechanisms are ex-ante inefficient. Assuming the reduction of compound lotteries axiom, social welfare deteriorates by first randomizing over these binary lotteries. Efficient full ex-ante equality is achieved if agents satisfy the compound independence axiom.

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1 Introduction

Ten thousand children need to be allocated into ten schools, each accommodating one thousand of them. The schools are not the same, and parents may rank them in different ways. However, if all children are considered equal, then a social lottery seems to be the best solution, where each student has an equal chance to attend each of the ten schools.[1] This procedure is egalitarian — everyone gets the same lottery — and feasible. But is it efficient? Specifically, is there no other procedure such that ex-ante, before people know their allocated school, they will get a better lottery?

If individual preferences over the schools are not the same, then this procedure may be inefficient — for example, if each school is ranked best by exactly 1000 parents. It is true that if all individuals are expected utility maximizers and have the same preferences over lotteries (and in particular, over the schools), then this procedure leads to an efficient allocation. This is also the case if all have the same quasi-concave preferences over lotteries. But if preferences are quasi-convex, and a mixture of two indifferent lotteries is inferior to the mixed lotteries, then we show that this procedure is never efficient, regardless of whether individual preferences are the same or not. Such preferences are implied by some well known alternatives to expected utility theory (for example, Tversky and Kahneman's [35] Cumulative Prospect Theory, where risk aversion implies quasi-convexity. See discussion below).

We analyze first an economy where N types of goods, with k units each, need to be allocated, one for each of Nk agents. All agents have strict preferences over the basic goods, and continuous, monotone (with respect to first-order stochastic dominance, based on their ranking of the goods), and strictly quasi-convex preferences over lotteries. Agents' preferences over the goods and over lotteries are not necessarily the same. Our first result (Theorem 1) shows that any feasible and ex-ante Pareto efficient allocation must give all but 'not too many' agents binary lotteries, where the proportion of agents who hold non-binary lotteries vanishes as k increases.[2] In particular, even if all preferences are the same, some identical agents necessarily receive different lotteries. For the case of identical preferences, we establish existence

[1] For example, divide the students into ten groups $A_1, \ldots, A_{10}$ of size 1000 each. Choose with probability $\frac{1}{10}$ one of the ten permutations $\sigma_1, \ldots, \sigma_{10}$ of $(1, \ldots, 10)$, where $\sigma_j(i) = (i + j - 1) \pmod{10} + 1$, $j = 1, \ldots, 10$.

[2] As we show in Section 2.1, this result, with a small caveat, essentially also holds when individuals have different expected utility preferences.

of a feasible and efficient solution, in which all the lotteries used are equally attractive (Theorem 2). We also derive an upper bound on the number of lotteries used.

We then consider a continuum economy with the same mass of agents and goods, where each type has its own quasi-convex preferences over lotteries. In this part we are looking for no-envy allocations, that is, allocations of lotteries where no person prefers to receive somebody else's lottery. We show that under a mild condition on preferences, a feasible and efficient allocation for the continuum economy with strictly quasi-convex preferences yields all agents a binary lottery. The set of no-envy such allocations is not empty. Moreover, if all agents have the same preferences, then equality with such lotteries is obtained (Theorem 3).

The last part of the paper discusses possible merits of random allocations of the binary lotteries among individuals with identical preferences. The need for such an extra layer of randomization may be due to lack of confidence in policy makers' integrity or willingness of the allocating agencies to demonstrate they are unbiased. We show how individual preferences over two-stage lotteries imply different answers to this question. If they simplify such lotteries by multiplying probabilities of the two stages, this extra randomization will reduce participants utilities. But if decision makers instead satisfy the compound independence axiom, according to which if they prefer q to q' they will prefer to replace q' with q in any compound lottery that includes the former as an outcome, then such randomizations will not change agents' welfare.

Our analysis depends on the assumption that individual preferences over lotteries are quasi-convex. This is, for example, the case with the popular family of rank-dependent utilities models (Quiggin [27]), which also includes Tversky and Kahneman's [35] Cumulative Prospect Theory, where risk aversion implies quasi-convexity. Other models which can exhibit quasi-convexity include quadratic utility (Chew, Epstein, and Segal [7]), and Köszegi and Rabin's [19] models of reference-dependence. In addition, Machina [20] pointed out that quasi-convexity occurs if, as is common in many applications such as insurance purchasing, before the lottery is resolved, agents can take actions that affect their final utility. If the optimal action depends on the probabilities, the induced maximum expected utility will be convex in the probabilities, meaning that even if the underlying preferences are expected utility, induced preferences over the 'optimal' lotteries will be quasi-convex.

The experimental evidence on quasi-convexity versus quasi-concavity is mixed. Most of the experimental literature that documents violations of expected utility (e.g., Coombs and Huang [8]) found either preference for randomization or aversion to it. Camerer and Ho [6] find support for quasi-convexity over gains and quasi-concavity over losses. An example of behavior that distinguishes between the two attitudes to mixture is the probabilistic insurance problem of Kahneman and Tversky [17]. They showed that in contrast with experimental evidence, any risk averse expected utility maximizer must prefer probabilistic insurance to regular insurance. Sarver [30] pointed out that this result readily extends to the case of quasi-concave preferences. In contrast, quasi-convex preferences can accommodate aversion to probabilistic insurance together with risk aversion (for example, risk-averse rank-dependent utility; see Segal [31]). Sarver further illustrates that quasi-convex preferences are consistent with increasing marginal willingness to pay for insurance at some levels of coverage; another plausible property that in most models requires violation of risk aversion. In the context of group decision making, Dillenberger and Raymond [11] show that quasi-convexity of preferences in the individual level is equivalent to the consensus effect: individuals tend to conform to the choices of others in group decisions, compared to choices made in isolation.

The idea of using lotteries to allocate indivisible goods is not new (see, for example, Diamond [9], Hylland and Zeckhauser [16], and Rogerson [28]). Moreover, the possible existence of an optimal solution that induces each individual to face a binary lottery was already discussed in Hylland and Zeckhauser [16], under expected utility preferences. Our approach differs from these works. We show that in a large economy with quasi-convex preferences, any ex-ante efficient solution *must* use only binary lotteries. Also, as long as individuals simplify compound lotteries by multiplying the probabilities , randomizing among these binary lotteries (thus giving identical people the same ex-ante lottery) is always suboptimal.

In this paper we employ a strong notion of ex-ante efficiency, which takes into consideration individuals' preferences over lotteries. Two weaker notions of efficiency were previously studied, ordinal efficiency and ex-post efficiency, both only depend on individuals' (ordinal) ranking of the final goods. As we remark in Section 2.1, our results imply that many of the popular allocation mechanisms used in the literature are ex-ante inefficient. For example, random serial dictatorship, that assigns the order of individuals using uniform distribution, is inefficient as is typically implies that each individual faces a lottery with more than two elements in its support. Note that unlike in standard expected utility, this inefficiency does not rely on cardinal information which can be used to assess the intensity of preferences over the basic goods, but on the ordinal property of the preferences over lotteries themselves, namely that they are quasi-convex in probabilities.

The paper is organized as follows. Section 2 lays out the basic problem in a finite environment. Section 3 studies the case of a continuum economy. In Section 4 we discuss the benefit of a pre-randomization over the allocation lotteries. Section 5 concludes with a further discussion on binary lotteries. All proofs are in the Appendix.

2 Finite Economies

Consider an economy with Nk individuals and with k units of each of $N \geqslant 3$ basic goods $x_1, \ldots, x_N$. Denote by $q = (q_1, \ldots, q_N)$ the lottery $(x_1, q_1; \ldots; x_N, q_N)$ that yields x_i with probability q_i, $i = 1, \ldots, N$. Each member n of society has preferences $\succeq_n$ over such lotteries, which are assumed to be continuous, strictly monotonic (with respect to first-order stochastic dominance), and strictly quasi-convex in probabilities. This last assumption captures a dislike of probabilistic mixtures of lotteries: $q \sim q' \Longrightarrow q \succ \alpha q + (1-\alpha) q'$ for all $\alpha \in (0, 1)$.

A solution is a list of N-dimensional probability vectors $q^1, \ldots, q^{Nk}$, where q^n is the lottery faced by person j. We require for all $n = 1, \ldots, Nk$,

$$\sum_{i=1}^{N} q_i^n = 1 \tag{1}$$

That is, the probability that person n will get one of the items is 1. Also, for $i = 1, \ldots, N$,

$$\sum_{n=1}^{N} q_i^n = k \tag{2}$$

This condition means that with probability 1, each of the k items of each good will be allocated to someone. The last equation implies

$$\frac{1}{Nk} \sum_{n=1}^{Nk} q^n = \left(\frac{1}{N}, \ldots, \frac{1}{N} \right) \tag{3}$$

That is, the average lottery faced by the participants is a uniform distribution over the N goods. Obviously, this distribution is feasible. The sum of its components must be 1, as the original lottery satisfies eq. (1). And if the average lottery is not uniform, then the original allocation is not feasible as it must violate eq. (2).

Any solution q specifies the probability distribution over final outcomes for each individual. The Birkhoff–von Neumann Theorem ([4],[39]) guarantees that for any q there is always a (social) lottery over all possible permutations of the allocations of the final outcomes that induces the marginal probabilities of q.[3]

2.1 Ex-Ante Efficiency

We first characterize solutions that are feasible, that is, satisfy equations (1) and (2), and are ex-ante Pareto efficient, in the sense that there is no other solution in which some individuals are strictly better off and no one is worse off.[4] As preferences are continuous over a compact domain, feasible efficient allocations exist. We show that in such allocations, and without any further assumptions on individuals' preferences, all but 'not too many' individuals obtain either a degenerate lottery or a lottery with positive probabilities on two goods only.

Definition 1 A lottery q^n is binary if $q_i^n > 0$ for *no more than* two outcomes.

Theorem 1 Suppose preferences are strictly quasi-convex. Let q be a feasible and Pareto efficient solution. Then for any three goods x_r, x_s, x_t, there is at most one person n such that q_r^n, q_s^n, $q_t^n > 0$.

This result implies that to detect violation of ex-ante efficiency, it is enough to observe an allocation in which two individuals receive lotteries that put positive probabilities on the same three goods. The exact probabilities are inconsequential. To illustrate the main argument of the theorem, suppose that two agents m and n agree on their ranking of three goods

[3]We assume throughout that each agent is indifferent between all units of the same good, so that we can confine our attention to the allocation of the goods themselves. This would not be the case if, for example, we were to allocate seats in a given flight and travelers prefer sitting in a window or an isle seat.

[4]Formally, there is no solution $\tilde{q}$ such that $\tilde{q}^n \succeq q^n$ for all n and $\tilde{q}^m \succ q^m$ for some m.

$x_r \succ x_s \succ x_t$ and that they both receive lotteries with positive probabilities on these three goods, as in Figure 1 in the appendix. To make both agents better off, transfer probabilities from one agent to another as in the right-hand side of the figure, a violation of the efficiency assumption. The same intuition extends also to cases where individuals' ordinal rankings of the goods are *not* identical. Note that this intuition applies also to agents with different expected utility preferences. See Footnote 9 below.

Corollary 1 The number of individuals who hold non-binary lotteries in any feasible and efficient allocation is bounded above by $\binom{N}{3}$.

The number of subsets of $\{1, \ldots, N\}$ where no two elements have an intersection with more than two numbers is bounded above by $\binom{N}{3}$, which is the case where all subsets have three elements each.[5] Since the number of individuals who hold non-binary lotteries is bounded above by $\binom{N}{3}$ while the total population size is Nk, their fraction becomes arbitrarily small as k increases.

There are many popular mechanisms that can be used to allocate objects among a group of agents. One example that is broadly used and is easy to implement is random serial dictatorship. Randomly order the Nk individuals and let them choose in their turn the best good still available according to their personal ranking. It is well known that using this mechanism, the ultimate ex-post allocation of goods among agents is Pareto optimal (see, for example, Abdulkadiroğlu and Sönmez [3]). Theorem 1 implies, however, that ex-ante this mechanism is typically inefficient. To illustrate, suppose all individuals have the same ranking over the basic goods and that each individual has a probability $\frac{1}{Nk}$ to be the i^{th} in the order. Then, each individual will perceive this as a uniform lottery over all the goods (with probability $\frac{1}{N}$ each), which, according to the theorem, is inefficient. This argument is also valid if individuals don't have the same (ordinal) preferences over the goods, in which case the ex-ante lottery induced by random serial dictatorship for each agent is not necessarily uniform, yet typically has more than two goods in its support.[6] It thus follows that with quasi-convex preferences, random serial dictatorship is typically inefficient ex-ante.

[5] This bound may be tighter under further assumptions on individuals' preferences. See, for example, the case of same preferences in Section 2.2.

[6] An extreme situation is where for each good i there are exactly k people who rank it first in their ordinal preferences. In this case the (degenerate) lottery is ex-ante efficient, but there is no need for a mechanism in the first place.

This suggests a broader point. There are known results that imply the equivalence of different randomized mechanisms and random serial dictatorship (Abdulkadiroğlu and Sönmez [3]; see also Pathak and Sethuraman [25]), in the sense that they induce the same ex-ante probability distribution over the final goods. But then those seemingly identical mechanisms are also typically ex-ante inefficient. If social planners know the individuals' preferences over lotteries, and in particular that they are strictly quasi-convex, they can improve the agents' welfare ex-ante.[7] Importantly, this argument only relies on simple, observable information: strict quasi-convexity of preferences and the size of the supports of the lotteries that are used, rather than on the intensity of preferences over the goods or the weights given to each of them in the corresponding lotteries.[8]

While for exposition purposes we confine our attention to the case of strict quasi-convex preferences, Theorem 1 generically also holds under expected utility, which is linear (and hence also weakly quasi-convex) in probabilities. Suppose all individuals are expected utility maximizers. Hylland and Zeckhauser [16] use competitive equilibrium with equal incomes to show the existence of a solution in which almost all agents receive a binary lottery.[9] Our result holds without relying on any market mechanism.

Also assuming expected utility, Bogomolnaia and Moulin [5] show how random serial dictatorship, which uses uniform distribution to rank agents, is not necessarily even ordinally efficient; it may induce for each agent a distribution over the goods that is stochastically dominated, with respect to that agent's ordinal preferences, by another feasible distribution. Their suggested

[7]Note that we ignore here the question of strategy-proofness, that is, how to guarantee that agents truly reveal their preferences. We are instead focusing only on the properties of the induced allocation (of lotteries) for any given set of preferences.

[8]Abdulkadiroğlu, Che, and Yasuda [2] point out that cardinal information allows the social planner to take into consideration preference intensities. These can be used to improve individuals' welfare over mechanisms that randomly break ties between agents with identical ordinal preferences over the goods.

[9]Under expected utility, if all agents have the same preferences over lotteries then there are many efficient solutions, including interior ones. Our results are thus more prominent once preferences are cardinally different. More precisely, for any three goods x_r, x_s, and x_t, take two expected utility agents m and n with Bernoulli utility u_m and u_n, respectively. If $\frac{u_m(x_s)-u_m(x_t)}{u_m(x_r)-u_m(x_s)} \neq \frac{u_n(x_s)-u_n(x_t)}{u_n(x_r)-u_n(x_s)}$, that is, if the slopes of their indifference curves in the corresponding probability triangles are not the same, then any allocation that gives both agents lotteries with positive probabilities on these three goods is not efficient. The proof is identical to the one given in the appendix for Theorem 1.

probabilistic serial mechanism (which is ordinally efficient) is typically not ex-ante efficient. It is also worth noting that their solution implies that agents with the same ordinal preferences must receive the same lottery over goods. In our case, even if all agents have the same cardinal preferences (and are strictly quasi-convex), *necessarily* not all of them receive the same lottery, as otherwise, the same binary lottery to all will not allocate all available goods.

2.2 Same Preferences

When all individuals have the same preferences, it is natural to require that a just mechanism will offer them the exact same outcome. But since, by Theorem 1, efficient allocations of lotteries with quasi-convex preferences are inconsistent with such a requirement, we instead impose equality in the sense that identical agents receive equally attractive outcomes. That is, if $\succeq_1 = \ldots = \succeq_{Nk} = \succeq$, then $q^1 \sim \ldots \sim q^{Nk}$.

The next result uses the floor function (or greatest integer function), which gives for any real number x the greatest integer less than or equal to x.

Theorem 2 Suppose that $\succeq_1 = \ldots = \succeq_{Nk} = \succeq$. Then:

1. There exist feasible and efficient solutions that satisfy equality.

2. The number of different binary lotteries used in any optimal solution is bounded above by M=floor($\frac{N^2}{4}$).

3. An optimal solution yields all but at most M=floor($\frac{N^2(N-2)}{8}$) agents a binary lottery.

The number of binary lotteries used in any optimal solution is bounded above by the number of non-dominated binary lotteries that can simultaneously be used. Suppose, without loss of generality, that all agents agree that $x_1 \succ x_2 \succ \ldots \succ x_N$. If one of the lotteries used involves outcomes x_i and x_j, with $i < j$, then since all lotteries on two outcomes better than x_i dominate it, and all lotteries on outcomes inferior to x_j are dominated by it, such lotteries cannot be part of the optimal solution.

Similarly, the bound on the number of agents who hold non-binary lotteries (which refines, for $N > 4$, the $\binom{N}{3}$ bound from the general case of Theorem 1) is the number of non-dominated lotteries with three possible outcomes that can simultaneously be used. Note that many individuals may hold the same binary lottery, but only one individual can hold any non-binary lottery. The actual number of different lotteries, binary and non-binary, used in an optimal solution depends on the individuals' preferences.

3 Continuum Economies

Consider a continuum economy with a unit mass $\mathcal{A}$ of N equally sized (with respect to the Lebesgue measure μ) types of agents $\mathcal{A}_1, \ldots, \mathcal{A}_N$. There is a unit mass $\mathcal{B}$ of N goods $x_1, \ldots, x_N$ to be allocated among them, where the mass of each unit is $\frac{1}{N}$.[10] Each of the individuals of type i has strictly quasi-convex preferences $\succeq_i$ over lotteries over the N goods.

Our aim in this paper is to analyze possible mechanisms for the allocation of goods which are desired by all, as otherwise there is no need for a compromise. Our analysis therefore fits best a situation where everyone has the same preferences over the N goods (even if not the same preferences over lotteries over these goods). Nevertheless, our mathematical results hold on a wider range of preferences, with the only restriction that all agents agree that a certain good, say x_1, is best. That is, for all $i = 1, \ldots, N$ and $j = 2, \ldots, N$, $x_1 \succ_i x_j$, but there are no other restrictions on the way individuals rank the outcomes $x_2, \ldots, x_N$.

With a little abuse of notation, a point q in the $(N-1)$-dimensional probability simplex Δ^{N-1} represents the lottery $(x_1, q_1; \ldots; x_{N-1}, q_{N-1}; x_N, 1 - \sum_{i=1}^{N-1} q_i)$ and we now denote by $q^a \in \Delta^{N-1}$ the lottery obtained by person a. An allocation is a measurable function $f : \mathcal{A} \to \Delta^{N-1}$. The allocation f is feasible if $\int_{\mathcal{A}} f_i(a) d\mu = \frac{1}{N}$, $i = 1, \ldots, N-1$ (this is the analogue condition to eq. (3) of Section 2). It is efficient if there is no allocation g such that $\forall i$ and $\forall a \in \mathcal{A}_i$, $g(a) \succeq_i f(a)$, and a positive mass of agents strictly prefer their outcome under g to their outcome from f. To simplify the presentation, we'll use the term "all" for "all but a zero measure of agents." We are interested in

[10] In fact, we can assume J types of goods, and that both the N types of individuals, as well as the J types of goods, are not of same size. However if the sizes are rational numbers, we can assume without loss of generality that $J = N$ and the sizes of the different goods are the same; and if they are irrational, we'll obtain our results using continuity, where the economy is the limit of economies with rational sizes. We therefore assume throughout $J = N$ and that the sizes of the types of agents and of the goods are all $\frac{1}{N}$. See Footnote 12 below for a further generalization.

characterizing allocations that are efficient and satisfy the following No-Envy criterion.

No-Envy For all a and b, $q^a \succeq_a q^b$.

No-Envy postulates that in the allocation of lotteries, no individual would prefer to replace their lottery with that of any other agent.[11] Clearly, if $\succeq_1 = \ldots = \succeq_N = \succeq$, then No-Envy implies equality, in the sense that for all $a, b \in \mathcal{A}$, $f(a) \sim f(b)$.

No-Envy is appealing on a normative grounds. Furthermore, in a standard (convex) economy, it is compatible with the Efficiency requirement (see, for example, Varian [37]). But in a non-convex economy as ours, it is not guaranteed that the two coexist (see, for example, Vohra [38] and Maniquet [22]). We show however that in the present context, the continuum economy guarantees the existence of no-envy allocations.

Theorem 3 A feasible and efficient allocation for the continuum economy with strictly quasi-convex preferences yields all agents a binary lottery. The set of no-envy such allocations is not empty, and if all agents have the same preferences, then equality with such lotteries is obtained.

We offer here an outline of the proof. The first step shows, similarly to the proof of Theorem 1, that efficient allocations must yield all agents a binary lottery. Next, we start from an allocation where everyone is facing the lottery that gives them an equal chance for each of the goods and employ a known technique of demand-sets convexification (see Mas-Colell, Whinston, and Green [24, Section 17.I]) to obtain a competitive market equilibrium prices and allocations. Given these prices, all agents will maximize their utility along the same budget set, so No-Envy is guaranteed. Competitive equilibria are feasible and efficient, hence the claim of the theorem.

There is however one issue that requires special attention in which our analysis of the market equilibrium differs from the literature. Formally, the lottery $(x_1, q_1; \ldots; x_N, 1-\sum_{i=1}^{N-1} q_i)$ is represented as the vector $(q_1, \ldots, q_{N-1})$ in the $N-1$-dimensional simplex. This is different from the standard model, where the domain of preferences is not bounded from above. To see why this may create a problem, consider Example 1 in the Appendix with $N = 3$. The preferences of this example are monotonic in the probabilities q_1 and q_2, but

[11] The definition is again in the ex-ante sense, before agents know the realization of the lotteries they receive.

they do not satisfy monotonicity with respect to first order stochastic dominance, and equilibrium does not exist. We show in the proof of Theorem 3 that this stronger version of monotonicity eliminates the existence problem.

Remark 1 Let T be the number of lotteries used in the proposed solution. Then for $h = 1, 2, ..., T$ there is a continuum of agents who receive the same binary lottery, say $(x^h, \rho^h; y^h, 1-\rho^h)$ for some outcomes x^h, y^h and $\rho^h \in [0, 1]$. The implementation of this, so that the fraction of the people in this group that receives x^h is ρ^h, can be guaranteed by using the appropriate law of large numbers for a continuum of independent random variables. Such approach appears, for example, in Sun [34], and we adopt here his measure theoretic framework.[12]

4 Ex-ante Lotteries

If preferences are strictly quasi-convex, then giving two identical agents the same interior outcome must be inefficient, as moving in opposite directions along a supporting plane of the indifference curve will make both better off. Instead of equality in outcomes, allocation mechanisms will seek a weaker notion of equality, where identical agents will be indifferent between their respective outcomes. This is indeed the conclusion from Theorem 3, where everyone receives a binary lottery, but not the same one.

But indifference between the outcomes does not imply indifference to the procedures used to allocate these outcomes. A person may be indifferent between two seemingly identical objects of art left by his grandparents. Yet realizing that at least one of them must be a faked copy of the original, he will not trust his cousin, a museum curator, to choose first. In the context of the school allocation problem, parents may suspect the social planner of having some private information regarding the schools which will imply better lotteries for some families favored by the authorities.

There is a simple way to avoid such potential mistrust: Everyone will face the same lottery P over the set of the binary lotteries. The learned cousin

[12] We assumed that there are N blocks of agents so that the analysis of the continuum will parallel the finite case. If there is a continuum of types where the measure of each type is zero, then as in Mas-Colell, Whinston, and Green [24, p. 629] the actual allocation doesn't require the analysis of this remark, as almost all agents will have a unique lottery in their demand set.

may know which of the two vases is Ming and which is a modern counterfeit, but she will not be able to use this information if the allocation is dictated by the outcome of a fair coin. Similarly, even if the social planner favors some families, inside information about the schools becomes useless if the binary lotteries of Theorem 3 are allocated by a lottery.[13] Given that all individuals will face the exact same lottery, this procedure guarantees full equality in the ex-ante stage.

The effectiveness of this procedure crucially depends on the agents' attitude towards multi-stage lotteries. Denote the relevant binary lotteries $q^{(1)}, \ldots, q^{(T)}$. If agents care only about the overall probability distribution over final outcomes, then they will perceive a compound lottery over lotteries as a simple lottery over final outcomes, where the probability of each x_i is $\sum_j P(q^{(j)})q^{(j)}(x_i)$. But then, if preferences over simple lotteries are strictly quasi-convex, all individuals will be worse off compared to their initially held lottery.

Suppose, however, that individuals do not reduce compound lotteries using the laws of probability, and instead satisfy the compound independence axiom (Segal [32], Dillenberger [10]). This axiom prescribes that if a person is indifferent between receiving either q or q' for sure, then they will be indifferent to replacing q with q' in any compound lottery that has q in its support. Since, by construction, equality implies that all agents are indifferent between all lotteries in the suggested allocation, they will also be indifferent to any lottery over them. In other words, compound independence guarantees full ex-ante equality among agents without reducing their welfare. The experimental support for compound independence, and the lack of such support for the reduction of compound lotteries axiom,[14] suggest that this is indeed a fair procedure to follow.

[13] The emphasize here is on a real randomization rather than an imaginary randomization that each agent may entertain about his possibility to receive any of the objects. Only the former will remove agents' concerns for unfairness or of a biased use of planer's private information in the allocation decision.

[14] See, among others, Halevy [12], Abdellaoui, Klibanoff, and Placido [1], Harrison, Martínez-Correa, and Swarthout [13], and Masatlioglu, Orhun, and Raymond [23].

5 Concluding Remarks

The use of binary lotteries is pervasive in economics. Many experimental works are conducted with choices among such lotteries (or between them and sure outcomes), where the main rationale for using binary lotteries is that they are easily interpretable. The Binary Lottery Procedure, which pays subjects in binary lottery tickets instead of monetary amounts, is often employed to induce risk neutrality of subjects in a belief elicitation task (see, for example, Harrison, Martínez-Correab, Swarthout, and Ulm [14] and Hossain and Okui [15]).[15] Some recent theoretical papers use simplicity criteria to argue for the attractiveness of binary lotteries in terms of minimizing complexity costs (for example, Puri [26]), and of binary acts, that are always 'well-understood' and can be used as a tool for making difficult comparisons (Valenzuela-Stookey [36]).

In our setting, that (almost) everyone should receive a binary lottery follows mathematically from the assumption that all individual preferences are quasi-convex. As argued above, this gives us a simple necessary condition that can be used to assess whether an allocation of lotteries is ex-ante efficient. But as a social mechanism, binary lotteries have another independent attraction of their own. When facing a lottery over a set of outcomes on which they do not have full information, people may wait till they know what outcome they won before learning more about it. But as it is quite natural for people to look for information about the potential outcomes before the lottery is played, it is clearly better for them to face a lottery with fewer outcomes.

In Section 4 we suggested another layer of social randomization over the binary lotteries that will be used. If people reduce lotteries by multiplying the probabilities then they will probably need to evaluate all N outcomes. But if they use the compound independence axiom, then they view the first stage as a lottery over lotteries and will defer evaluating the outcomes till the next stage, when they'll face a lottery over two outcomes only.

[15] This method was first introduced by Smith [33] and Roth and Malouf [29]. There is an on going debate about its descriptive accuracy (see Kirchkamp, Oechssler, and Sofianos [18] and references therein).

Appendix

Proof of Theorem 1: Suppose that for $a = n, m$, $q_r^a, q_s^a, q_t^a > 0$. If the two individuals do not have the same ordinal preferences over the three goods, for example, if $x_r \succ_n x_s$ but $x_s \succ_m x_r$, then transfer ε probability of x_r from person m to n and ε probability of x_s from n to m to obtain a feasible allocation which is strictly preferred to the original one by n and m and indifferent to the original one by everyone else.

Suppose now that for $a = m, n$, $x_r \succ_a x_s \succ_a x_t$ and as before, that $q_r^a, q_s^a, q_t^a > 0$. For $a = m, n$, let $\bar{q}^a = q_r^a + q_s^a + q_t^a$. The two modified Machina-Marschak triangles below depict probability allocations over the three outcomes for individuals m and n that do not change the sums of these probabilities (see Figure 1 below). All the changes in this proof are sufficiently small so that they can be done without violating eqs. (1) and (2). In both panels, the probability of x_t is measured on the horizontal axis and that of x_r on the vertical one. The only values of q that will be changed are those of q_i^a for $a = m, n$ and $i = r, s, t$. We will therefore deal with the induced preferences over the above triangles and ignore the rest of the probabilities. To simplify notation, we write (q_t^a, q_r^a) for $(q_t^a, \bar{q}^a - q_t^a - q_r^a, q_r^a)$, which by itself stands for $(q_t^a, \bar{q}^a - q_t^a - q_r^a, q_r^a, q_{-(t,s,r)}^a)$.

Without loss of generality, one of the supporting slopes to the indifference curve of person n through (q_t^n, q_r^n) is weakly steeper than one of the supporting slopes to the indifference curve of person m through (q_t^m, q_r^m) (such slopes exist by the quasi-convexity of the preferences). Let τ be a slope between these two values. Since preferences are strictly quasi-convex, we get that for a sufficiently small $\varepsilon > 0$, $(q_t^m + \varepsilon, q_r^m + \tau\varepsilon) \succ_m (q_t^m, q_r^m)$ and $(q_t^n - \varepsilon, q_r^n - \tau\varepsilon) \succ_n (q_t^n, q_r^n)$. Observe that eqs. (1) and (2) are still satisfied and everyone else is indifferent between the new and the old lotteries. ■

Proof of Theorem 2: Let q be a feasible egalitarian solution in which two individuals, m and n, receive units of the goods $x_r \succ x_s \succ x_t$ with positive probabilities, for example, a lottery q that yields everybody the lottery $(\frac{1}{N}, \ldots, \frac{1}{N})$. As in the proof of Theorem 1 (see figure 1), transfer probabilities from one person to the other to make both of them strictly better off.

Suppose that person m is now better off than person n. That is, $(q_t^m + \varepsilon, q_r^m + \tau\varepsilon, q_{-t,r}^m) \succ (q_t^n - \varepsilon, q_r^n - \tau\varepsilon, q_{-t,r}^n)$. Transfer probability of r from person m to n (and the same probability of s from n to m) to equate their utilities,

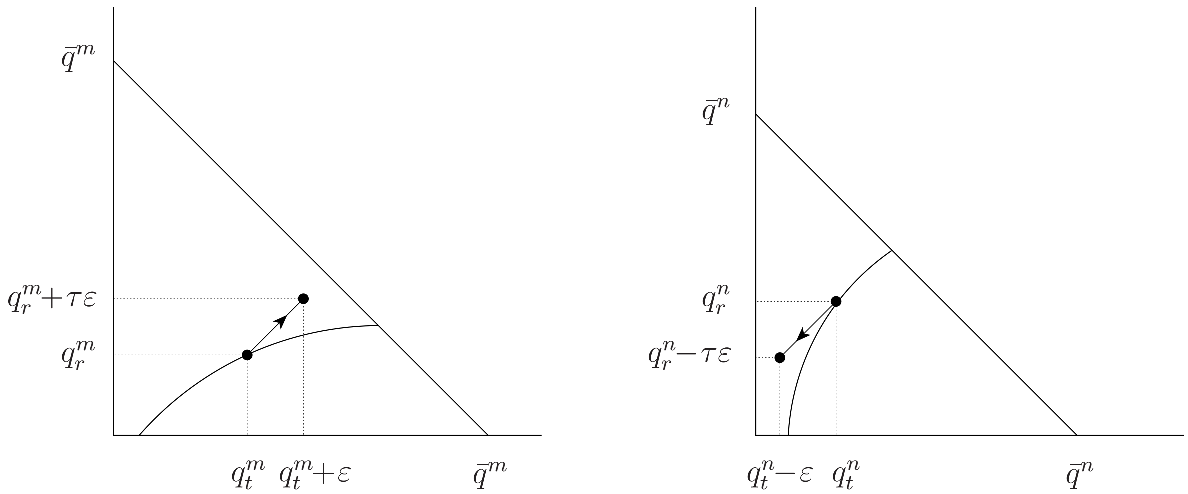

Figure 1: Changes in the allocations of individuals m and n

which are still higher than their original utilities under p, hence higher than the utility of everyone else. Likewise, if person n is better off than person m, we'll transfer probability of s from n to m and probability of t from m to n.

Let V be a continuous representation of the common $\succeq$. The next step in the proof shows that we can utilize this improvement in the utilities of m and n to benefit everyone else while maintaining equality. Since for all $b \neq m, n$, $q^m \sim q^n \succ q^b$ and individuals m and n both receive non-degenerate lotteries, there are for each $a = m, n$ and $b \neq m, n$ goods $i(ab)$ and $j(ab)$ such that $x_{i(ab)} \succ x_{j(ab)}$ and $q^a_{i(ab)}$, $q^b_{j(ab)} > 0$. Transfer now probability of $x_{i(ab)}$ from persons $a = m, n$ to person $b \neq m, n$ in exchange for probability of $x_{j(ab)}$ while maintaining equal utility to m and n and equal utility to everyone else, until the decreased utility of the formers match those of the rest. Continuity implies the possibility of this procedure. This new allocation satisfies eqs. (1) and (2). It satisfies equality and by monotonicity everyone prefers it to the original one, hence ex-ante efficiency is violated.

The set of solutions satisfying equality is not empty, for example, $q^1 = \ldots = q^{Nk} = (\frac{1}{N}, \ldots, \frac{1}{N})$. Let $v = \sup\{V(q) : q$ is a solution satisfying equality$\}$ and for $h = 1, \ldots$, let $q^h = (q^{1,h}, \ldots, q^{Nk,h})$ such that $V(q^h) \to v$.

Since all probabilities are between 0 and 1, it follows by standard arguments that there is a subsequence of q^h, without loss of generality the sequence itself, such that for all $n = 1, \ldots, Nk$, $q^{n,h} \to q^{n,*}$. The vector $q^* = (q^{1,*}, \ldots, q^{Nk,*})$ satisfies eqs. (1) and (2), hence it is a solution. Since V is continuous it satisfies equality, and as by the continuity of V, $V(q^{n,*}) = v^n$, it follows by the definition of v that q^* is optimal.

We next establish the bound M on the number of possible binary lotteries. Relabel the basic goods so that all agents agree that $x_1 \succ x_2 \succ \ldots \succ x_N$. Consider the set $B := \{(x_i, x_j) : i \leqslant \frac{N}{2}, j > \frac{N}{2}\}$. Note that for any pair $(x_k, x_l) \notin B$, there is $(x_i, x_j) \in B$ with either (i) $j \leqslant l, k$ with at least one strict inequality; or (ii) $i \geqslant l, k$, with at least one strict inequality. This means that either any lottery between x_i and x_j is strictly better than any lottery between x_k and x_l or the other way around, and hence such lotteries cannot be given to different agents while maintaining equality. On the other hand, this is not the case for any $(x_i, x_j), (x'_i, x'_j) \in B$, as $x'_i < x_j$ and $x'_j > x_i$. The cardinality of B is floor($\frac{N^2}{4}$):= $\max\{m \in \mathbb{Z} : m \leq \frac{N^2}{4}\}$, where $\mathbb{Z}$ is the set of all integers. To show that there is no other such set of 'non-dominated' pairs with greater cardinality, we note that, mathematically, the bound M cannot be greater than the bound on the number of edges in any graph on N vertices with no triangle. Mantel's theorem (Mantel [21]) states that such graph contains at most floor($\frac{N^2}{4}$) edges. Indeed, the set B described above can be represented as an N-vertex complete balanced bipartite graph that has no triangle subgraph and has exactly floor($\frac{N^2}{4}$) edges.

Establishing the bound M on the number of possible non-binary lotteries is similar. For the same relabeling of the goods as above, let B^* be a maximal set of non-binary lotteries. We show first that there is ℓ^* such that for all $(x_i, x_j, x_k) \in B^*$ such that $i < j < k$, either $i, j \leqslant \ell^*$ and $k > \ell^*$, or $i \leqslant \ell^*$ and $j, k > \ell^*$. Suppose not. Then for every ℓ either there is $(x_i, x_j, x_k) \in B^*$ such that $i, j, k \leqslant \ell^*$, or there is $(x_i, x_j, x_k) \in B^*$ such that $i, j, k > \ell^*$. Let $\bar{\ell}$ be the highest index for which there is $(x_i, x_j, x_k) \in B^*$ such that $i, j, k > \bar{\ell}$. Clearly $i = \bar{\ell} + 1$ (otherwise $\bar{\ell}$ is not the highest such index). Therefore there is $(x_{i'}, x_{j'}, x_{k'}) \in B^*$ such that $i', j', k' \leqslant \bar{\ell} + 1$, but then all lotteries with support $(x_{i'}, x_{j'}, x_{k'})$ are strictly preferred to all lotteries with support (x_i, x_j, x_k), so no-envy cannot be satisfied.

The maximal number or triplets given ℓ^* is

$$\binom{\ell^*}{2} \times (N - \ell^*) + \ell^* \times \binom{N - \ell^*}{2} = \frac{\ell^*(N - \ell^*)(N - 2)}{2}$$

This expression is maximized at $\ell^* = \frac{N}{2}$, where it is equal to floor$(\frac{N^2(N-2)}{8})$. $\blacksquare$

Example 1 Consider a continuum economy as in Section 3 with $N = 3$. The preferences $\succeq_1$, $\succeq_2$, and $\succeq_3$ over $\Delta^2 = \{(q_1, q_2) \in \Re^2_+ : q_1 + q_2 \leqslant 1\}$ can be represented by $V_1 = V_2 = 3q_1 + q_2$ and $V_3 = 6.25q_1^2 + q_2^2$. The initial lottery held by each person is represented by the point $(\frac{1}{3}, \frac{1}{3}) \in \Delta^2$. Let the price of q_2 be 1, and denote the price of q_1 by π. The *convexified* demand correspondences of the various agents are given by

$$D_1(\pi) = D_2(\pi) = \begin{cases} (1,0) & \pi \leqslant \frac{1}{2} \\ (\frac{1+\pi}{3\pi}, 0) & \frac{1}{2} < \pi < 3 \\ \{(\frac{3+5\alpha}{18}, \frac{5(1-\alpha)}{6}) : \alpha \in [0,1]\} & \pi = 3 \\ (\frac{\pi-2}{3\pi-3}, \frac{2\pi-1}{3\pi-3}) & \pi > 3 \end{cases}$$

$$D_3(\pi) = \begin{cases} (1,0) & \pi \leqslant \frac{1}{2} \\ (\frac{1+\pi}{3\pi}, 0) & \frac{1}{2} < \pi < 5 \\ \{(\frac{2\alpha}{5}, 1-\alpha) : \alpha \in [0,1]\} & \pi = 5 \\ (0,1) & 5 < \pi < 6.8 \\ (\frac{8\alpha}{29}, \frac{29-8\alpha}{29}) : \alpha \in [0,1] & \pi = 6.8 \\ (\frac{\pi-2}{3\pi-3}, \frac{2\pi-1}{3\pi-3}) & \pi > 6.8 \end{cases}$$

Clearly, there is no π such that $\frac{1}{3}[D_1(\pi) + D_2(\pi) + D_3(\pi)] = (\frac{1}{3}, \frac{1}{3})$. $\square$

Proof of Theorem 3: We show first that an efficient solution yields everyone a binary lottery. Suppose that q is an efficient solution with $\gamma > 0$ mass of individuals receiving non-binary lotteries. Without loss of generality, they all receive with positive probabilities each of the three outcomes x_r, x_s, x_t where $r > s > t$. That is, $\mu\{a : f_i(a) > 0,\, i = r, s, t\} > 0$. As μ is σ-additive, it follows that $\mu(A) \geqslant 0$, where $A = \{a : f_i(a) > \varepsilon,\, i = r, s, t\}$ for some $\varepsilon > 0$.

For every $a \in A$, let D_a be the triangle $\{(q_t, q_r) \in \Re^2_+ : q_t + q_r \leqslant \bar{q}_a = f_r(a) + f_s(a) + f_t(a)\}$. Let τ_a be the slope of a supporting line to the indifference curve in D_a through $(f_t(a), f_r(a))$. Let τ^* be such that $\mu(a : \tau_a > \tau^*)$, $\mu(a : \tau_a < \tau^*) \leqslant \frac{1}{2}\mu(A)$. Divide A into two sets A_1 and A_2 such that $\mu(A_1) = \mu(A_2) = \frac{1}{2}\mu(A)$, for all $a \in A_1$, $\tau_a \geqslant \tau^*$ and for all $a \in A_2$, $\tau_a \leqslant \tau^*$. We now follow the procedure described in the proof of Theorem 1, where individuals m and n are replaced with A_1 and A_2.

Let $\Pi^{N-1} = \{(\pi_1, \ldots, \pi_{N-1}) \in \Re_+^{N-1} : \sum_{i=1}^{N-1} \pi_i = 1\}$ be a prices simplex. For $\pi \in \Pi^{N-1}$, let $D_i(\pi) = \{q \in \Delta^{N-1} : \pi \cdot q \leqslant \frac{1}{N}$ and $\pi \cdot q' \leqslant \frac{1}{N} \Longrightarrow q \succeq_i q'\}$, $i = 1, \ldots, N$, and let $D_i^*(\pi) = \text{Conv}(D_i(\pi))$. These are the convexified demand sets of the various types given prices π. Observe that since the preferences $\succeq_i$ are strictly quasi-convex, the set $D_i(\pi)$ is a finite set of binary lotteries. In the continuum economy, these lotteries can be allocated to the type-i individuals in such proportions to obtain any point in $D_i^*(\pi)$.

Suppose that for some $q \in D_i(\pi)$, $\pi \cdot q < \frac{1}{N}$. If $q = (1, 0, \ldots, 0) := \delta_1$, then since for all i, x_1 is the best outcome, it follows that for all i, $D_i(\pi) = \delta_1$ and π cannot be a Walrasian equilibrium price-vector. Otherwise, there is $\alpha \in (0, 1]$ such that $\pi \cdot [\alpha\delta_1 + (1-\alpha)p] = \frac{1}{N}$. By monotonicity with respect to first-order stochastic dominance, $\alpha\delta_1 + (1-\alpha)q \succ_i q$, a contradiction to the definition of $D_i(\pi)$. It thus follows that $D_i(\pi) = \{q \in \Delta^{N-1} : \pi \cdot q = \frac{1}{N}$ and $\pi \cdot q' \leqslant \frac{1}{N} \Longrightarrow q \succeq_i q'\}$. Clearly the correspondences $D_i(\pi)$ (and therefore $D_i^*(\pi)$) are upper hemi-continuous, hence there exists an equilibrium vector π and allocations q_i^* in $D_i^*(\pi)$, $i = 1, \ldots, N$, such that $\sum_{i=1}^{N} q_i^* = (\frac{1}{N}, \ldots, \frac{1}{N})$.

These allocations are efficient, feasible, and since all agents face the same "price" vector π, they satisfy no-envy. The first part of the proof implies that all agents receive a binary lottery, hence the claim of the theorem. ■

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