Formation of Teams in Contests: Tradeoffs Between Inter- and Intra-Team Inequalities*

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Abstract

We consider a team contest in which players make efforts to compete with other teams for a prize, and players of the winning team divide the prize according to a prize-sharing rule. This prize-sharing rule matters in generating members' efforts and attracting players from outside. Assuming that players differ in their abilities to contribute to a team and their abilities are observable, we analyze which team structure realizes by allowing players to move across teams. This inter-team mobility is achieved via head-hunting: a team leader can offer one of the positions to an outside player. We say that it is a successful head-hunting if the player is better off by taking the position, and the team's winning probability is improved. A team structure is stable if there is no successful head-hunting opportunity. We show that if all teams employ the egalitarian sharing rule, then the complete sorting of players according to their abilities occurs, and inter-team inequality becomes the largest. In contrast, if all teams employ a substantially unequal sharing rule, there is a stable team structure with a small inter-team inequality and a large intra-team inequality. This result illustrates a trade-off between intra-team inequality and inter-team inequality in forming teams.

Keywords: team contest, CES effort aggregator function, prize sharing rule, head-hunting, stable team structure, intra-team inequality, inter-team inequality

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1 Introduction

In team contests players have to exert joint effort in order to compete with other teams for a prize. Each player's performance is typically affected by internal team characteristics such as the team composition (the ability or productivity of one's teammates), the agreed-upon prize sharing rule, the complementarity between individual efforts, as well as the presence of free-riding incentives. The resulting teammates' efforts are then aggregated to team effort which, when measured against that of other teams, determines the team's winning probability according to a Tullock contest success function. Thus, the relative strength and composition of opposing teams can affect not only the equilibrium effort choice of each player but the equilibrium team composition in the first place.

In this paper we focus on a specific type of player mobility across teams, namely mobility achieved through head-hunting. We assume that in an attempt to improve their equilibrium winning probability, teams can extend an offer to any player. This offer must specify the new recruit's relative position on the team, characterized by a predetermined value for their share of the prize. When players consider the decision of potentially accepting such an offer, they have to weigh multiple costs and benefits: (1) what is the share of the prize they would receive on the other team? (2) how much effort would they have to exert there? (3) what is the new team's winning probability, and how important would their new position be in affecting the team's performance? (4) how does their (publicly known) ability compare to that of their new teammates, and could a potential transfer lead to increased free-riding incentives? (5) what happens to the seat they vacated on their current team (if they had one), and how will this affect the competition as a whole? When a player finds an offer acceptable, and if their new team's equilibrium winning probability is increased as a result of them joining, then we say that a successful head-hunting occurs. A team structure (a matching between teams and players) that allows no successful head-hunting will be defined as head-hunting-proof (or stable).

The goal of this work is to study the types of stable team structures that might result from common reward allocation rules. We assume that all teams have the same fixed capacity and that they all use the same common prize allocation rule. This sharing rule might be imposed as a part of the rules of the contest (such as the Kaggle example below), or it might just be the result of a long-established social norm in each industry. We distinguish between different allocation rules according to how equally they treat team members. At one end of the spectrum we consider the egalitarian rule which divides the prize equally among the players. At the other end are rules that treat players unequally, giving higher-ranked members substantially higher shares. Before proceeding with an overview of our findings, we present several examples that help illustrate the use of such allocation rules and the resulting team structures.

The first example comes from the website Kaggle, a platform hosting a variety of data science and machine learning competitions. Each competition is self-contained, with a predetermined prize, and there is a global cap of eight players per team (although each competition often has a lower team size limit). Anyone is allowed to participate and submit a solution, and joining a team is left entirely in the hands of the contestants. Committing to a team occurs at the time of registration for each contest and results in a binding agreement. When a team wins a competition, the prize is equally divided between team members regardless of their contribution (the egalitarian rule is enforced by Kaggle). It should be noted that all solutions are evaluated and ranked, not only the winning one. Each player is then granted a score representing how well their team did in the contest. The score is publicly known and can help players find teams in future contests. It has been observed that over time many of the teams have ended up sorted by ability - some of the highest-ranked contestants have joined teams together, winning or scoring high in multiple competitions. The same seems to be true at the middle- and lower-end of scores as well. The continued evolution of team formation at Kaggle is very reminiscent of head-hunting and has led to an outcome in which the resulting inter-team inequality stands in contrast to the implicitly enforced intra-team equality.

The second example comes naturally from team sports leagues such as the NBA. The drafts in such leagues are often designed to help weaker teams by giving them priority in picking new players and thus ensuring a degree of inter-team equality. As weaker teams can get the highest-ability players, this type of system typically leads to matching outcomes with a wide distribution of ability within each team (at least to start with). But this comes hand-in-hand with a very high salary inequality within teams as well. Just among the top 30 rookie contracts the variation in 2019 in the NBA for example was between \$13 and \$51 million. In a way, the high degree of intra-team inequality is a prerequisite for achieving relative inter-team equality.

A perhaps better example illustrating the emergence of inter-team equality in the presence of intra-team team inequality comes from hierarchical reward systems in many industries requiring white-collar jobs. Such systems are typically based on the rationale that high differentiation (in pay or recognition) between ranks within an organization promotes competition for promotion, which can result in increases in on-the-job performance. However, even though such structures are designed as an incentive mechanism within a company, they often lead recruiters to hire industry specialists (known as head-hunters) who are able to target candidates from competing organizations. These specialists do not typically engage in actual recruitment but instead focus on identifying high-quality employees for specific high-level positions.[1] The resulting recruitment process leads high-ability individuals into the highest-ranking jobs across competing organizations which inadvertently results in a relatively even distribution of talent across companies.

The way in which we allow the intuitive structure from the examples above to transpire in this work is by explicitly modeling worker heterogeneity. In particular, players differ in their *observable* abilities and hence in their ability to contribute to a given team. Furthermore,

[1]Some US higher education institutes hire head-hunting companies to conduct their search for new deans of their schools. For example, in conducting a dean's search for a business school of a university, a head-hunting company often approaches department chairs of business schools and others elsewhere to ask about their interests in playing such a role.

team output is the result of aggregated individual effort inputs, which we model via a CES aggregator function, allowing for different levels of effort complementarity. Players' efforts are *not observable* or *not contractable*—thus players' efforts contribute to the winning probability of their team but do not affect their shares of the winning prize. The shares that players receive are allowed to be heterogeneous based on the positions they are assigned to, and we explicitly focus on the rate at which lower positions are discounted relative to higher positions within each team. It should be noted that by combining the approaches by Konishi and Pan (2020, 2021) and Simeonov (2020), we can explicitly solve for player's equilibrium payoffs, which makes it possible to discuss head-hunting as a well-defined process of attracting better candidates.

The main result of this work is to show that the tradeoffs between intra- and inter-team inequalities are not coincidental. We show that when the egalitarian rule is used within each team, then complete ability sorting across teams is the only stable team structure. Alternatively, we consider hierarchical prize allocation rules in which a common discount factor for rewards is used. For high discount factors, we show that the cyclical allocation of players across teams is stable. For intermediate discount factors, both the cyclical and complete sorting by ability can coexist, and more generally, a combination of cyclical assignment and ability sorting can occur in a stable team structure.

Much of the rationale behind these results originates from our key Lemma 3 in Section 4 below. It would be instructive to diverge with a brief discussion of Lemma 3 before proceeding with the model. Consider in particular a scenario with two teams: a strong team A with high average team ability and a high equilibrium chance of winning and a weaker team B with lower average team ability and low chance of success. Suppose, however, that the stronger team A currently has hired a relatively low-ability player for a certain position (player 1), while the same corresponding position on team B is filled by a player of higher skill (player 2). As long as the shares attached to these positions are the same (they are filling equivalent positions), then Lemma 3 states that team A will have a higher winning probability and player 2 will get a higher expected payoff by switching from team B to team A. It should be intuitively clear that team A only has to benefit by hiring a more capable player - by replacing player 1 with player 2, team A improves its winning probability. Player 2, on the other hand, receives the same fraction of the prize in both cases, so moving to the stronger team increases her probability of getting that share in the first place.

In many ways Lemma 3 is the embodiment of the tradeoff between intra- and inter-team inequalities. If all teams use the egalitarian rule to split the rewards equally, then high-ability players will not be happy on lower-ability teams. Not only might they receive an unfair share given their skill, but their presence on such teams might exacerbate free-riding by their lower-ability teammates. It would make sense for those high-ability players to welcome head-hunting by high-ability teams instead. Thus, a team matching structure characterized by a high degree of ability sorting seems like a natural outcome here. What incentives then would be sufficient to break a high-ability team so that some of its players are willing to join other, perhaps weaker teams? Clearly, there must be a significant difference in compensation between group members to open the possibility for such an occurrence. Only then would a high-ability player find it viable to join a higher position on a weaker team instead of keeping a lower position on a more successful team. High inequality within teams seems to become a necessary prerequisite for achieving a more even distribution of talent across teams.

The rest of the paper is organized as follows. The following section presents a brief review of related literature. Section 2 describes the model and assumptions. Section 3 presents the equilibrium player and team efforts in general team contests. In Section 4, we proceed with the discussion of stability and the main results regarding the tradeoffs between intra- and inter-team inequalities, and Section 5 concludes.

1.1 Relations to the Literature

Broadly, this paper belongs to the theory of coalition formation with externalities. Players' payoffs depend not only on which coalition they belong to but also on other coalitions. Hart and Kurz (1983), Bloch (1996), Yi (1997), Ray and Vohra (1999), and Ray (2008) provide a general analysis of coalition formation games with externalities across coalitions. As specific economic applications, Bloch (1995), Yi (1996), and Ray and Vohra (2001) consider cartel structures, customs unions, and public good provision groups, respectively. Our paper belongs to this literature, but there are some differences: in our game, there is a membership quota for each team, and prize-sharing rules within a team are predetermined, but shares can be heterogeneous. Thus, each position of a team can be heterogeneous for players, and players care about which position of a team they will be assigned to. This is a new feature of our model in the coalition formation literature.

More specifically, this paper belongs to the literature on group contests and prize-sharing rules. Assuming individual efforts are contractable, Nitzan (1991) analyzes how the combination of an egalitarian and a relative-effort-sharing rules affects members' incentives for players in large and small groups. Lee (1995) and Ueda (2002) endogeneize group sharing rules in this class. Esteban and Ray (2001) and Nitzan and Ueda (2011) show that Olson's (1973) group size paradox disappears if the prize among the members can be allocated into public and private benefits and if private benefits can be allocated by an endogenously chosen relative-effort-sharing rule, respectively. Based on the line of group contest research above, Baik and Lee (1997, 2001) endogenize the alliance formation in Nitzan's (1991) game with endogenous group sharing rules and analyze two- and multiple-alliance cases, respectively. They use open-membership games to describe alliance formation. Bloch et al. (2006) generalize the model substantially to analyze the stability of the grand alliance in different alliance formation games. Sanchez-Pages (2007a,b) explores different types of stability concepts in alliance formation in contests where efforts are perfect substitutes. These papers assume alliance members can write a binding contract of sharing rules in the case of the alliance's winning. In contrast, following Esteban and Sakovics (2003), Konishi and Pan (2020, 2021) analyze equilibrium alliance structures in homogenous player alliance formation games without side payments when members' efforts are complementary with each other by using a CES aggregator function.[2] The current paper extends Konishi and Pan (2020,2021), allowing for heterogeneous abilities and unequal sharing rules using the approach by Simeonov (2020) and Kobayashi, Konishi, and Ueda (2023).[3] Unlike in Nitzan (1991) and Nitzan and Ueda (2011), individual efforts are unobservable or noncontractable, allowing for free-riders as in Esteban and Ray (2001). For more complete surveys of the literature on group contests, see Konrad (2009) and Fu and Wu (2019).

Our stability notion, head-hunting-proofness, is close to pairwise stability in matching literature due to the presence of team membership quotas. Gale and Shapley (1962) introduce the celebrated two-sided matching problem and its solution concept, pairwise stable matching. In their domain, the pairwise stability is equivalent to the core despite its simplicity. Imamura, Konishi, and Pan (2021) introduce externalities across matched pairs to the two-sided matching problems and show that their pairwise stable matching via swapping preserves nice properties.[4] Our head-hunting-proofness can be interpreted as a team structure that is immune to pairwise deviations by a team leader who cares about the winning probability of the team and a player who cares about his/her private benefits.[5]

Lastly, our intra- and inter-team inequality result is closely related to two papers in hedonic game literature. Hedonic games are characteristic function form games (thus no externalities across coalitions), in which a player's payoff is solely determined by the coalition she joins.[6] Banerjee, Konishi, and Sonmez (2001) show that under their top coalition property, an assortative coalition structure is the unique coalition structure core allocation. Morelli and Park (2016) consider hedonic games in which a player must consider both i) the expected power of the coalition and ii) her position in the vertical structure within the coalition, and show that cyclical assignment allocations are core stable. Our model generates these two allocations by changing the sharing rules in the same model with externalities.[7]

[2]In contest games, Kolmar and Rommeswinkel (2013) consider a group contest played by exogenously formed groups using a CES effort-aggregator function when group members have heterogeneous abilities.

[3]When team memberships are fixed, Simeonov (2020) and Kobayashi et al. (2023) consider the optimal sharing rules and task-assignment rules to maximize the aggregated team effort under CES effort aggregator functions and constant elasticity effort cost functions.

[4]Imamura and Konishi (2021) provide additional support for pairwise stability via swapping by introducing farsighted agents. They show that the pairwise stability via swapping is equivalent to the largest consistent set in Chwe (1994) in the pairs competition problem introduced in Imamura, Konishi, and Pan (2021).

[5]Our stability notion assumes that players are somewhat naive. In the theory of coalition formation with externalities across coalitions, typical solution concepts assume quite sophisticated players (see Bloch 1997, Ray 2008, and Ray and Vohra 2014 for surveys).

[6]See Banerjee, Konishi, and Sonmez (2001) and Bogomolnaia and Jackson (2002).

[7]The readers should be reminded that the solution concept of this paper differs from their core stability. It is hard to formulate group deviations without adding a lot of structures to the game in the presence of widespread externalities.

2 The Model

There are potentially $j = 1, 2, ..., J$ teams, and there are M positions in a team. Let (m, j) stand for the mth position in j team. Player $i = 1, ..., N$ is characterized by her *ability* a_i. We assume that $a_1 \geq a_2 \geq ... \geq a_N$. With some abuse of notations, we also let M, J, and N stand for the set of positions, teams, and players, respectively. A *membership profile* is $\varphi = (\varphi_{mj})_{m\in M, j\in J}$ where $\varphi_{mj} \in N \cup \{\emptyset\}$ for all $m \in M$ and $j \in J$. We assume a player can only belong to a team. Therefore, a membership profile is *feasible* if $\varphi_{mj} \neq \varphi_{m'j'}$ for all $(m, j) \neq (m', j')$. Let $N_j = \{i \in N | \varphi_{mj} = i$ for some $m \in M\} \subset N$ be the set of players in team j under φ.

We will consider our team stability problem in a team contest framework in two stages. In stage 1, a team structure φ is determined, and in stage 2, an actual team contest occurs given φ. Membership profile φ is formed in stage 1, by players' foreseeing the resulting outcomes in stage 2. So, we will first describe the team contest problem in stage 2, and our stability notion in stage 1 will be introduced in Section 4.[8]

Given a feasible membership profile φ, players compete with each other as a team for a prize, which value is V. In this contest, team members $i \in N_j$ choose their effort levels e_i simultaneously and non-cooperatively. The members' efforts in team j are aggregated by a CES function $X_j = (\sum_{m\in M} a^{\sigma}_{\varphi_{mj}} e^{\sigma}_{\varphi_{mj}})^{\frac{1}{\sigma}}$, where $0 < \sigma < 1$.[9] This CES aggregator function becomes a linear function (perfect substitutes) when $\sigma = 1$, and becomes a Cobb-Douglas function when $\sigma = 0$ in the limit. Teams' aggregate effort vector $(X_1, ..., X_J)$ determines each team's winning probability. The winning probabilities of teams are determined by a Tullock-style contest: team j's "winning probability" is given by

$$P_j = \frac{X_j}{\sum_{k=1}^{J} X_k}. \tag{1}$$

After the winning team gets the prize, it will distribute the prize to its team members by a *common fixed sharing rule* that is considered as a social norm. This common sharing rule is $\theta = (\theta_1, ..., \theta_m, ..., \theta_M)$ with $\theta_m \in [0, 1]$ and $\sum_{m\in M} \theta_m = 1$, in which θ_m stands for the prize share that the player in position m of a team. Without loss of generality, we rank positions in a team by its shares, that is, $\theta_1 \geq \theta_2 \geq ... \geq \theta_M$. The effort cost function is common for all players: player i's effort is common and linear $c_i(e_i) = e_i$. Therefore, the expected payoff of the player in the position m of team j, or, equivalently, the player i such that $\varphi_{mj} = i$ is

$$U_{\varphi_{mj}} = \theta_m P_j V - e_{\varphi_{mj}}.$$

Each member of a team decides his/her effort level to maximize his/her expected payoff. We

[8]Our solution concept is a hybrid of cooperative and noncooperative games. In stage 2, team contests are played noncooperatively, whereas in stage 1, we consider an allocation that is immune to head-hunting.

[9]Kolmar and Rommeswinkel (2013) call this CES function a group impact function.

assume that team j members regard the other groups' aggregate effort X_{-j} as given and consider a Nash equilibrium of team j's effort contribution game as the best response of group j to the other groups' aggregate effort X_{-j}.

3 Team Contest

3.1 Equilibrium Analysis

For the time being, let's assume that all teams and all players make positive efforts. If so, the first-order condition of any player i with $\varphi_{mj} = i$ is

$$\frac{\partial U_{\varphi_{mj}}}{\partial e_{\varphi_{mj}}} = \theta_m \frac{(\sum_{m'=1}^{M} a^{\sigma}_{\varphi_{m'j}} e^{\sigma}_{\varphi_{m'j}})^{\frac{1}{\sigma}-1} a^{\sigma}_{\varphi_{mj}} e^{\sigma-1}_{\varphi_{mj}} X_{-j}}{\left(\left(\sum_{m'=1}^{M} a^{\sigma}_{\varphi_{m'j}} e^{\sigma}_{\varphi_{m'j}}\right)^{\frac{1}{\sigma}} + X_{-j}\right)^2} V - 1 = 0.$$

By using $X_j^{1-\sigma} = \left(\sum_{m'=1}^{M} a^{\sigma}_{\varphi_{m'j}} e^{\sigma}_{\varphi_{m'j}}\right)^{\frac{1}{\sigma}-1}$, this can be rewritten as

$$(1 - P_j)\frac{V}{X} X_j^{1-\sigma} a^{\sigma}_{\varphi_{mj}} e^{\sigma-1}_{\varphi_{mj}} \theta_m - 1 = 0.$$

From this expression, we have

$$e^{1-\sigma}_{\varphi_{mj}} = X_j^{1-\sigma} \left[(1 - P_j)\frac{V}{X}\right] a^{\sigma}_{\varphi_{mj}} \theta_m$$

and

$$e_{\varphi_{mj}} = X_j \left[(1 - P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}} \left(a^{\sigma}_{\varphi_{mj}} \theta_m\right)^{\frac{1}{1-\sigma}} \tag{2}$$

Here, note that if $X_j = 0$ then $e_i = 0$ for all $i \in N_j$. We raise this to the power of σ and then multiply it by $a^{\sigma}_{\varphi_{mj}}$,

$$a^{\sigma}_{\varphi_{mj}} e^{\sigma}_{\varphi_{mj}} = X_j^{\sigma} \left[(1 - P_j)\frac{V}{X}\right]^{\frac{\sigma}{1-\sigma}} a^{\frac{\sigma}{1-\sigma}}_{\varphi_{mj}} \theta_m^{\frac{\sigma}{1-\sigma}}$$

is obtained (the power of a_i is calculated by $\frac{\sigma^2}{1-\sigma} + \sigma = \frac{\sigma}{1-\sigma}$). Therefore, we may sum up with respect to all positions m in team j and then raise it to the power of $\frac{1}{\sigma}$,

$$X_j = X_j \left[(1 - P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}} \left(\sum_{m=1}^{M} a^{\frac{\sigma}{1-\sigma}}_{\varphi_{mj}} \theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1}{\sigma}}$$

or

$$1 = \left[\frac{X_{-j} V}{X^2}\right]^{\frac{1}{1-\sigma}} \left(\sum_{m=1}^{M} a^{\frac{\sigma}{1-\sigma}}_{\varphi_{mj}} \theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1}{\sigma}}$$

is obtained. Thus, we have

$$\frac{1}{\left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}} = \frac{X_{-j}}{X^2} \times V. \tag{3}$$

Summarizing this, we present the following lemma.

Lemma 1. Suppose that team j makes a positive aggregate effort in equilibrium. Then, we have

$$\frac{1}{A_j(\varphi_j)} = \frac{X_{-j}}{X^2} \times V, \tag{4}$$

where $A_j(\varphi_j) = \left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$ stands for the *productivity of team j*.

Summing the above up over all active teams, we have

$$\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)} = \frac{(J-1)}{X} V$$

or

$$X = \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}.$$

Substituting this back into (3) and recalling $\frac{X_{-j}}{X} = 1 - P_j$, we obtain

$$1 - P_j = \frac{\frac{1}{A_j(\varphi_j)}}{V} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} = \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}$$

and

$$P_j = 1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}. \tag{5}$$

Equation (5) says that the equilibrium winning probability of team j is an increasing function of $A_j(\varphi_j)$, the productivity of team j, which is only determined by φ_j and θ_ms.

Equations (4) and (5) together imply that X_j is

$$X_j = P_j X = \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}\right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}$$

We summarize these results in a proposition.

Proposition 1. Suppose that all J teams make positive efforts in equilibrium. Then, for all $j = 1, ..., J$, equilibrium aggregate effort and winning probability of team j are written as

$$X_j = \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}\right\} \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}},$$

and

$$P_j = 1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}},$$

where $A_j(\varphi_j) = \left(\sum_{m=1}^{M} a_{\varphi_{mj}}^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$. The total effort of all teams X can be written as

$$X = \frac{(J-1)\,V}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}.$$

Remark 1. Note that there are possible coordination failures in this model since we assume $\sigma \in (0,1)$ and a linear effort cost function.[10] A team member may not make an effort if her teammates are not making an effort due to the complementarity of production. As a result, there may exist two equilibria—with positive efforts and without effort. *We will choose a positive effort equilibrium whenever it exists to avoid this coordination problem.* With this refinement, we have a unique equilibrium.

Using the results, we can explicitly calculate each player's equilibrium payoff in stage 2 (see Appendix for the proof).

Proposition 2. Given a feasible membership assignment φ, suppose that all J teams make positive efforts in equilibrium. Then, for any team memberships φ_js and their sharing rules θ_js, the player i's equilibrium payoff with $i = \varphi_{mj}$ is written as

$$U_{\varphi_{mj}} = V \times \underbrace{\theta_m}_{\text{share}} \underbrace{\left[1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}}\right]}_{\text{team's winning probability}} \underbrace{\left[1 - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} \left(\frac{a_i^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}}{\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'}^{\frac{\sigma}{1-\sigma}}}\right)\right]}_{\text{net benefits by taking effort disutility into account}}$$

Remark 2. Note that the contents of the bracket is positive since $\frac{a_i^{\frac{\sigma}{1-\sigma}} \theta_m^{\frac{\sigma}{1-\sigma}}}{\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'}^{\frac{\sigma}{1-\sigma}}} < 1$ and $1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}} = P_j$. That is, $U_{\varphi_{mj}} \geq V \times \theta_m P_j^2 > 0$ must hold as long as team j makes positive efforts.

3.2 Share Function Approach and Active/Inactive Teams

So far, we assumed that all teams are active in equilibrium in the sense that their aggregate effort is positive. However, since we assume that each player's marginal cost of making an effort is a positive constant, it might not be the case if there is a team in which players have

[10] A CES production function can produce a positive output even if some inputs are zero for $\sigma \in (0,1)$ ($\sigma = 0$ corresponds to a Cobb-Douglas production function, in which all inputs need to be positive to have a positive output). A linear cost function means a constant marginal cost. Thus, if the marginal cost exceeds the marginal product, then a team player does not make an effort.

much lower abilities than other teams. So, to complete the equilibrium analysis, we will apply a method called the "share function" approach that is systematically analyzed in Cornes and Hartley (2005), by rewriting the second-stage competition as a Tullock contest with J individual players with heterogeneous marginal costs.[11] Cornes and Hartley (2005) considered a J player (individual) Tullock contest with heterogenous constant marginal costs $w_1 \leq w_2 \leq .... \leq w_J$, in which player $j = 1, ..., J$ exerts effort X_j with $w_j > 0$. Her winning probability is specified by $P_j = \frac{X_j}{\sum_{k=1}^{J} X_k}$, and her payoff is

$$u_j = \frac{X_j}{\sum_{k=1}^{J} X_k} V - w_j X_j.$$

The payoff function is strictly concave in X_j, and the first-order condition is

$$\frac{\left(\sum_{k \neq j} X_k\right)}{\left(\sum_{k=1}^{J} X_k\right)^2} V - w_j = \frac{X_{-j}}{X^2} V - w_j = 0, \tag{6}$$

for $j = 1, ..., J$. Then, $X_j > 0$ is a unique best response to X_{-j} if and only if

$$X_j^2 + 2X_{-j}X_j + X_{-j}^2 - \frac{X_{-j}}{w_j} V = 0.$$

Noting that some players may have too high a marginal cost for an interior solution, player j's best response to X_{-j} is

$$\beta_j(X_{-j}) = \max\left\{-X_{-j} + \sqrt{\frac{X_{-j}V}{w_j}}, 0\right\}.$$

We define player j's *replacement function* following Cornes and Hartley (2005): *a replacement function* $r_j(X)$ is a function of total effort X such that $r_j(X)$ is the best response to $X - r_j(X)$: i.e., $r_j(X) = \beta_j(X - r_j(X))$. Thus we obtain

$$r_j(X) = \max\left\{X - \frac{w_j X^2}{V}, 0\right\}.$$

Let group j's share function be $s_j(X) = \frac{1}{X} r_j(X)$:

$$s_j(X) = \max\left\{1 - \frac{w_j X}{V}, 0\right\}.$$

Note that $s_j(X)$ is a decreasing function in X. Let $s(X) = \sum_k s_k(X)$. This is a decreasing function as well. Order players by $w_1 \leq w_2 \leq .... \leq w_J$. The share function $s(X)$ is a piece-wise linear function with kinks at $\hat{X}^{n_j} = \frac{V}{w_j}$ for each $j = 1, ..., J$. Figure 1 depicts share functions for

[11] Esteban and Ray (2001) and Ueda (2002) used the same method in Cornes and Hartley (2005).

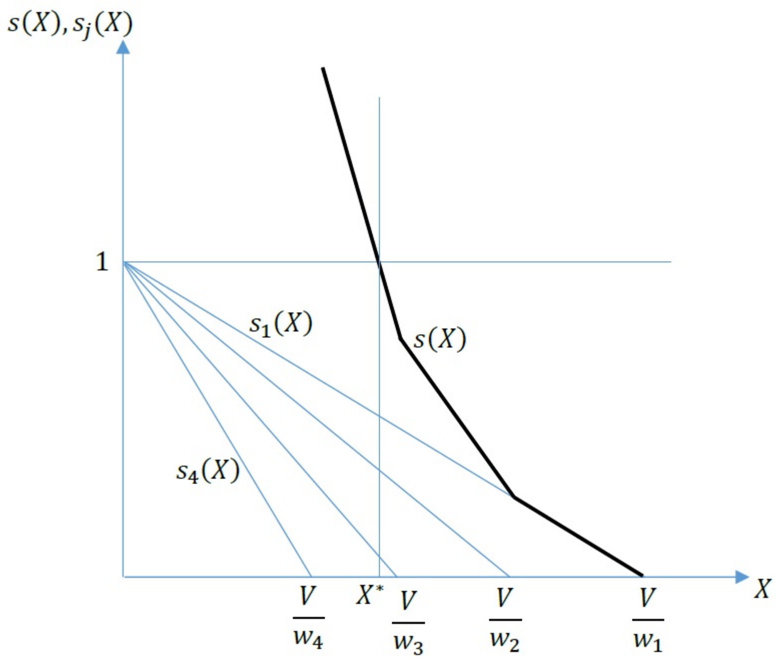

Figure 1: An example with $J = 4$ and $w_1 < w_2 < w_3 < w_4$. Teams 1, 2, and 3 are active. Team 4 is inactive.

$j = 1, ..., J$ and $s(X)$. The equilibrium for the artificial contest is a total effort, X^*, for which every group's optimal share sums up to 1. Clearly, there exists a *unique equilibrium* X^* defined by $\sum_k s_k(X^*) = 1$. Moreover, at the equilibrium X^*, $s_j(X^*)$ is also the winning probability of player j. As is easily seen from Figure 1, if $\hat{X}^{n_j} = \frac{V}{w_j} < X^*$, then $s_j(X^*) = 0$ must hold, which means only those groups with smaller marginal costs are *active, i.e., exert positive efforts.* The following lemma summarizes the result of this Tullock game with heterogeneous marginal costs $(J, (w_j)_{j=1}^J)$.

Lemma 2. [Cornes and Hartley, 2005] *A Tullock game with heterogeneous marginal costs* $(J, (w_j)_{j=1}^J)$ *has a unique equilibrium* X^* *at* $\sum_j s_j(X^*) = 1$. *Moreover, there exists* j^* *such that, for each* $j = 1, ..., j^*$, $X_j = X^* - \frac{w_j(X^*)^2}{V}$, *and for each* $j = j^* + 1, ..., J$, $\hat{X}^{n_j} \leq X^*$ *(or* $\sum_k s_k(\hat{X}^{n_j}) \geq 1$*) and* $X_j = 0$ *hold.*

Note that the set of (*interior*) first-order conditions for the Tullock contest (6) is identical to the set of first-order conditions (4) for the original game by setting $w_j = \frac{1}{A_j(\varphi_j)}$:

$$\frac{\left(\sum_{k \neq j} X_k\right)}{\left(\sum_{k=1}^J X_k\right)^2} V - w_j = \frac{X_{-j}}{X^2} V - \frac{1}{A_j(\varphi_j)} = 0,$$

for $j = 1, ..., J$. Note that the team with a lower marginal cost w_j is the team with higher productivity $A_j(\varphi_j)$. Given a team structure, we obtain the following explicit solutions by considering a special case of Kolmar and Rommeswinkel (2013).

Theorem 1. *Given a team profile φ, there exists a unique equilibrium in the inter-team contest for any partition of players $\pi = \{N_1, ..., N_J\}$ characterized by the share function $s(X^*) = 1$. There is $j^* \in \{1, ..., J\}$ such that $P_j^* = s_j(X^*) > 0$ (active teams) for all $j \leq j^*$ ($\hat{X}_j > X^*$), while $P_j^* = s_j(X^*) = 0$ (inactive teams) for all $j > j^*$ ($\hat{X}_j \leq X^*$). Then, team j's* winning probability is

$$P_j^* = 1 - \frac{(j^* - 1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}},$$

player $i \in N_j$ of team $j = 1, ..., J$ obtains payoff

$$U_{\varphi_{mj}} = \begin{cases} V \times \theta_m P_j^* \left[1 - \frac{(j^*-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}} \left(\frac{a_{\varphi_{mj}}\theta_m}{A_j(\varphi_j)}\right)^{\frac{\sigma}{1-\sigma}}\right] & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}.$$

Moreover, the equilibrium total efforts are

$$X^* = \frac{j^* - 1}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}},$$

and

$$(j^* - 1)\frac{1}{A_j(\varphi_j)} < \sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}$$

holds for all $j = 1, ..., j^*$.[12]

4 Stable Team Structures

In this section, we will consider the stability of a team structure generated by a given φ. We will consider a simple concept of *head-hunting*: given $\varphi_{mj} = i' \in N$, a team j offers this position (m, j) to another player i by replacing the incumbent player φ_{mj} by player i. A head-hunting is *successful* if (i) team j's winning probability improves, and (ii) player i who received the offer is better off by switching positions. However, we need to define this concept carefully due to externalities across teams. We will do so step by step.

A player $i \in N$ is *employed* under φ if i belongs to one of active teams ($i = \varphi_{mj}$ for some $m = 1, ..., M$, and some $j = 1, ..., j^*$), and is *unemployed* under φ if $i \in N$ does not belong to active teams ($i \neq \varphi_{mj}$ for any $j = 1, ..., j^*$). The sets of employed and unemployed players under φ are denoted by $E(\varphi)$ and $UE(\varphi)$, respectively. *We say that a* ***successful head-hunting of an unemployed player*** *is a pair of a team j's position (m, j) and player i such that (i) team j's winning probability increases by replacing player φ_{mj} by player $i \in UE(\varphi)$, and (ii) player i's payoff increases.* Condition (ii) is trivially satisfied since we know that unemployed players

[12]For $j > j^*$, we have $(j^* - 1) \times \frac{1}{A_j(\varphi_j)} \geq \sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}$, and team j is inactive ($X_j = 0$ and $P_j^* = 0$).

get a zero payoff (Theorem 1). We have the following result (see Appendix for the proof).

Proposition 3. *Suppose that φ is immune to a successful head-hunting of unemployed workers. Then, $a_{i'} \geq a_i$ for all $i' \in E(\varphi)$ and all $i \in UE(\varphi)$.*

This proposition implies that if we are concerned about stable allocations, then we can focus on the highest $M \times j^*$ ability players. Given that the highest ability $M \times j^*$ players are employed initially, and if a head-hunting of an employed player takes place, a vacant position in the head-hunted team and a newly unemployed player (fired by the head-hunting team) are generated. If players are totally myopic, and head-hunting decisions are made based on this resulting team structure, there are successful head-huntings that are unreasonable. The following casual example illustrates the point.

Example 1. *Suppose that there are three two-person teams (pairs) of players. The common sharing rule is egalitarian so that both members of a team get 50% share. Player 1 is the highest ability one, and player 2 is the second, and so on: player 6 is the lowest ability player. Now, consider an assortative matching of the players: a great team (players 1 and 2), a very good team (players 3 and 4), and a poor team (players 5 and 6). In this case, the great team has the highest winning probability, and the very good team has the second highest winning probability. The poor team has little chance to win. If players are myopic, there is a successful head-hunting from this intuitively very stable ability-sorted team structure. The great team may kick out player 2, and head-hunt player 3. In this case, players 2 and 4 are left alone, and there are effectively only two teams: a semi-great team with players 1 and 3 and a poor team with players 5 and 6. The former team's winning probability jumps up close to one without having a serious rival team. This is a successful head-hunting.*

In the above example, players 2 and 4 were unemployed after head-hunting. However, it is natural to think that these two players form a team in reaction to the head-hunting. Since the newly unemployed worker has the highest ability, it is best for the team with the vacancy to make an offer to the newly unemployed player. Thus, it is natural to assume that when team j head-hunts a player who is currently employed by team k, then team j's fired player is employed by team k. Team j expects that team k would hire the player fired by j, and decide if this head-hunting is profitable.[13] Formally, we say:

Definition 1. *Let φ is a feasible allocation, and assume that $E(\varphi) = \{1, ..., Mj^*\}$ where j^* is the number of active teams (highest Mj^* ability players are employed). Consider swapping players $i = \varphi_{mj}$ and $h = \varphi_{\ell k}$ for $j, k \leq j^*$, and let $\varphi' = (\varphi_{j'})_{j'=1}^M$ be the resulting allocation, where (i) $\varphi'_{j'} = \varphi_{j'}$ for all $j' \neq j, k$, (ii) $\varphi'_j = (\varphi_{1j}, ..., \varphi_{m-1j}, h, \varphi_{m+1j}, ..., \varphi_{Mj})$, and (iii) $\varphi'_k = (\varphi_{1k}, ..., \varphi_{\ell-1k}, i, \varphi_{\ell+1k}, ..., \varphi_{Mk})$. This swapping is a **successful head-hunting of an***

[13] Knuth (1976) asked if a sequence of myopic pairwise deviation processes via swapping would lead to a stable matching in the context of the marriage problem in which all players are acceptable to all players. In marriage problems with externalities, Imamura, Konishi, and Pan (2023) and Imamura and Konishi (2023) discussed the desirability of pairwise deviation via swapping to define pairwise stable matching.

employed player for *j if (a) $P_j(\varphi') > P_j(\varphi)$ and (b) $U_h(\varphi') > U_h(\varphi)$. We say that φ is* ***stable*** *if φ has no successful head-hunting of neither employed nor unemployed players.*

Remark 3. Note that with the above definition of successful head-hunting, head-hunting team j is better off in the Pareto sense except for the former member i who was asked to go. This is because $A(\varphi'_j) > A(\varphi_j)$ implies all team member's payoff goes up (by Theorem 1). Thus, our successful head-hunting implies that the head-hunting team j *unanimously* accepts player h's taking position m. Alternatively, we can define a successful head-hunting by giving priorities to the team leaders' preferences who simply want to maximize their teams' winning probabilities. If a team leader can assign team members to M positions freely, she assigns them to the positions by their abilities in descending order: $a_{\varphi_{1j}} \geq a_{\varphi_{2j}} \geq ... \geq a_{\varphi_{Mj}}$. Starting from any membership profile φ_j, if player h is head-hunted for position m from team k, then she would fire φ_{Mj} instead of φ_{mj} by rearranging players as $\varphi'_{mj} = h$, and $\varphi'_{\tilde{m}j} = \varphi_{\tilde{m}-1j}$ for all $\tilde{m} = m+1, ..., M$. In this case, players $\varphi_{mj}, ..., \varphi_{M-1j}$ may not be better off by player h's joining the team. We can modify our stability concept by using this definition of a successful head-hunting. Our Propositions 4 and 5 are robust to this modification of the definition of stability.[14]

By Proposition 3, if $E(\varphi) = \{1, ..., Mj^*\}$ holds, then there is no successful head-hunting from $UE(\varphi)$, so Definition 1 does not need to consider unemployed players. We will first investigate when swapping $i = \varphi_{mj}$ and $h = \varphi_{\ell k}$ is a successful head-hunting for team j. The following key lemma argues that when two players in two different teams have the same share, and the player in the stronger team has a lower ability, then swapping two players is a successful head-hunting (see Appendix for the proof).

Lemma 3. *Let $i = \varphi_{mj}$ and $h = \varphi_{\ell k}$ with $\theta_{mj} = \theta_{\ell k}$ for $\ell, k \leq j^*$. Suppose that $A_j(\varphi_j) \geq A_k(\varphi_k)$ and $a_i < a_h$. Then, swapping i and h is a successful head-hunting for team j.*

Now, we will consider an **egalitarian sharing rule**: $\theta = (\frac{1}{M}, ..., \frac{1}{M})$: i.e., all positions in a team are the same with equal share. Repeatedly applying Lemma 3, we obtain the following result.

Proposition 4. *Suppose that all teams use the egalitarian sharing rule. Then, under any stable φ, we have a complete ability sorting of players: i.e., by ordering teams by their winning probabilities ($P_1(\varphi) \geq P_2(\varphi) \geq ... \geq P_J(\varphi)$), we have $a_{\varphi_{mj}} \geq a_{\varphi_{\ell j+1}}$ for all $j = 1, ..., \min\{j^*, J-1\}$, and all $m, \ell = 1, ..., M$.*

Another interesting team structure is generated by a **cyclical assignment of players over J teams** is $\varphi = (\varphi_j)_{j=1}^J$ such that $a_{\varphi_{11}} \geq ... \geq a_{\varphi_{1J}} \geq a_{\varphi_{21}} \geq ... \geq a_{\varphi_{2J}} \geq a_{\varphi_{31}} \geq ... \geq a_{\varphi_{3J}} \geq ... \geq a_{\varphi_{M1}} \geq ... \geq a_{\varphi_{MJ}}$. That is, players are ordered by their abilities from the highest to

[14] Since Lemma 3 below holds even if the stability concept is modified, Proposition 4 obviously holds as it is. See the discussion in Appendix. Regarding Proposition 5, the statement does not change, but the bound $\bar{\mu}$ will be reduced by the modification.

the lowest, and the top J players are assigned to position 1 of each team, then next J players are assigned to position 2 of each team, and so on and so forth. This means that team j is composed of players of abilities a_j, a_{j+J}, ..., $a_{j+(M-1)J}$ for all $j = 1, ..., J$. In an interesting coalition formation game, Morelli and Park (2016) showed this allocation to be group-stable. If θ_ms are heterogeneous enough, we can show that a cyclical assignment of players over J teams is a stable team structure. To simplify the exposition, we assume that all J teams are active under the cyclical assignment.

We will consider a special family of θs which satisfies $\theta_{m+1} = \mu\theta_m$ for all $m = 1, ..., M-1$ for $\mu \in [0,1]$. We may call this rule a **hierarchical sharing rule**. Let $\theta : [0,1] \to \Delta^M$ be such that

$$\theta_m(\mu) = \frac{\mu^{m-1}}{1+\mu+...+\mu^{M-1}} = \frac{(1-\mu)\,\mu^{m-1}}{1-\mu^M}$$

for all $m = 1, ..., M$. If $\mu = 0$, $\theta_1 = 1$ with $\theta_m = 0$ for all $m = 2, ..., M$, which is a monopolization rule, while if $\mu = 1$ then it is the egalitarian rule $\theta_m = \frac{1}{M}$ for all $m = 1, ..., M$. The next proposition shows that the hierarchical sharing rule supports the cyclical assignment allocation for μ small enough (see Appendix for the proof).

Proposition 5. *Consider hierarchical sharing rules. There is $\bar{\mu} \in (0,1)$ such that for all $\mu \in [0, \bar{\mu}) > 0$, the cyclical assignment of players over J active teams is stable if $\frac{\theta_{m+1}}{\theta_m} = \mu$ holds for all $m = 1, ..., M-1$.*

Under the egalitarian sharing rule, intra-team inequality is minimized, while inter-team ability sorting occurs. Thus, inequalities of teams' productivities $A_1(\varphi_1), ..., A_J(\varphi_J)$ are highly unequal, resulting in a small number of active teams, and the competitiveness of team contests is limited. In contrast, under the above hierarchical sharing rule, intra-team inequality is high, while the inequality in teams' productivities is kept small due to high ability players being spread over all teams. A large number of active teams can be supported, and the team contest becomes highly competitive.

The following example shows that complete ability-sorting and cyclical assignment allocations can coexist (see Appendix for the details).

Example 2. *Let $\sigma = \frac{1}{2}$ and $V = 1$, and let $a_1 = a_2 = a_3 = a$, $a_4 = a_5 = a_6 = \nu a$, and $a_7 = a_8 = a_9 = \nu^2 a$. With $\mu < 1$, we have $\theta_1 = \frac{1}{1+\mu+\mu^2}$, $\theta_2 = \frac{\mu}{1+\mu+\mu^2}$, and $\theta_3 = \frac{\mu^2}{1+\mu+\mu^2}$. A cyclical assignment allocation is described by $\varphi_1 = (1,4,7)$, $\varphi_2 = (2,5,8)$, and $\varphi_3 = (3,6,9)$, and a complete sorting allocation is described by $\varphi_1 = (1,2,3)$, $\varphi_2 = (4,5,6)$, and $\varphi_3 = (7,8,9)$. When $\mu = 0.9$ and $\nu = 0.8$, both the complete ability sorting and cyclical assignment allocations are stable.*

There can be combinations of these two types of allocations.

Example 3. Let $\sigma = \frac{1}{2}$ and $V = 1$, and let $a_i = a \times (0.9)^{i-1}$ for $i = 1, ..., 6$, $a_i = (0.9)^6$ for $i = 7, 8, 9$, $a_i = (0.9)^7$ for $i = 10, 11, 12$, and $a_i = (0.9)^8$ for $i = 13, 14, 15$. Let $M = 5$,

$\theta_1 = 0.5$, $\theta_2 = 0.2$, and $\theta_3 = \theta_4 = \theta_5 = 0.1$. There is a stable allocation $\varphi_1 = (1, 4, 7, 8, 9)$, $\varphi_2 = (2, 5, 10, 11, 12)$, and $\varphi_3 = (3, 6, 13, 14, 15)$, which is a combination of cyclical assignment and ability sorting allocations. Their winning probabilities are: $P_1 = 0.369$, $P_2 = 0.332$, and $P_3 = 0.299$. This sharing rule assigns hierarchical shares but treats lower ranks equally. The resulting allocation reveals that high ability players are spread over teams while low ability players are ability sorted across teams. This pattern may mimic corporates' worker ability distributions.

Finally, we illustrate how our results can be extended to the case with different categories of positions and different skill types of workers. So far, we assumed that all players belong to the same category and that teams' positions are all symmetric. However, teams may have different categories of positions, and players may have different skill sets.[15] For example, in a baseball team, there are different positions, such as a pitcher, a catcher, infielders, and outfielders, and players have different skill sets that are suitable for these positions. To describe different roles of positions and different types of players in a simple manner, we partition the set of positions M and the player set N into K categories each. Again with an abuse of notations, let the set of categories be $K = \{1, ..., K\}$, and $M \equiv \cup_{k\in K} M_k$ with $M_k \cap M_\ell = \emptyset$ for any $k \neq \ell$, and $N \equiv \cup_{k\in K} N_k$ with $N_k \cap N_\ell = \emptyset$ for any $k \neq \ell$. That is, M_k and N_k represent the set of positions for category k of each team and the set of category k players, respectively. Let team j's assignment function $\varphi_j : M \to N$ be such that for all $k \in K$, all $m \in M_k$, $\varphi_{mj} = i$ implies $i \in N_k$. With φ_j, a modified team j's effort aggregator function can be described by

$$X_j = \left(\sum_{k\in K} b_k \left(\sum_{m\in M_k} a^\sigma_{\varphi_{mj}} e^\sigma_{\varphi_{mj}} \right) \right)^{\frac{1}{\sigma}},$$

where $0 < \sigma < 1$, and $b_k > 0$ is the contribution weight for category k positions' production for all $k \in K$. We also partition vector θ into k categories: $\theta = (\theta_k)_{k\in K}$ with $\theta_k = (\theta_{1k}, ..., \theta_{M_k k})$ for all $k \in K$.

We can obtain counterparts of Propositions 4 and 5 at no cost by applying Proposition 3 within each category. That is, if θ_k is egalitarian, type k players sort out by their abilities, and if θ_k is highly hierarchical then type k players are spread over teams in a cyclical assignment manner. It is because the above production function preserves additive structure in the first parenthesis, and the effect of swapping players in a category is additively separable from other categories.

[15] Imamura, Konishi, and Pan (2023) analyze pair formation problems of two-sided matching problems with externalities such as pairs figure skating and oligopolistic joint ventures. In these problems, the role plays of players are different (say, male and female, and product development and marketing), and they introduce categories of players.

5 Concluding Remarks

When agents decide to join a new team environment, they face a lot of tradeoffs and uncertainties. As members of a new group, they seek the prize collectively, but their positions within the group and their stakes of winning can often be very unequal. Some of the considerations they have to account for include their relative position on the team both in terms of their ability to contribute and their relative compensation, the effort they would have to exert and its effect on their teammates, its effect on the team's probability of success and free-riding incentives, as well as its effect on competing teams' decisions. We have constructed a model of a collective group contest that allows us to account for all these effects while permitting free mobility of players across teams via head-hunting. Our results show that across the resulting stable team structures there is a significant tradeoff between inter-team inequality and intra-team inequality.

If all teams use the egalitarian rule to split the rewards equally, then high-ability players end up being head-hunted by high-ability teams, leading to a very equal distribution of ability within teams but a very unequal distribution of ability across teams. Conversely, if the goal of a contest organizer is to design a very competitive environment with equally matched teams, then a substantial degree of inequality within teams is a necessary prerequisite toward such a goal.

An interesting direction of research could address the question of how stable team allocations would look like when different sharing rules can coexist. Unfortunately, it is very hard to obtain a general existence result of a stable team structure for an arbitrary set of sharing rules. This is due to integer problems and the externalities across teams: even for our positive results, we needed to use head-hunting-proofness via swappings by specifying what happens after a head-hunting (also see Imamura, Pan, and Konishi, 2023, and Imamura and Konishi 2023). An alternative way to avoid this problem is to assume a large number of atomless teams so that head-hunting activities or the entries of new teams with nonexisting sharing rules have no impact on the rest of the teams following the approach introduced by Kaneko and Wooders (1986).[16] In a companion paper, Konishi, Pan, and Simeonov (2023) consider an oligopolistic market with finite player types and production teams formed by a continuum of atomless players when each team has only finite players. This paper proves that there is a free entry equilibrium that is a stable team structure in this idealized large market from which no new team can improve when multiple sharing rules can coexist. Utilizing their result, we can consider the limit of replica problems of our team contest with a continuum of players and uniform prizes. We can formulate subsequent games after teams are formed in the following manner: formed teams are randomly assigned into (a large number of) K-team leagues, in each of which they play a team contest for an identical prize: that is, there will be a continuum of ex ante identical

[16] Assuming finite arbitrary types of atomless players, Kaneko and Wooders (1986) proved the nonemptyness of *f-core*, which is immune to invasions by newly formed team membership profiles with new sharing rules. See Konishi and Simeonov (2023) as well.

team contests, although ex post each contest is played by a set of different teams. Thus, each player's expected payoff is calculated as a weighted expected payoff of each possible draw of team profile in the contests—they are playing contests with a distribution of team types in their league.[17] In the team formation stage, each player decides which position is available for her to take based on her expected payoff comparison, and a potential team manager can enter the market by offering a nonexisting sharing rule in the market if possible. This is so much stronger equilibrium concept, and the resulting allocation is strongly stable. We are planning to explore the properties of this free entry equilibrium in such large replica contests.

Appendix

We collect most proofs here.

Proof of Proposition 2. We compute the equilibrium effort level first. Recalling (2), we obtain

$$\begin{aligned}
e_{\varphi_{mj}} &= X_j \left[(1-P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}} \left(a^{\sigma}_{\varphi_{mj}}\theta_m\right)^{\frac{1}{1-\sigma}} \\
&= \left\{1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\}\frac{(J-1)\,V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\left[\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\frac{V}{\frac{(J-1)V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}}\right]^{\frac{1}{1-\sigma}}\left(a^{\sigma}_{\varphi_{mj}}\theta_m\right)^{\frac{1}{1-\sigma}} \\
&= \left\{1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\}\frac{(J-1)\,V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\left[\frac{1}{A_j(\varphi_j)}\right]^{\frac{1}{1-\sigma}}\left(a^{\sigma}_{\varphi_{mj}}\theta_m\right)^{\frac{1}{1-\sigma}}
\end{aligned}$$

This implies that player i's payoff is written as

$$\begin{aligned}
U_{\varphi_{mj}} &= P_j\theta_m V - e_{\varphi_{mj}} \\
&= \left\{1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\}\theta_m V - \left\{1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\}\frac{(J-1)\,V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\left[\frac{1}{A_j(\varphi_j)}\right]^{\frac{1}{1-\sigma}}\left(a^{\sigma}_{\varphi_{mj}}\theta_m\right)^{\frac{1}{1-\sigma}} \\
&= \left\{1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\}\left[\theta_m V - \frac{(J-1)\,V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\left[\frac{1}{A_j(\varphi_j)}\right]^{\frac{1}{1-\sigma}}\left(a^{\sigma}_{\varphi_{mj}}\theta_m\right)^{\frac{1}{1-\sigma}}\right] \\
&= \left\{1-\frac{(J-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\}\left[\theta_m V - \frac{(J-1)\,\frac{1}{A_j(\varphi_j)}V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\left[\frac{1}{A_j(\varphi_j)}\right]^{\frac{\sigma}{1-\sigma}}\theta_m\left(a^{\sigma}_{\varphi_{mj}}\theta^{\sigma}_m\right)^{\frac{1}{1-\sigma}}\right] \\
&= V\times\theta_m\left[1-\frac{(j^*-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*}\frac{1}{A_k(\varphi_k)}}\right]\left[1-\frac{(j^*-1)\,\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*}\frac{1}{A_k(\varphi_k)}}\left(\frac{a_i^{\frac{\sigma}{1-\sigma}}\theta_m^{\frac{\sigma}{1-\sigma}}}{\sum_{m'=1}^{M}a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}}\theta_{m'}^{\frac{\sigma}{1-\sigma}}}\right)\right]
\end{aligned}$$

We completed the proof.■

[17] Thus, each team's winning probability is affected by the distribution of other team types, and we need widespread externalities in Konishi, Pan, and Simeonov (2023).

Next, we present a useful lemma for the following proofs.

Lemma A1. *Suppose that team $j = 1, 2, ..., j^*$ are active, and team $j = j^* + 1, ..., J$ are inactive under assignment φ. Let $k \geq j^* + 1$ and $h \leq j^*$, then we have*

$$\frac{j^* - 2}{\sum_{j' \neq k,h}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^* - 1}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^*}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_k(\varphi_k)}}.$$

Proof of Lemma A1. By Theorem 1, since team k is inactive we have

$$\frac{j^* - 1}{A_k(\varphi_k)} \geq \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}.$$

Then

$$\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_k(\varphi_k)} \leq \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{j^* - 1} \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} = \frac{j^*}{j^* - 1} \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}.$$

Rearranging the above inequality yields $\frac{j^*-1}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^*}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_k(\varphi_k)}}$. The remaining part can be proved in a similar way.■

Proof of Proposition 3. Suppose not. Then, there are $\varphi_{mj} \in E(\varphi)$ and $i \in UE(\varphi)$ such that $a_{\varphi_{mj}} < a_i$. Let the number of active teams under φ be j^*. From Proposition 1, we know $P_j = 1 - \frac{(j^*-1)\frac{1}{A_j(\theta_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\theta_k)}}$ and $A_j(\varphi_j) = \left(\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'j}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$. Thus, if team j replaces φ_{mj} by i, the new membership profile denoted by φ' yields a higher productivity $A_j(\varphi'_j) = \left(\sum_{m'=1}^{M} a_{\varphi'_{m'j}}^{\frac{\sigma}{1-\sigma}} \theta_{m'j}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} > A_j(\varphi_j)$. If the number of active teams does not change by this swapping, team j's winning probability increases since $A_j(\varphi'_j) > A_j(\varphi_j)$. Moreover, player i obtains a positive payoff by being employed, so she has an incentive to join. This is a contradiction with φ's being stable.

So far, we have considered the case where j^* does not change. We check if this consideration changes the above result. From Theorem 1, we know that team j^*+1 is inactive if $\frac{1}{A_{j^*+1}(\varphi_{j^*+1})} \geq \frac{1}{j^*-1}\left[\sum_{k=1,k\neq j}^{j^*} \frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j)}\right]$. Since this swapping of players i and φ_{mj} yields $A_j(\varphi'_j) > A_j(\varphi_j)$, team $j^* + 1$ will still be inactive after the swapping. Thus, the number of active teams does not increase by the swapping. It is possible for the number of active teams to decrease. Let P'_j stand for team j's winning probability under φ'. Suppose that team j^* becomes inactive after the swapping and observe that

$$\begin{aligned}
P_j' &= 1 - \frac{(j^*-2)\frac{1}{A_j(\varphi_j')}}{\sum_{k=1,k\neq j}^{j^*-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j')}} \\
&> 1 - \frac{(j^*-2)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1,k\neq j}^{j^*-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j)}} \\
&> 1 - \frac{(j^*-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1,k\neq j}^{j^*-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_{j^*}(\varphi_{j^*})} + \frac{1}{A_j(\varphi_j)}}.
\end{aligned}$$

The last inequity holds because of $\frac{1}{A_{j^*}(\varphi_{j^*})} < \frac{1}{j^*-2}\left[\sum_{k=1,k\neq j}^{j^*-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j)}\right]$ (team j^* is active under φ) and Lemma A1.■

Proof of Lemma 3. First, we consider the case where the swapping does not change the number of active teams j^*. Note that $A_j(\varphi_j) \geq A_k(\varphi_k)$ means $\tilde{A}_j(\varphi_j) = \sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}}\theta_{m'j}^{\frac{\sigma}{1-\sigma}} \geq \tilde{A}_k(\varphi_k) = \sum_{m'=1}^{M} a_{\varphi_{m'k}}^{\frac{\sigma}{1-\sigma}}\theta_{m'k}^{\frac{\sigma}{1-\sigma}}$. Let $\theta = \theta_{mj} = \theta_{\ell k}$, and let $\tilde{a}_i \equiv a_i^{\frac{\sigma}{1-\sigma}}\theta_{\ell k}^{\frac{\sigma}{1-\sigma}} = a_i^{\frac{\sigma}{1-\sigma}}\theta^{\frac{\sigma}{1-\sigma}}$ and $\tilde{a}_h = a_h^{\frac{\sigma}{1-\sigma}}\theta_{mj}^{\frac{\sigma}{1-\sigma}} = a_h^{\frac{\sigma}{1-\sigma}}\theta^{\frac{\sigma}{1-\sigma}}$. Clearly, $\tilde{a}_h > \tilde{a}_i$ holds, since $a_h > a_i$. The immediate consequence of this is $\tilde{A}_j(\varphi_j') > \tilde{A}_j(\varphi_j) \geq \tilde{A}_k(\varphi_k) > \tilde{A}_k(\varphi_k')$, which implies $A_j(\varphi_j') > A_j(\varphi_j) \geq A_k(\varphi_k) > A_k(\varphi_k')$. Note that

$$\frac{\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{\frac{1}{A_j(\varphi_j')}}{\sum_{s\neq j,k}\frac{1}{A_s(\varphi_s)} + \frac{1}{A_j(\varphi_j')} + \frac{1}{A_k(\varphi_k)}} \geq \frac{\frac{1}{A_j(\varphi_j')}}{\sum_{s\neq j,k}\frac{1}{A_s(\varphi_s)} + \frac{1}{A_j(\varphi_j')} + \frac{1}{A_k(\varphi_k')}}.$$

The first inequality is due to $\frac{1}{A_j(\varphi_j')} < \frac{1}{A_j(\varphi_j)}$, and the second one is due to $\frac{1}{A_k(\varphi_k')} > \frac{1}{A_k(\varphi_k)}$. Since $P_j(\varphi) = 1 - \frac{(j^*-1)\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}}$, we conclude $P_j(\varphi') > P_j(\varphi)$. So, condition (a) for a successful head-hunting is satisfied. Now, let's move on to (b). From Theorem 1, we know

$$U_h(\varphi) = V \times \theta P_k(\varphi)\left[1 - \frac{(j^*-1)\frac{1}{A_k(\varphi_k)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}}\left(\frac{a_h\theta}{A_k(\varphi_k)}\right)^{\frac{\sigma}{1-\sigma}}\right]$$

and

$$U_h(\varphi') = V \times \theta P_j(\varphi')\left[1 - \frac{(j^*-1)\frac{1}{A_j(\varphi_j')}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'}')}}\left(\frac{a_h\theta}{A_j(\varphi_j')}\right)^{\frac{\sigma}{1-\sigma}}\right].$$

Since $P_j(\varphi') > P_j(\varphi) \geq P_k(\varphi)$, $\frac{\frac{1}{A_k(\varphi_k)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}} > \frac{\frac{1}{A_j(\varphi_j')}}{\sum_{s\neq j,k}\frac{1}{A_s(\varphi_s)}+\frac{1}{A_j(\varphi_j')}+\frac{1}{A_k(\varphi_k')}} = \frac{\frac{1}{A_j(\varphi_j')}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'}')}}$, and $A_j(\varphi_j') > A_k(\varphi_k)$, we conclude $U_h(\varphi') > U_h(\varphi)$. Thus, (b) is satisfied, too.

Second, we consider the case where an inactive team, team j^*+1, is activated by the swapping. By Lemma A1, we have $(j^*-1)\frac{1}{A_{j^*+1}(\varphi_{j^*+1})} \geq \sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}$ and $(j^*-1)\frac{1}{A_{j^*+1}(\varphi_{j^*+1})} <$

$\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi'_{j'})}$. These two inequalities imply that

$$\frac{(j^*-1)\frac{1}{A_k(\varphi_k)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{(j^*-1)\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^*\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{j^*}\frac{1}{A_{j'}(\varphi_{j'})}+\frac{1}{A_{j^*+1}(\varphi_{j^*+1})}}$$
$$> \frac{j^*\frac{1}{A_j(\varphi'_j)}}{\sum_{j'\neq j,k}\frac{1}{A_{j'}(\varphi_{j'})}+\frac{1}{A_j(\varphi'_j)}+\frac{1}{A_k(\varphi'_k)}+\frac{1}{A_{j^*+1}(\varphi_{j^*+1})}}$$

For the same reason as the previous case, conditions (a) and (b) are satisfied.

Third, it is also possible for the number of active teams to decrease, but it simply increases P_j, and team j benefits from that. Thus, the conclusion does not change. Hence, we conclude $U_h(\varphi') > U_h(\varphi)$.[18]■

Proof of Proposition 5. Given $\theta(\mu) = (\theta_1(\mu), ..., \theta_M(\mu))$, we will check the stability of a cyclical assignment of players over J teams. Let's assign players over teams from the highest ability to the lowest cyclically and denote the resulting assignment as φ. For notational simplicity, we drop μ from $\theta(\mu)$. First, let $T_m = \{(m-1)J + j' | j' = 1, ..., J\}$ be the set of players occupying position m for each team in a cyclical assignment. Note that any player in T_m will not be head-hunted for a higher position. Therefore, we may focus on player $\varphi_{mj} = i = (m-1)J + j \in T_m$ who belongs to team j and considers taking position $m+1$ of another team $k \neq j$. By Theorem 1 and Remark 2, we have

$$\begin{aligned} U_i(\varphi;\mu) &> \theta_m \left(P_j(\varphi)\right)^2 \\ &= \theta_m \left(1-(J-1)\frac{\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi_{j'})}}\right)^2 \end{aligned}$$

If player i moves take position $m+1$ of team s, players i and $\varphi_{(m+1)k} = mJ + k$ are swapped. Denote the new assignment as φ', and player i's payoff at position $m+1$ of team k is

$$\begin{aligned} U_i(\varphi';\mu) &= \theta_{m+1}P_k(\varphi')\left[1-(J-1)\frac{\frac{1}{A_k(\varphi'_k)}}{\sum_{j'=1}^{J}\frac{1}{A_{j'}(\varphi'_{j'})}}\left(\frac{a_i\theta_{m+1}}{A_k(\varphi'_k)}\right)^{\frac{\sigma}{1-\sigma}}\right] \\ &< \theta_{m+1}P_k(\varphi') \\ &= \theta_{m+1}\left(1-(J-1)\frac{\frac{1}{A_k(\varphi'_k)}}{\sum_{j'\neq k,j}\frac{1}{A_{j'}(\varphi_{j'})}+\frac{1}{A_k(\varphi'_k)}+\frac{1}{A_j(\varphi'_j)}}\right) \end{aligned}$$

where $A_k(\varphi'_k)$ and $A_j(\varphi'_j)$ are the resulting productivity after swapping. Let $\bar{\nu} = \min_{m=1,...,M-1} \frac{a_{(m+1)J}}{a_{(m-1)J+1}} \leq 1$, which is the widest player ability difference ratio of two adjacent ranks. This implies that $a_{mJ+k} \in [\bar{\nu}a_i, a_i]$.

[18] By the above proof, we only need to show that $\tilde{A}_j(\varphi'_j) > \tilde{A}_j(\varphi_j) \geq \tilde{A}_k(\varphi_k) > \tilde{A}_k(\varphi'_k)$ after the swap, which is immediate from the alternative definition of a successful head-hunting in Remark 3. Therefore, Lemma 3 holds under the alternative definition, too.

Since $A_j(\varphi_j) = \left(\sum_{\ell=1}^{M} a_{j+(\ell-1)J}^{\frac{\sigma}{1-\sigma}} \theta_\ell^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$, $A_k(\varphi_k') < \left(\left(\frac{1}{\bar{\nu}} a_{mJ+k}\right)^{\frac{\sigma}{1-\sigma}} \theta_{m+1}^{\frac{\sigma}{1-\sigma}} + \sum_{\ell \neq m+1}^{M} a_{k+(\ell-1)J}^{\frac{\sigma}{1-\sigma}} \theta_\ell^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} < \frac{1}{\bar{\nu}} A_k(\varphi_k)$. Similarly, $A_j(\varphi_j') > \left(\bar{\nu} a_i + \sum_{\ell \neq m}^{M} a_{j+(\ell-1)J}^{\frac{\sigma}{1-\sigma}} \theta_\ell^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} > \bar{\nu} A_j(\varphi_j)$ hold. Thus, we have

$$\begin{aligned}
U_i(\varphi';\mu) &< \theta_{m+1}\left(1-(J-1)\frac{\frac{1}{A_k(\varphi_k')}}{\sum_{j'\neq k,j}\frac{1}{A_{j'}(\varphi_{j'})}+\frac{1}{A_k(\varphi_k')}+\frac{1}{A_j(\varphi_j')}}\right) \\
&< \theta_{m+1}\left(1-(J-1)\frac{\frac{\bar{\nu}}{A_k(\varphi_k)}}{\sum_{j'\neq k,j}\frac{1}{A_k(\varphi_k)}+\frac{1}{A_k(\varphi_k)}+\frac{1}{\bar{\nu}A_j(\varphi_j)}}\right) \\
&< \theta_{m+1}\left(1-(J-1)\frac{\frac{\bar{\nu}^2}{A_k(\varphi_k)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right) < \frac{1}{\bar{\nu}^2}\theta_{m+1}P_k(\varphi)
\end{aligned}$$

and

$$U_i(\varphi;\mu) > \theta_m\left(1-(J-1)\frac{\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right)^2 = \theta_m\left(P_j(\varphi)\right)^2$$

Hence, $U_i(\varphi;\mu) \geq U_i(\varphi';\mu)$ holds if

$$\frac{1}{\bar{\nu}^2}\theta_{m+1}P_k(\varphi) \leq \theta_m\left(P_j(\varphi)\right)^2$$

or

$$\mu = \frac{\theta_{m+1}}{\theta_m} \leq \bar{\nu}^2\frac{\left(P_j(\varphi)\right)^2}{P_k(\varphi)}.$$

Note that the RHS of the inequality above is also a function of μ. Let $\psi_{kj}(\mu) \equiv \bar{\nu}^2\frac{(P_j(\varphi))^2}{P_k(\varphi)} - \mu$. We have $\psi_{kj}(0) > 0$. If $\psi_{kj}(\mu) > 0$ for all $\mu \in [0,1]$, let $\bar{\mu}_{kj} = 1$. If $\psi_{kj}(1) < 0$, then there is a solution $\psi_{kj}(\mu) = 0$ in interval $[0,1]$ since $U_i(\varphi;\mu)$ and $U_i(\varphi';\mu)$ are continuous. Let $\bar{\mu}_{kj}$ be the smallest of these solutions, and let $\bar{\mu} \equiv \min_{k,j=1,\ldots,J}\bar{\mu}_{kj}$. Then, for all $\mu \in [0,\bar{\mu}]$, $U_i(\varphi';\mu) \geq U_i(\varphi';\mu)$ holds for all players i. Hence, the cyclical assignment over J teams is stable for $\mu < \bar{\mu}$.■

Calculations for Example 2. Let's start with the complete sorting allocation. We need to check if player 3 in team 1's position 3 is head-hunted by team 2 for the top position. Under complete ability sorting, we have

$$\begin{aligned}
A_1(\theta) &= a(\theta_1+\theta_2+\theta_3) = a \\
A_2(\theta) &= \nu a \\
A_3(\theta) &= \nu^2 a
\end{aligned}$$

Thus, if all teams are active, we have

$$P_1 = 1 - \frac{2\times\frac{1}{A_1}}{\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}} = 1 - \frac{2}{1+\frac{1}{\nu}+\frac{1}{\nu^2}} = 1 - \frac{2\nu^2}{1+\nu+\nu^2}$$

and

$$u_3^{sort} = \frac{\mu^2}{1+\mu+\mu^2}\left(1-\frac{2\nu^2}{1+\nu+\nu^2}\right)\left(1-\frac{2\mu^2\nu^2}{(1+\nu+\nu^2)(1+\mu+\mu^2)}\right)$$

If player 3 moves to team 2, we have

$$\begin{aligned}
A_1' &= \left(\frac{1}{1+\nu+\nu^2}+\frac{\mu}{1+\nu+\nu^2}+\frac{\nu\mu^2}{1+\nu+\nu^2}\right)a \\
A_2' &= \left(\frac{1}{1+\nu+\nu^2}+\frac{\nu\mu}{1+\nu+\nu^2}+\frac{\nu\mu^2}{1+\nu+\nu^2}\right)a \\
A_3' &= v^2 a
\end{aligned}$$

Then, we have

$$P_2' = 1-\frac{2\times\frac{1}{A_1'}}{\frac{1}{A_1'}+\frac{1}{A_2'}+\frac{1}{A_3'}} = 1-\frac{2\frac{1}{1+\nu\mu+\nu\mu^2}}{\frac{1}{1+\mu+\nu\mu^2}+\frac{1}{1+\nu\mu+\nu\mu^2}+\frac{1}{\nu^2(1+\mu+\mu^2)}}$$

and

$$u_3' = \frac{1}{1+\mu+\mu^2}\left(1-\frac{2\frac{1}{1+\nu\mu+\nu\mu^2}}{\frac{1}{1+\mu+\nu\mu^2}+\frac{1}{1+\nu\mu+\nu\mu^2}+\frac{1}{\nu^2(1+\mu+\mu^2)}}\right)\left(1-\frac{2\left(\frac{1}{1+\nu\mu+\nu\mu^2}\right)^2}{\frac{1}{1+\mu+\nu\mu^2}+\frac{1}{1+\nu\mu+\nu\mu^2}+\frac{1}{\nu^2(1+\mu+\mu^2)}}\right)$$

The complete sorting is stable if $u_3^{sort} \geq u_3'$:

$$\begin{aligned}
&\mu^2\left(1-\frac{2\nu^2}{1+\nu+\nu^2}\right)\left(1-\frac{2\mu^2\nu^2}{(1+\nu+\nu^2)(1+\mu+\mu^2)}\right) \\
\geq\ &\left(1-\frac{2\frac{1}{1+\nu\mu+\nu\mu^2}}{\frac{1}{1+\mu+\nu\mu^2}+\frac{1}{1+\nu\mu+\nu\mu^2}+\frac{1}{\nu^2(1+\mu+\mu^2)}}\right)\left(1-\frac{2\left(\frac{1}{1+\nu\mu+\nu\mu^2}\right)^2}{\frac{1}{1+\mu+\nu\mu^2}+\frac{1}{1+\nu\mu+\nu\mu^2}+\frac{1}{\nu^2(1+\mu+\mu^2)}}\right)
\end{aligned}$$

Second, we consider the cyclic assignment. In this case, as long as player 2 in team 2 does not want to join team 1 to work with player 1, it is head-hunting-proof. Player 2's payoff in team 2 is

$$u_2^{cyc} = \frac{1}{1+\mu+\mu^2}\times\frac{1}{3}\left(1-\frac{1}{3}\frac{1}{1+\mu\nu+\mu^2\nu^2}\right)$$

If she moves to team 1, her payoff is

$$\begin{aligned}
u_2' &= \frac{\mu}{1+\mu+\mu^2}\left(1-\frac{2\frac{1}{1+\mu+\mu^2\nu^2}}{\frac{1}{1+\mu+\mu^2\nu^2}+\frac{1}{1+\mu\nu+\mu^2\nu^2}+\frac{1}{\nu+\mu\nu+\mu^2\nu^2}}\right) \\
&\quad\times\left(1-\frac{2\frac{1}{1+\mu+\mu^2\nu^2}}{\frac{1}{1+\mu+\mu^2\nu^2}+\frac{1}{1+\mu\nu+\mu^2\nu^2}+\frac{1}{\nu+\mu\nu+\mu^2\nu^2}}\frac{\mu}{1+\mu+\mu^2 v^2}\right)
\end{aligned}$$

We need $u_2^{cyc} \geq u_2'$ for the cyclical assignment allocation to be stable:

$$\begin{aligned} & \frac{1}{3}\left(1-\frac{1}{3}\frac{1}{1+\mu\nu+\mu^2\nu^2}\right) \\ \geq\ & \mu\left(1-\frac{2\frac{1}{1+\mu+\mu^2\nu^2}}{\frac{1}{1+\mu+\mu^2\nu^2}+\frac{1}{1+\mu\nu+\mu^2\nu^2}+\frac{1}{\nu+\mu\nu+\mu^2\nu^2}}\right) \\ & \times\left(1-\frac{2\frac{1}{1+\mu+\mu^2\nu^2}}{\frac{1}{1+\mu+\mu^2\nu^2}+\frac{1}{1+\mu\nu+\mu^2\nu^2}+\frac{1}{\nu+\mu\nu+\mu^2\nu^2}}\frac{\mu}{1+\mu+\mu^2 v^2}\right) \end{aligned}$$

We will show that both conditions are satisfied under $\mu = 0.9$ and $\nu = 0.8$. Player 3 in the complete ability sorting allocation is $u_3^{sort} = 0.3247$, while if she takes the highest position in team 2, she obtains $u_3' = 0.29222$. Thus, she does not have an incentive to move, and the complete ability sorting allocation is stable. Now, the highest ability player under a cyclical assignment allocation is $u_2^{cyc} = 0.28369$, while if she moves to a second-ranked position, she gets $u_2' = 0.26949$. Thus, she has no incentive to deviate.■

References

[1] Baik, K.H., and S. Lee (1997): Collective Rent Seeking with Endogenous Group Sizes, European Journal of Political Economy 13, pp. 121-130.

[2] Baik, K.H., and S. Lee (2001): Strategic Groups and Rent Dissipation, Economic Inquiry 39, pp. 672-684.

[3] Banerjee, S., H. Konishi, and T. Sonmez (2001), Simple Coalition Formation Games, Social Choice and Welfare 18, 135-153.

[4] Bloch, F. (1995), "Endogenous Structures of Association in Oligopolies," The RAND Journal of Economics 26, 537-556.

[5] Bloch, F. (1996), "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division, Games and Economic Behavior 14, 90-123.

[6] Bloch, F. (1997), "Non-Cooperative Models of Coalition Formation in Games with Spillovers," C. Carraro, and D. Siniscalco in, eds., New Directions in the Economic Theory of the Environment, 311-52, Cambridge: Cambridge University Press.

[7] Bloch, F., S. Sanchez-Pages, and R. Soubeyran (2006), When Does Universal Peace Prevails? Secession and Group Formation in Contests, Economics of Governance 7, 3-29.

[8] Bogomolnaia, A., and M.O. Jackson (2002), The Stability of Hedonic Coalition Structures, Games and Economic Behavior 38, 201-230.

[9] Chade, H., and J. Eeckhout (2020), Competing Teams. Review of Economic Studies 87.3, 1134-1173.

[10] Chwe, M.S.Y. (1994), Farsighted Coalitional Stability, Journal of Economic Theory 63, 299–325.

[11] Cornes, R. and R. Hartley (2005), Asymmetric Contests with General Technologies, Economic Theory 26-4, 923 – 946.

[12] Esteban, J., and D. Ray (2001), Collective Action and the Group Size Paradox, American Political Science Review 95-3, 663 - 672.

[13] Esteban, J. and J. Sakovics (2003): Olson v.s. Coase: Coalitional Worth in Conflict, Theory and Decisions , 55, 339-357.

[14] Fu, Q., and Z. Wu (2019): Contests: Theory and Topics. A survey on theoretical studies of contests as an invited contribution to Oxford Research Encyclopedia of Economics.

[15] Gale, D. and L.S. Shapley (1962), College Admissions and the Stability of Marriage, American Mathematical Monthly 69, 9-15.

[16] Hart, S. and M. Kurz (1983), Endogenous Formation of Coalitions, Econometrica 51-4, 1047-1064.

[17] Imamura, K., and H. Konishi (2023), Assortative Matching with Externalities and Farsighted Players, *Dynamic Games and Applications* 13 (special issue on Group Formation and Farsightedness), 497-509.

[18] Imamura, K., H. Konishi, and C.-Y. Pan (2023), Stability in Matching with Externalities: Pairs Figure Skating and Oligopolistic Joint Ventures, *Journal of Economic Behavior and Organization* 205, 270-286.

[19] Kaneko, M., and M.H. Wooders (1986), The core of a game with a continuum of players and finite coalitions: The model and some results, Mathematical Social Sciences 2, 105-137.

[20] Knuth, D.E., (1976), *Marriages Stables*, Montreal: Les Presses de l'Universite de Montreal.

[21] Kobayashi, K., H. Konishi, and K. Ueda (2021): Share Function Approach to Joint Production Problems (draft).

[22] Kolmar, M. and Rommeswinkel, H. (2013), Contests with group-specific public-goods and complementarities in Efforts, Journal of Economic Behavior and Organization 89: 9-22.

[23] Konishi, H., and C.-Y. Pan (2020): Sequential Formation of Alliances in Survival Contests, International Journal of Economic Theory 16, 95-105.

[24] Konishi, H. and C.-Y. Pan (2021), Endogenous Alliances in Survival Contests, Journal of Economic Behavior and Organization 189, 337-358.

[25] Konishi, H., C.-Y. Pan, D. Simeonov (2023), Competing Teams in a Large Market: Free Entry Equilibrium with (Sub-)Optimal Contracts, Working Paper.

[26] Konishi, H., and D. Simeonov (2023), Nonemptiness of the f-Core Without Comprehensiveness, Working Paper.

[27] Konrad, K.A., (2009): Strategy and Dynamics in Contests. Oxford, UK: Oxford University Press.

[28] Lee, S. (1995): Endogenous Sharing Rules in Collective-Group Rent-Seeking, Public Choice 85, 31-44.

[29] Morelli, M., and I.-U. Park (2016), Internal Hierarchy and Stable Coalition Structures, Games and Economic Behavior 96: 90-96.

[30] Nitzan, S. (1991), Collective Rent Dissipation, Economic Journal 101, 1522-1534.

[31] Nitzan, S. and Ueda, K. (2011), "Prize sharing in collective contests," European Economic Review 55: 678-687.

[32] Olson, M. (1965), The logic of collective action, Harvard University Press, Cambridge, MA, US.

[33] Ray, D., and R. Vohra, (1999), "A Theory of Endogenous Coalition Structures," Games and Economic Behavior 26, 286-336.

[34] Ray, D., and R. Vohra, (2001), "Coalitional Power and Public Goods," Journal of Political Economy 109, 1355-1384.

[35] Ray, D. (2008), A Game-Theoretic Perspective on Coalition Formation, Oxford University Press, Oxford.

[36] Ray, D., and R. Vohra (2014), "Coalition Formation," Handbook of Game Theory vol. 4, pp. 239-326.

[37] Rosenthal, R.W., (1972), Cooperative games in effectiveness form, Journal of Economic Theory 5, 88-101.

[38] Sanchez-Pages, S. (2007a), Endogenous Coalition Formation in Contests, Review of Economic Design 11, 139-63.

[39] Sanchez-Pages, S. (2007b), Rivalry, Exclusion, and Coalitions, Journal of Public Economic Theory 9, 809-30.

[40] Simeonov, D., “Essays in Contest Theory” Ph.D. thesis, Boston College, August 2020.

[41] Tullock, G., (1980), “Efficient Rent Seeking,” Buchanan, J.M., Tollison, R.D., Tullock, G. (Eds.), Toward a Theory of the Rent-Seeking Society, Texas A&M University Press, College Station, TX, pp. 97-112.

[42] Ueda, K. (2002): Oligopolization in Collective Rent-Seeking, Social Choice and Welfare, 19, 613-626.

[43] Yi, S.-S. (1997), Stable Coalition Structures with Externalities, Games and Economic Behavior, 20(2), pp.201-237.