$\forall$ or $\exists$?

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Abstract

This paper shows that in some axioms regarding the mixture of random variables, the requirement that the conclusions hold for all values of the mixture parameter can be replaced by requiring the existence of only one non-trivial value of the parameter, which needs not be fixed. This is the case for the independence, betweenness, and the mixture symmetry axioms.

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1 Introduction

Typical mixture axioms for preferences over random variables state that "For all random variables and for all values of a mixing parameter, if some preferences hold, then other preferences hold as well." For example, the betweenness axiom (Chew [1], Dekel [3]) states that for all random variables F and G and for all $\alpha \in [0,1]$, if $F \sim G$ then $F \sim \alpha F + (1-\alpha)G$. Obviously, it is very easy to refute the behavioral validity of such axioms — all one needs is one value of the parameter for which they are violated.[1] But what

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[1] For a detailed discussion of the behavioral foundations of decision models and further references, see Wakker [8].

happens if "for all values of a mixing parameter" is replaced with "for some values," or even with "there exists a value?" The first result of this paper shows that with the standard assumptions of completeness, transitivity, and continuity, the independence and the betweenness axioms are satisfied for all α if for each set of relevant distributions there is one non-trivial α for which these axioms are satisfied. The bulk of the paper is devoted to the proof of a similar result for the mixture symmetry axiom.

Quadratic utility, one of the early alternatives to expected utility theory, was suggested by Machina [7, fn. 45]. It was axiomatized by Chew, Epstein, and Segal [2] (henceforth CES) and was extended to social choice theory by Epstein and Segal [4]. The key axiom in CES is strong mixture symmetry: If $F \sim G$, then for all $\alpha \in [0,1]$, $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$. If the decision maker is indifferent between F and G and decides which of them to play by flipping a biased coin, then indifference follows between the option of playing F if Heads, G if Tails, and the option of playing G is Heads, F if Tails. Moreover, this holds for any biased coin. Theorem 4 in CES states that if, in addition, preferences are either quasi-concave or quasi-convex, then they can be represented by a quadratic functional. The current paper replaces both requirements with weaker axioms. Instead of quasi-concavity (or quasi-convexity) I require only that preferences along chords have a single extreme, and the "for all α" in the strong mixture symmetry axiom is replaced with "there exists α." Moreover, this α may vary from one pair of lotteries to another. It turns out that these weaker assumptions still imply the quadratic representation.

The claims of this paper show that instead of assuming betweenness, independence, or mixture symmetry for *all* values of α, it is enough to assume the *existence* of such a value. Obviously, such axioms can never be directly contradicted by experimental methods. The theories predicted by these axioms can be checked in experiments, but in order to obtain these theories more axioms are needed, usually completeness, transitivity, and continuity. As many suspect, it may well be that these three axioms (and especially the first two) are violated by most of the experimental results.

2 Betweenness and Independence

Let $\mathcal{F}$ be the set of distributions over $[0, a]$, $a \in (0, \infty]$. Consider a complete and transitive preference relation $\succeq$, satisfying continuity and monotonicity with respect to first-order stochastic dominance, and let $V : \mathcal{F} \to \Re$ represent it. For $F, G \in \mathcal{F}$, let $[F, G] = \{\alpha F + (1-\alpha)G : \alpha \in [0, 1]\}$ and $(F, G) = \{\alpha F + (1-\alpha)G : \alpha \in (0, 1)\}$. For $F \neq G$, the line through F and G is the set $L_{F,G} = \{H : F \in [H, G] \text{ or } G \in [H, F]\}$.

I/WI: Independence/Weak Independence For all $F, G, H \in \mathcal{F}$, if $F \sim G$ then for all [for at least one value of] $\alpha \in (0, 1)$, $\alpha F + (1-\alpha)H \sim \alpha G + (1-\alpha)H$.

B/WB: Betweenness/Weak Betweenness For all $F, G \in \mathcal{F}$, if $F \sim G$ then for all [for at least one value of] $\alpha \in (0, 1)$, $\alpha F + (1-\alpha)G \sim F$.

It is of course well known that the requirement "for all" in these two axioms can be replaced with "for $\alpha = \frac{1}{2}$," or in fact, for any fixed value of α (see Herstein and Milnor [6]). The weak versions discussed here only require the existence of one value of α, but this value may depend on the underlying distributions.

Theorem 1. Assuming continuity, **WB** implies **B** and **WI** implies **I**.

Proof: **WB** implies **B**: Using a method introduced by Hardy, Littlewood, and Pòlya [5, Observation 88 in §3.7],[2] suppose that there are F, G and α_0 such that $\alpha_0 F + (1-\alpha_0)G \nsim F$. Let $\alpha^* = \sup_\alpha\{\alpha < \alpha_0 : \alpha F + (1-\alpha)G \sim F\}$ and $\alpha_* = \inf_\alpha\{\alpha > \alpha_0 : \alpha F + (1-\alpha)G \sim F\}$. By continuity, $F^* := \alpha^* F + (1-\alpha^*)G \sim F_* := \alpha_* F + (1-\alpha_*)G \sim F$, hence by **WB** there is $\alpha \in (0, 1)$ such that $\beta F^* + (1-\beta)F_* \sim F^* \sim F$, a contradiction.

WI implies **I**: Let $H = G$ in the definition of **WI** to obtain that it implies **WB**, hence **B**. Let $F \sim G$ and consider their mixtures with an arbitrary H. If $F \sim H \sim G$, then by **B**, for all $\alpha \in (0, 1)$, $\alpha F + (1-\alpha)H \sim H \sim \alpha G + (1-\alpha)H$. Suppose wlg that $F \sim G \succ H$. Again by **B**, for all α and $D = F, G$, $D \succeq \alpha F + (1-\alpha)H \succeq H$. Otherwise, if for example, for some $\alpha' \in (0, 1)$, $F' := \alpha' F + (1-\alpha')G \succ F$, then by continuity there is $\alpha'' \in (0, \alpha')$

[2] I am grateful to Peter Wakker for this reference.

such that $F'' := \alpha'' F + (1-\alpha'')G \sim F$, a violation of **B**, as $F' \in [F, F'']$. It follows therefore by continuity that for all $\alpha \in (0,1)$ there is $\beta \in (0,1)$ such that $\alpha F + (1-\alpha)H \sim \beta G + (1-\beta)H$.

Let $F \sim G \succ H$. By **WI**, there is an decreasing sequence α_n such that $\alpha_n F + (1-\alpha_n)H \sim \alpha_n G + (1-\alpha_n)H$. Let $\bar{\alpha} = \lim_{n\to\infty} \alpha_n$ (it exists as $\{\alpha_n\}$ is a decreasing and bounded sequence). By **WI**, there is $\alpha < \bar{\alpha}$ such that $\alpha F + (1-\alpha)H \sim \alpha G + (1-\alpha)H$. Choose therefore a sequence such that $\bar{\alpha} = 0$.

Suppose now that for a certain $\tilde{\alpha} \in (0,1)$ there is $\tilde{\beta} \in (0,1)$, $\tilde{\beta} \neq \tilde{\alpha}$, such that $\tilde{F} := \tilde{\alpha} F + (1-\tilde{\alpha})H \sim \tilde{G} := \tilde{\beta} G + (1-\tilde{\beta})H$. As before, there is a sequence $\beta_n \downarrow 0$ such that for all n, $\beta_n \tilde{F} + (1-\beta_n)H \sim \beta_n \tilde{G} + (1-\beta_n)H$. By construction, the line L_n through $\alpha_n F + (1-\alpha_n)H$ and $\alpha_n G + (1-\alpha_n)H$ and $\tilde{L}_n$ through $\beta_n \tilde{F} + (1-\beta_n)H$ and $\beta_n \tilde{G} + (1-\beta_n)H$ are not parallel. Wlg, H is in the interior of a probability triangle (see Machina [7]) containing also F and G. Otherwise, let $H_n \to H$ where for every n, H_n is in the interior of the triangle formed by F, G, H. The limit of the intersection points of L_n and $\tilde{L}_n$ is H, therefore these intersection points are in the triangle, a violation of transitivity, see Figure 1. ■

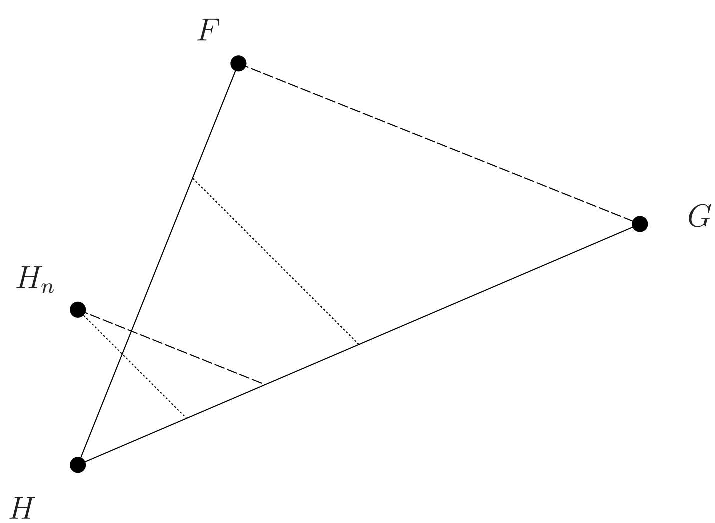

Figure 1: Wide-dash: α-lines, dense-dash: β-lines

3 Mixture Symmetry

This section deals with some variants of the mixture symmetry axiom. The first two are taken from Chew, Epstein, and Segal [2], who show in their Theorem 1 that assuming continuity and monotonicity, they are equivalent.

MS: Mixture Symmetry For all $F, G \in \mathcal{F}$, if $F \sim G$ then for all $\alpha \in (0, \frac{1}{2})$ there exists $\beta \in (\frac{1}{2}, 1)$ such that $\alpha F + (1-\alpha)G \sim \beta F + (1-\beta)G$.

SMS: Strong Mixture Symmetry For all $F, G \in \mathcal{F}$, if $F \sim G$, then for all $\alpha \in [0, 1]$, $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$.

WMS: Weak Mixture Symmetry For every $F, G \in \mathcal{F}$, if $F \sim G$, then there exists $\alpha \in (0, \frac{1}{2})$ such that $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$.

Note that in the definition of weak mixture symmetry, α may depend on F and G. We get **MS** $\Longleftrightarrow$ **SMS**$\Longrightarrow$**WMS**.

Quasi concavity and convexity of preferences play a crucial role in the analysis of quadratic functions.

SP / SD: Single Peak / Deep on $[\boldsymbol{F}, \boldsymbol{G}]$ Let $F \sim G$, $F \neq G$. There is $\beta \in (0, 1)$ such that the preferences $\succeq$ over $\alpha F + (1-\alpha)G$ are strictly increasing [decreasing] in α on $[0, \beta]$ and strictly decreasing [increasing] in α on $[\beta, 1]$.

SE: Single Extreme on $[\boldsymbol{F}, \boldsymbol{G}]$ Let $F \sim G$ such that there is no $\alpha \in (0, 1)$ for which $F \sim \alpha F + (1-\alpha)G$. Then $\succeq$ is either **SP** or **SD** on $[F, G]$.

SE: Single Extreme For every $F \sim G$, $\succeq$ satisfies **SE** on $[F, G]$.

SQC: Strict Quasi-Concavity For all $F \neq G$ and $\alpha \in (0, 1)$, $F \succeq G$ implies $\alpha F + (1-\alpha)G \succ G$.

SQX: Strict Quasi-Convexity For all $F \neq G$ and $\alpha \in (0, 1)$, $F \succeq G$ implies $F \succ \alpha F + (1-\alpha)G$.

NL: Non-Linearity For all $F \neq G$, $F \sim G$, there is $H \in [F, G]$ such that $F \not\sim H$.

Clearly **SQC** implies **SP** and **SQX** implies **SD** on $[F, G]$ for all F and G and both **SP** and **SD**, and hence **SE**, imply **NL**.[3] However, neither **SQC** nor **SQX** is implied by **SE**. For example, let $\succeq$ on $\Re^2_+$ be represented by

$$V(p,q) = \begin{cases} \frac{2p+q+\sqrt{4pq-3q^2}}{4} & q \leqslant p \\ \frac{p^2+q^2}{2q} & q > p \end{cases}$$

(see Figure 2).

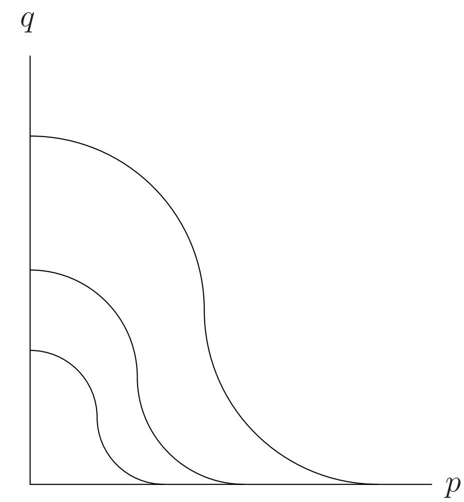

Figure 2: **SE** and **NL** together do not imply **SQC** or **SQX**

Definition 1. A preference relation $\succeq$ is quadratic if can be represented by

$$V(F) = \int\int \varphi(x,y)\mathrm{d}F(x)\mathrm{d}F(y)$$

For some continuous, monotonic, and symmetric function $\varphi : \Re^2_+ \to \Re$. For finite lotteries $(x_1, p_1; \ldots; x_n, p_n)$ this function becomes

$$V(x_1, p_1; \ldots; x_n, p_n) = \sum_i \sum_j p_i p_j \varphi(x_i, x_j)$$

Theorem 2. Let the continuous preference relation $\succeq$ satisfy **SE**. Then the following three conditions are equivalent.

[3] Although I never assume **NL** without assuming **SE**, it is sometimes illustrative to use it directly.

1. It satisfies **WMS**

2. It satisfies **SMS**

3. It can be represented by a quadratic function.

and in all three cases, it either satisfies **SQC** or it satisfies **SQX**.

The theorem is proved through a sequence of claims. The arguments are presented in the interiors of probability triangles $\{p\bar{F} + q\bar{G} + (1 - p - q)\delta_0 : p, q \geqslant 0, p + q \leqslant 1\}$ where $\bar{F}$, $\bar{G}$, and δ_0 are not on the same line L. By continuity, the claims apply to the boundaries of the triangles as well. Observe that since the set of outcomes is (a subset of) $\Re_+$, monotonicity with respect to first-order stochastic dominance implies that preferences are increasing in p and q.

Claim 1. Assume **SE**. Let $\mathcal{I}$ be an indifference curve of $\succeq$ and let L be a line. If $|\mathcal{I} \cap L| \geqslant 3$, then there exists an indifference curve $\mathcal{I}'$ and $F^*, G^*, H^* \in \mathcal{I}' \cap L$, such that $(F^*, G^*) \cap \mathcal{I}'$ and $(G^*, H^*) \cap \mathcal{I}'$ are empty.

Proof: By **NL** there exists $D \in L \backslash \mathcal{I}$ (see Figure 3). By the continuity of $\succeq$, there are $G, H \in L \cap \mathcal{I}$ such that $D \in (G, H)$ and $(G, H) \cap \mathcal{I} = \varnothing$ (see the first part of the proof of Theorem 1 above). Wlg, $G \succ D$ and there is $F \in L \cap \mathcal{I}$ such that $G \in (F, H)$. If $(F, G) \cap \mathcal{I} = \varnothing$, we are through, and the three desired points are F, G, H. Otherwise, there is in $(F, G) \cap \mathcal{I}$ a sequence $G_n \to G$. Assume that there exists $F' \in (F, G)$ such that $G \succ F'$. If not, that is, if for all $F' \in (F, G)$, $F' \succeq G$, then start over by choosing $\bar{D}, \bar{G}, \bar{H} \in (F, G)$ such that $\bar{D} \in (\bar{G}, \bar{H})$, $\bar{G}, \bar{H} \in \mathcal{I}$, $\bar{D} \succ \bar{G}$, and there exists $F' \in (F, \bar{G})$ such that $F' \succ \bar{G}$. The proof then continues with the opposite preference signs.

Since all points in (G, H) are inferior to G, as is F', it follows by continuity that there is an indifference curve $\mathcal{I}'$, sufficiently close to $\mathcal{I}$, and three points $\tilde{F}, G^*, H^* \in \mathcal{I}'$ such that $[G^*, H^*] \subset (G, H)$ and $\tilde{F} \in (F, G)$. By **SE** on $[G, H]$, $(G^*, H^*) \cap \mathcal{I}' = \varnothing$ and for all $F'' \in [G, G^*)$, $G^* \succ F''$. Moreover, there is $F^* \in [\tilde{F}, G]$ such that $F^* \in \mathcal{I}'$ and $(F^*, G] \cap \mathcal{I}' = \varnothing$. Otherwise, there is a sequence $\tilde{F}_n \to G$ such that for all n, $\tilde{F}_{n+1} \in (\tilde{F}_n, G) \cap \mathcal{I}'$, a violation of continuity, as $G \notin \mathcal{I}'$. It follows that $F^* \sim G^*$, for all $D' \in (F^*, G]$, $F^* \succ D'$ and for all $D' \in [G, G^*)$ or in (G^*, H^*), $G^* \succ D'$, hence F^*, G^*, H^* satisfy the requirements of the claim. ■

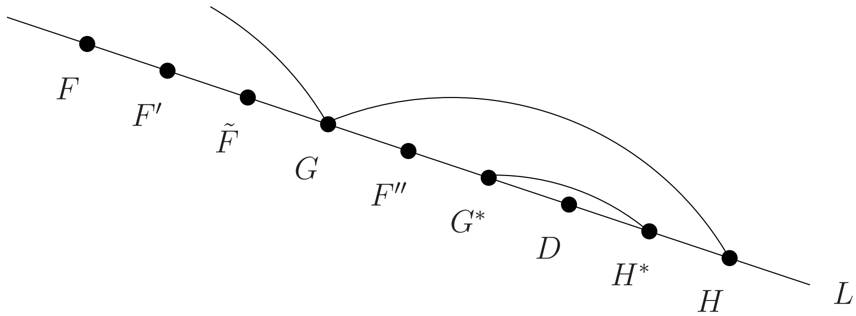

Figure 3: Distributions along L

Claim 3 below proves that the conclusion of Claim 1 contradicts our assumptions, and therefore the "if" part of the last claim is empty.

Claim 2. Let $F \sim G$. If $\succeq$ satisfies **WMS** and **SE** on $[F, G]$, then for all $\alpha \in [0,1]$, $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$.

Proof: Assume **SP** on $[F, G]$ (the proof for **SD** is similar) and that $F \neq G$ (otherwise the claim is trivial). Let

$$\bar{\alpha} = \sup\left\{\alpha \in (0, \tfrac{1}{2}) : \alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G\right\} \tag{1}$$

By continuity, $F_0 := \bar{\alpha}F + (1-\bar{\alpha})G \sim G_0 := (1-\bar{\alpha})F + \bar{\alpha}G$. We want to show that $\bar{\alpha} = \frac{1}{2}$. Suppose that $\bar{\alpha} < \frac{1}{2}$. Then by **WMS** there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\begin{aligned}
&\alpha F_0 + (1-\alpha)G_0 \sim (1-\alpha)F_0 + \alpha G_0 \implies \\
&\alpha[\bar{\alpha}F + (1-\bar{\alpha})G] + (1-\alpha)[(1-\bar{\alpha})F + \bar{\alpha}G] \sim \\
&\qquad (1-\alpha)[\bar{\alpha}F + (1-\bar{\alpha})G] + \alpha[(1-\bar{\alpha})F + \bar{\alpha}G] \implies \\
&[\alpha\bar{\alpha} + (1-\alpha)(1-\bar{\alpha})]F + [\alpha(1-\bar{\alpha}) + (1-\alpha)\bar{\alpha}]G \sim \\
&\qquad [(1-\alpha)\bar{\alpha} + \alpha(1-\bar{\alpha})]F + [(1-\alpha)(1-\bar{\alpha}) + \alpha\bar{\alpha}]G
\end{aligned}$$

Let $\alpha_1 = \alpha\bar{\alpha} + (1-\alpha)(1-\bar{\alpha})$ and $\alpha_2 = \alpha(1-\bar{\alpha}) + (1-\alpha)\bar{\alpha}$ and observe that $\alpha_2 = 1 - \alpha_1$. Also, for $\alpha \in (0,1)$, $\alpha_1 > \bar{\alpha}$ iff $\bar{\alpha} < \frac{1}{2}$. Moreover, for $\bar{\alpha}, \alpha \in (0, \frac{1}{2})$, α_1 is decreasing in α and $\bar{\alpha}$, and at $\alpha = \bar{\alpha} = \frac{1}{2}$, $\alpha_1 = \frac{1}{2}$.

We obtain that $\frac{1}{2} > \alpha_1 > \bar{\alpha}$, yet $\alpha_1 F + (1-\alpha_1)G \sim (1-\alpha_1)F + \alpha_1 G$, in contradiction to the definition of $\bar{\alpha}$ (see eq. (1)). It thus follows that $\bar{\alpha} = \frac{1}{2}$.

Next we show that for all $\alpha \neq \frac{1}{2}$, $\frac{1}{2}F + \frac{1}{2}G \succ \alpha F + (1-\alpha)G$. Suppose not. Wlg, there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)G \succeq \frac{1}{2}F + \frac{1}{2}G$, and since $\succeq$ is **SP** on $[F,G]$, there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)G \succ \frac{1}{2}F + \frac{1}{2}G$. It follows that $\succeq$ is decreasing in α on $[\beta, 1]$ for some $\beta < \frac{1}{2}$, in contradiction to the above conclusion that $\bar{\alpha} = \frac{1}{2}$. It thus follows that $\succeq$ is increasing in α on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$.

Let $F_1 = \alpha F + (1-\alpha)G$ for some $\alpha \in (0, \frac{1}{2})$. By continuity and the last conclusion there is $\alpha' \in (\frac{1}{2}, 1)$ such that $F_1 \sim G_1 := \alpha' F + (1-\alpha')G$. Since $\succeq$ is **SP** on $[F_1, G_1]$ it follows as above that $\frac{1}{2}F_1 + \frac{1}{2}G_1 \succ \alpha F_1 + (1-\alpha)G_1$ for all $\alpha \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. But since $\frac{1}{2}F + \frac{1}{2}G \in [F_1, G_1]$, it must be the midpoint of this segment, hence $\alpha' = \alpha_1$. ■

Claim 3. Assume **WMS**, and let $G \in (F, H)$ such that $F \sim G \sim H$. If $\succeq$ satisfies **SE** on $[F, G]$, then it does not satisfy **SE** on $[G, H]$.

Proof: Let $G = \alpha_0 F + (1-\alpha_0)H$, and suppose wlg that $\alpha_0 \leqslant \frac{1}{2}$ and that $\succeq$ satisfies **SD** on $[G, H]$ (see Figure 4, where the indifference curve between H and G is depicted by the solid curve, and its two possible continuations to F are depicted by the dashed lines).

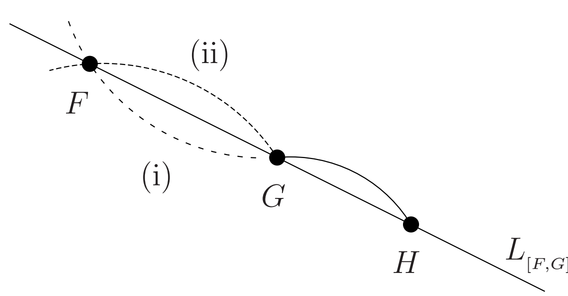

Figure 4: $G = \alpha_0 F + (1-\alpha_0)H$, $\alpha_0 \leqslant \frac{1}{2}$

(i) $\succeq$ satisfies **SP** on $[F, G]$: $F \sim H$, hence by **WMS** there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)H \sim (1-\alpha)F + \alpha H$. For $\alpha \leqslant \alpha_0$, $\alpha F + (1-\alpha)H \succ H \sim F \succ (1-\alpha)F + \alpha H$, therefore $\alpha_0 < \alpha < \frac{1}{2}$. In that case, both $\alpha F + (1-\alpha)H$ and $(1-\alpha)F + \alpha H$ are between F and G. By Claim 2,

$\beta F + (1-\beta)G \sim (1-\gamma)F + \gamma G$ iff $\beta = \gamma$. Let $\beta \in (0,1)$ such that $\alpha F + (1-\alpha)H = \beta F + (1-\beta)G$, hence

$$\begin{aligned}
&\alpha F + (1-\alpha)H = \beta F + (1-\beta)[\alpha_0 F + (1-\alpha_0)H] = \\
&[\beta + (1-\beta)\alpha_0]F + (1-\beta)(1-\alpha_0)H \implies \\
&\beta = \frac{\alpha - \alpha_0}{1-\alpha_0} \qquad (2)
\end{aligned}$$

On the other hand, by **SP** on $[F,G]$, the only points in $[F,G]$ to be indifferent to $\alpha F + (1-\alpha)H$ and $\beta F + (1-\beta)G$ are $(1-\alpha)F + \alpha H$ and $(1-\beta)F + \beta G$, which must be the same. We get

$$\begin{aligned}
&(1-\alpha)F + \alpha H = (1-\beta)F + \beta[\alpha_0 F + (1-\alpha_0)H] = \\
&[(1-\beta+\beta\alpha_0]F + \beta(1-\alpha_0)H \implies \\
&\beta = \frac{\alpha}{1-\alpha_0} \qquad (3)
\end{aligned}$$

Eqs. (2) and (3) imply $\alpha_0 = 0$, a contradiction, as $G \neq H$.

(ii) $\succeq$ satisfies **SD** on $[F,G]$: Here too, there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)H \sim (1-\alpha)F + \alpha H$. If $\alpha > \alpha_0$, a contradiction is created as above. Otherwise, creating a sequence of points as in the proof of Claim 2, we eventually get to points in $[F,G]$ that are indifferent to each other but are not in symmetrical position on this segment, a contradiction to the assumption that $\succeq$ on $[F,G]$ is **SD**. ∎

Claim 1 show that under **SE**, if an indifference curve $\mathcal{I}$ intersects line L in more than two points, then there is an indifference curve $\mathcal{I}'$ that intersects L at three points but not between them. Claim 3 shows that under **WMS**, such $\mathcal{I}'$ does not exist.

Conclusion 1. Let $\succeq$ satisfy **SE** and **WMS** and let $\mathcal{I}$ be an indifference curve of $\succeq$. Then for any line L, $|\mathcal{I} \cap L| \leqslant 2$.

Claim 4. If $\succeq$ satisfies **WMS** and **SE**, then it satisfies either **SQC** or **SQX**.

Proof: Suppose that there are two indifference curves $\mathcal{I}$ and $\mathcal{I}'$ with $F, G \in \mathcal{I}$ and $F', G' \in \mathcal{I}'$ such that $\succeq$ is **SP** on $[F,G]$ and **SD** on $[F',G']$. By continuity, $\mathcal{I}$ and $\mathcal{I}'$ can be assumed to be different indifference curves. Also by continuity, if such points exist then we can find such pairs that are not all on the same line. Therefore we can assume wlg that $[F, F'] \cap [G, G'] = \varnothing$, otherwise $[F, G'] \cap [G, F'] = \varnothing$ and the roles of G and G' are reversed. By assumption, $\frac{1}{2}F + \frac{1}{2}G \succ F \sim G$ while $F' \sim G' \succ \frac{1}{2}F' + \frac{1}{2}G'$. By continuity, for every $\alpha \in (0,1)$ there exist $\beta_\alpha \in (0,1)$ such that $\alpha F + (1-\alpha)F' \sim \beta_\alpha G + (1-\beta_\alpha)G'$. By continuity, there is α such that

$$\alpha F + (1-\alpha)F' \sim \beta_\alpha G + (1-\beta_\alpha)G' \sim \frac{1}{2}(\alpha F + (1-\alpha)F') + \frac{1}{2}(\beta_\alpha G + (1-\beta_\alpha)G')$$

Contradicting Conclusion 1 that a line can intersect an indifference curve at no more than two points. ■

Proof of Theorem 1: Obviously, **SMS** implies **WMS** and since we assume **SE**, by Claim 2 **WMS** implies **SMS**. By Claim 4, $\succeq$ is either **QCV** or **QCX**. By [2, Theorem 4], if $\succeq$ is either quasi-concave or quasi-convex, then it can be represented by a quadratic function iff it satisfies **SMS**. ■

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