# A Game of Hide and Seek in Networks*

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**Abstract**

We propose and study a strategic model of hiding in a network, where the network designer chooses the links and his position in the network facing the seeker who inspects and disrupts the network. We characterize optimal networks for the hider, as well as equilibrium hiding and seeking strategies on these networks.

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# 1 Introduction

This paper studies the design of a network in order to hide an object or a person. This question has a very long standing. According to Greek mythology, Daedalus invented the Labyrinth in order to hide the monstrous Minotaur.[1] Tunnels and underground chambers in Medieval castles and fortresses were built to hide treasures or prisoners. Underground fortifications were constructed in the XXth century to hide weapons and combatants. In modern days, criminals and terrorists build covert networks in order to hide leaders, money or secret instructions.

When an object or a person is being hidden, it must also be accessible for those who need it. The Minotaur cannot be sealed off in the Labyrinth, because every nine years, he receives a tribute of seven young boys and seven young girls from Athens. The medieval treasures and prisoners, the weapons and combatants of military forts also need to be recovered and freely moved. Leaders of criminal and terrorist organizations, secret plans and money must also be able to freely and efficiently move in the network. Hence, the design of networks to hide always involves a trade-off between security (the inviolability of the hiding place) and efficiency (the accessibility of hidden objects and persons). In this paper, we characterize the optimal network design as a function of this trade-off between security and efficiency.

We construct a zero-sum game with two players, a Hider and a Seeker. In the first stage of the game, the Hider designs a network which is observed both by the Hider and Seeker. In the second stage of the game, the Hider and Seeker simultaneously choose a location in the network (where the Hider hides and the Seeker seeks). The Seeker is able to observe any neighboring node to the location she seeks. If the Hider hides in any of the nodes observed by the Seeker, the Seeker wins. If the Seeker does not find the Hider, she is still able to disrupt part of the network, by taking away the node that she observes. The Hider then receives a payoff which is an increasing function of the size of the component in which she hides. The payoff in the zero-sum two-person game thus consists of two elements: (i) a benefit (to the Seeker) of capturing the hidden object or person and (ii) a benefit (to the Hider) of using a network connecting a given number of nodes.

We fully characterize the optimal network architecture chosen by the Hider. It can only take one of two forms: the optimal network is either a cycle (where all nodes are connected in a circle) or a special core-periphery network where half of the nodes form an interconnected core, and the other half are leaves, each connected to a single node in the core.[2] In addition, a subset of the nodes will remain isolated. The size of the subset of isolated nodes, and the choice between the circle and the core-periphery network for connected nodes depends on the parameters of the game, and in particular the shape of the function mapping the size of the network into the benefit of the Hider.

To understand this characterization of an optimal network, notice that any network which cannot be "disrupted" (in the sense that the network is not broken

[1] See Book 8 in Ovid's Metamorphosis.

[2] If the number of nodes in the core-periphery network is odd, the architecture is slightly different, with three orphaned nodes.

into different components if the Seeker fails to find the hidden object) must be two-connected, and hence contain a cycle. Now, adding links to the cycle only increases the sizes of the neighborhoods and hence the probability that the hidden object is discovered. Therefore, if the objective of the Hider is primarily to avoid disruption of the network, his optimal choice will be to form a cycle. Notice however that in a cycle, every agent has two neighbors, so the probability of discovery of the hidden object must be at least equal to $\frac{3}{n}$. In order to reduce this probability of discovery, while keeping the network connected, one has to allow for the possibility that some nodes only have degree one. In the core-periphery network where half of the nodes are leaves connected to one node in the core, the probability of discovery is reduced to the minimal value for a connected graph. In equilibrium, the Hider chooses to hide in any of the peripheral nodes, whereas the Seeker seeks in any of the core nodes. This uniform hide and seek strategy results in a probability of discovery equal to $\frac{2}{n}$, lower than in the cycle, but induces a larger disruption, as the size of the remaining component after the Seeker fails to find the object is equal to $n-2$ rather than $n-1$. In the main characterization Theorem, we show that no other network performs better than the cycle or the core-periphery network. The cycle is preferred when the Hider puts more weight on avoiding disruption and the core-periphery network is preferred when the Hider puts more weight on avoiding discovery of the hidden object.

While no real network has the exact architecture of a cycle or core-periphery network, our results echo some observations on the trade-off between security and efficiency in physical networks of military fortifications and human networks of criminals and terrorists.

Following the trench warfare of World War 1, the French army built the “Maginot line”, a system of underground fortifications to protect the border between Germany and France between 1929 and 1935.[3] The design of the underground tunnels struck a balance between separating blocks (where combatants could hide) and allowing for easy communication of men and materials. Figure 1 provides an example of the underground tunnels in three of the largest fortifications of the Maginot line: the Hackenberg, Mont des Welches and Fermont “gros ouvrages”. It shows that blocks are not directly connected to each other (echoing the fact that peripheral nodes are only connected to one node in the core and not to each other nor to a central node), while central areas (where men sleep and weapons and ammunition are stored) form a well-connected core in the middle of the “gros ouvrage”.

Morselli et al. (2007) illustrate the trade-off between security and efficiency using data on terrorist networks (Krebs (2002)’s map of the 9/11WTC terrorist cells) and criminal networks (a drug-trafficking network in Canada). They argue that terrorist networks are more likely to have longer average distances and fewer

[3] Ironically, the Maginot line proved useless during the German invasion of France in May 1940, as the German army simply by-passed the line of fortifications and entered France from Belgium and Luxembourg.

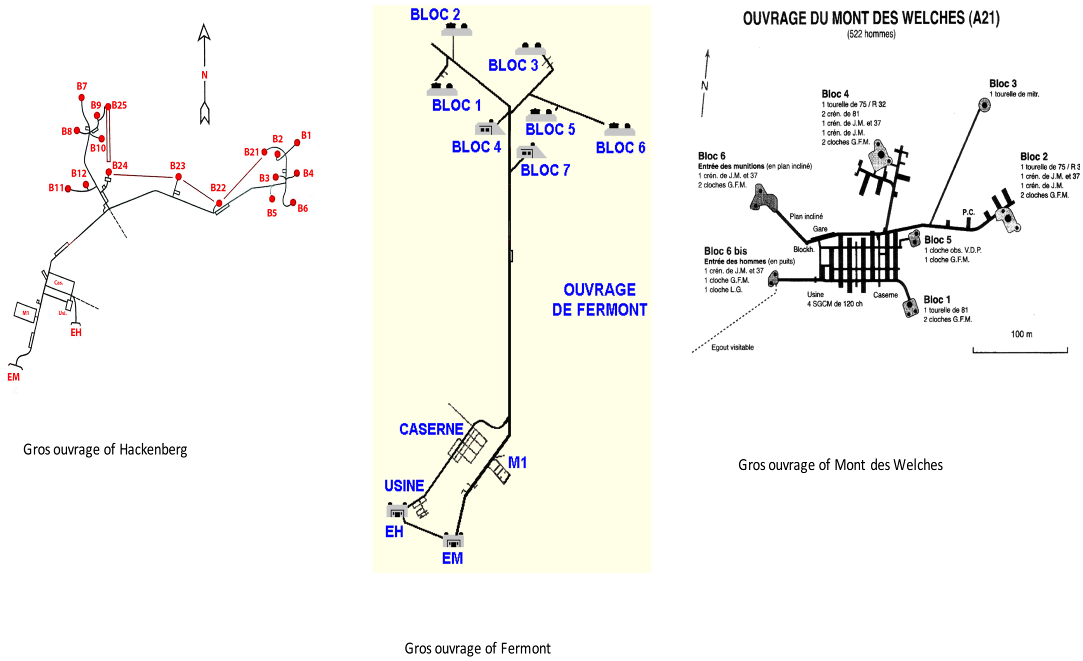
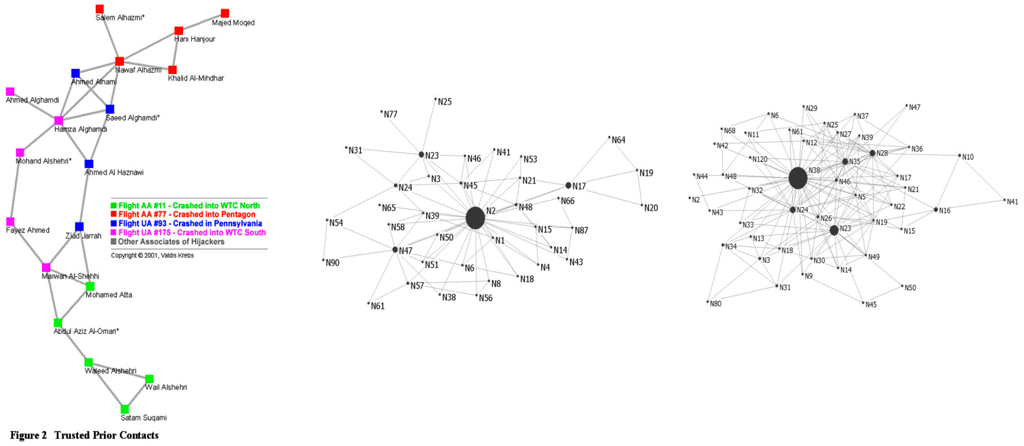



Figure 1: Three "Gros Ouvrages" of the Maginot Line

connections with no node assuming a central position, whereas criminal networks are more clustered and exhibit a core of nodes with high centrality. In addition they note that support nodes (which are not direct perpetrators of criminal or terrorist activities) help connect distant nodes in terrorist networks but not in criminal networks, where each support agent is attached to a single agent in the core. These two network architectures (long lines and core-periphery with clusters) can be related to the cycle and the core-periphery network we identify in our analysis. Figure 2 illustrates these network architectures, by reproducing the map of the 9/11 WTC terrorist network (Krebs (2002)) as well as the maps of two drug-trafficking mafia groups collected by Calderoni (2012).

## 2 Related literature

The related literature spans a variety of disciplines, with the earlier literature focusing more on the hiding and seeking. Perhaps, the first paper was by von Neumann(1953) who discusses a zero-sum game where $H$ chooses a cell of an exogenously given matrix, and cell in matrix, while $S$ simultaneously chooses a column or row in the matrix. $S$ "captures" $H$ if the cell chosen by $H$ lies in the row or column chosen by $S$. A related paper is Fischer (1993), who too analyses a similar zero sum game, where $H$ and $S$ simultaneously choose vertices of an exogenously given graph. $H$ is caught if $S$ chooses the same node as him or a node connected



The 9/11WTC terrorist network

The N'Drangheta network of cocaine trafficking (operation Chalonero)

The N'Drangheta network of cocaine trafficking (operation Stupor Mundi)

Figure 2: Three examples of terrorist and criminal networks

to the node chosen by him. Interestingly, the value of this "hide and seek game" on a fixed arbitrary network can been computed following Fisher (1991), using *fractional graph theory*. [4].

Computer scientists have also contributed to this literature. with Waniek et al. (2017) and Waniek et al. (2018) studiying a related, but different problem, of hiding in a network. They consider the leader of a terrorist or criminal organization, and ask the following question: How can a set of edges be added to the network in order to reduce the leader's measure of centrality in order to avoid detection? Waniek et al. (2017) show that, both for degree and closeness centrality, the problem is NP-complete. However, they also propose a procedure to build a new network from scratch around the leader (the "captain network") which achieves low levels of degree and closeness centrality but high values of diffusion centrality, where diffusion centrality is measured using the independent cascade and linear threshold diffusion models. Waniek et al. (2018) extend the analysis to betweenness centrality and to the detection of communities (rather than individuals) in the network. Notice, however, that these models are not fully strategic since $S$ does not best respond to $H$'s strategy.

Our paper is also related to a recent strand of the economics literature analyzing network design and attack and defense on networks. Baccara and Bar-Isaac (2008) study network design by an adversary (a criminal organization) taking the detection strategy of the defender as fixed. They highlight differences between two forms of

[4] (See also Theorem 1.4.1 in Scheinerman and Ullman (1997))

detection, one which depends on the cooperation between criminals and the other which does not. In both situations, they characterize the optimal network architecture of the criminal network, which either consists of isolated two-player cells (with independent detection) or an asymmetric structure with one agent serving as an information hub (with cooperation-based detection). Goyal and Vigier (2014) propose an alternative model of network design where the defender designs the network and chooses the distribution of defense across nodes before the attacker chooses to attack. Nodes are captured according to a Tullock contest function given the resources spent by the attacker and the defender. If a node is captured by the attacker, contagion occurs and the attacker starts attacking neighboring nodes while the defender loses his defense resources. The main message of Goyal and Vigier (2014) is that the defendant optimally forms a star and concentrates all the defenses at the hub. Dziubiński and Goyal (2013) analyze a related model, where the defender designs the network and chooses defense resources before the attacker attacks. As opposed to Goyal and Vigier (2014), contagion does not occur and the network structure only matters through the payoffs of the two-person zero-sum game between the defender and the attacker. The objective function of the defender is assumed to be increasing and convex in the size of components of the network, reflecting the fact that the defender wants to avoid disruption in the network. The analysis shows that the designer will either form a star and protect the hub, or not protect any node and choose to form a $(k+1)$-connected network when the attacker has $k$ units, so that the attacker will not be able to disrupt the network. In the same model, Dziubiński and Goyal (2017) study equilibrium strategies of the defender and attacker for any arbitrary network structure while Cerdeiro et al. (2017) consider decentralized defense decisions by the different nodes in the network.

The main difference between our paper and the literature on design, attack and defense stems from a difference in the game played by the defender and adversary once the network is given. The payoff of the players in our analysis is different from that in Goyal and Vigier (2014) and Dziubiński and Goyal (2013) as we assume a specific payoff when the hidden object or person is discovered by the adversary, in addition to the payoff arising from disruption of the network. Another difference comes from the timing of the game. We suppose that the hider and seeker simultaneously choose the nodes in which to hide and that they inspect, resulting in equilibria in mixed strategies as in Colonel Blotto games, whereas Goyal and Vigier (2014) and Dziubiński and Goyal (2013) assume that the defender and attacker move sequentially, allowing for pure strategy equilibria.

# 3 The Model

There are two players, a *Hider* ($H$) and a *Seeker* ($S$). The hider $H$ is, for instance, the leader of a covert organisation, which has a set of $n-1$ additional members. The interaction between $H$ and $S$ is modelled as a two-stage process, which is described below.

In the first stage, $H$ chooses a network of interactions amongst the members of the organisation. Formally, $H$ chooses a graph $G = \langle V, E \rangle$ where $V$ is a set of $n$ vertices, and $E$ is a set of undirected edges $E \subseteq \binom{V}{2}$. A typical edge $e \in E$ will be denoted $ij$, where $i, j \in V$.

Both players observe the chosen network at the beginning of the second stage. After observing the network $G$, players $H$ and $S$ *simultaneously* choose 1 and $k$ nodes respectively. The node chosen by the hider is his (hiding) position in the network. Let $K$ be the set of $k$ nodes chosen by $S$, and $N_G(K) = K \cup \{j \in V | i \in K, ij \in E\}$. That is, $N_G(K)$ is the set of nodes chosen by $S$ as well as all neighbours of $K$ in $G$. All nodes in $N_G(K)$ can be inspected by the seeker. The significance of inspection is that if the chosen position of $H$ is in $N_G(K)$, then $H$ is captured by $S$. In addition, the set $K$ is removed from the network, irrespective of whether $H$ is captured or not. The seeker uses his choice to capture the hider and to damage the network.

If caught, the hider gets payoff $-\beta$, where $\beta \geq 0$. Otherwise, his payoff is given by a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of the size of his component in the residual network. The payoff to the seeker is equal to minus the payoff of the hider (so the game is zero-sum). We assume $f$ to be strictly increasing with $f(0) = 0$. An example of function $f$ in line with these assumptions is the identity function, $f(x) = x$ for all $x \in \mathbb{R}_{\geq 0}$.

Formally, given a set of nodes $U \subseteq V$, let $\mathcal{G}(U)$ be the set of all undirected graphs over $U$ and let $\mathcal{G} = \bigcup_{U \subseteq V} \mathcal{G}(U)$ be the set of all undirected graphs that can be formed over $V$ or any of its subsets. A strategy of the hider is a pair $(G, h) \in \mathcal{G}(V) \times V$, where $G$ is the graph and $h$ is the hiding place chosen by $\mathsf{H}$ in $G$. A strategy of the seeker is a function $s : \mathcal{G}(V) \to \mathbf{K}(V)$ where $\mathbf{K}(V)$ is the set of all $k$-element subsets of $V$.

Before defining the payoffs we need to introduce a number of auxiliary notions. Given a set of nodes $U \subseteq V$ and a graph $G = \langle U, E \rangle$ over $U$, a maximal set of nodes $C \subseteq U$ such that any two nodes $i, j \in C$ are connected in $G$ is a *component* of $G$.[5] The set of all components of $G$ is denoted by $\mathcal{C}(G)$. In addition, given $i \in U$, let $C_i(G)$ be the be the component in $G$ containing $i$. Given a set of nodes $U \subseteq V$, a graph $G = \langle U, E \rangle$ over $U$, and a set of nodes $U' \subseteq U$, let $G[U'] = \langle U', E[U'] \rangle$ with $E[U'] = \{ij \in E : \{i, j\} \subseteq U'\}$ be the *subgraph of $G$ induced by $U'$*. Given a set $K \subset V$ let $G - K = G[U \setminus K]$ be the *residual network* obtained from $G$ by removing nodes in $K$ and all their links from $G$.

Given the strategy profile $((G, h), s)$, the payoff to the hider is

$$\Pi^{\mathsf{H}}(G, h, s) = \begin{cases} -\beta & \text{if } h \in N_G(s(G)) \\ f(|C_i(G - s(G))|) & \text{otherwise.} \end{cases} \tag{1}$$

where $f$ is strictly increasing with $f(0) = 0$.

The payoff to the seeker is $\Pi^{\mathsf{S}}((G, h), s) = -\Pi^{\mathsf{H}}((G, h), s)$.

The cycle network and the *core periphery* networks will be important in our analysis.

A *core-periphery* network over a set $V = P \cup C$ of $n$ nodes is a network defined as follows. There are $q \geq \lceil n/2 \rceil$ *core* nodes in set $C = \{\mathrm{c}_1, \ldots, \mathrm{c}_q\}$ and $m \leq \lfloor n/2 \rfloor$

[5] Two nodes $i, j \in U$ are connected in $G = \langle U, E \rangle$ if there exists a sequence of nodes $i_1, \ldots, i_l$ such that $i_0 = i$, $i_n = j$, and for all $k \in \{1, \ldots, l\}$, $i_{k-1} i_k \in E$.

*periphery* nodes in set $P = \{\mathrm{p}_1, \ldots, \mathrm{p}_m\}$. Nodes of the core are connected forming a graph containing a cycle over these nodes, while each periphery node, $\mathrm{p}_i$ with $1 \leq i \leq m$, is connected to core node $\mathrm{c}_i$. Nodes of the core which are not connected to a periphery node are called *orphaned*. A core-periphery network where $m = \lfloor n/2 \rfloor$, i.e. $m$ takes its maximal value, is called *maximal*.

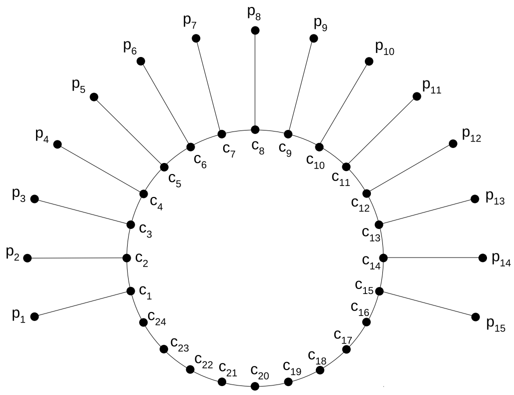



Figure 3: A core-periphery network over 39 nodes, with 15 periphery nodes and 9 orphaned core nodes.

## 4 The Characterization Result for $k = 1$

Our objective in this section is to provide optimal networks for the hider as well as to characterize the hiding and the seeking strategies on these networks when $k = 1$. As we show in our main result (Theorem 1) these networks consist of a number of singleton nodes and a connected component which is either a *cycle* or has a particular *core periphery* topology,

Whether a cycle or a core-periphery topology is better for the hider depends on the sign of the following expression

$$T(n, s) = (n - s - 3)f(n - s - 1) - (n - s - 2)f(n - s - 2). \tag{2}$$

We will show that the cycle topology is better if $T(n, s) \geq \beta$, while a core-periphery topology is better when $T(n, s) \leq \beta$.

The following lemma, which shows that an optimal network can never have any component containing just two or three nodes, will be used in the proof of the main theorem.

**Lemma 1.** *Suppose $G$ is an optimal network for $H$ whose non-singleton components are $\{g_1, g_2, \ldots, g_r\}$. Then, each component $g_i$ contains at least 4 nodes.*

*Proof.* Suppose the lemma is not true and some $g_i$ has exactly three nodes $n_1, n_2, n_3$. Following standard arguments, $\{n_1, n_2, n_3\}$ must have a non-empty intersection with the support of $H$'s optimal hiding strategy as well as $S$'s optimal seeking strategy, given $G$. Moreover, conditional on hiding in $\{n_1, n_2, n_3\}$, $H$ is caught with probability $\rho$, where $\rho$ is the total probability with which $S$ seeks in $\{n_1, n_2, n_3\}$. This is true because $S$ can search one node in $\{n_1, n_2, n_3\}$ that has two neighbours.

Let $G'$ be another network which coincides with $G$ everywhere except that $g_i$ is broken up into singleton nodes $n_1, n_2, n_3$. Moreover, suppose $H$'s hiding strategy coincide with that in $G$ everywhere on $G \setminus g_i$, while $H$ distributes the earlier probability weight on $g_i$ uniformly on the three nodes $n_1, n_2, n_3$. It is straightforward to check that $H$'s expected payoff in $G'$ is strictly higher than in $G$, contradicting optimality of $G$.

A similar argument rules out the optimal network containing two nodes.

□

**Remark 1.** *An implication of this lemma is that the optimal network will either be completely disconnected with $n$ singletons or will have at most $n - 4$ singletons. This implication will be used throughout the proof of the theorem.*

At this stage, we describe the main result of the paper somewhat informally. A formal statement comes towards the end of the section.

The optimal network for the hider as well as her hiding strategy will have the following features.

- The optimal network $G$ will have a certain number of singleton nodes $s$ (that will be determined) where $s \leq n - 4$ or $s = n$.
- If $T(n, s) \geq \beta$ and $s \neq n$, then $G$ has a cycle component over $n - s$ nodes.

- If $T(n,s) < \beta$, $n - s \geq 4$, then $G$ will have a maximal core periphery over $n - s$ nodes if $n - s$ is even, and a core periphery with 3 orphaned nodes if $n - s$ is odd

- The hider mixes between hiding in the singleton nodes and in the connected component with probabilities that will be determined. When hiding in the singleton nodes, he mixes uniformly across all these nodes. When hiding in the connected component, he mixes uniformly across all the nodes when it is a cycle, mixes uniformly across the periphery nodes when it is a maximal core-periphery network, and mixes between hiding in periphery nodes, mixing uniformly across them, and the middle orphaned node otherwise.

- The seeker mixes between seeking in the singleton nodes and in the connected component. When seeking in the singleton nodes, he mixes uniformly across all these nodes. When seeking in the connected component, he mixes uniformly across all the nodes when it is a cycle, mixes uniformly across the core nodes when it is a maximal core-periphery network, and mixes between seeking in the neighbours of periphery nodes, mixing uniformly across them, and the middle orphaned node otherwise.

To get some intuition behind the result, notice that the hider faces a tradeoff between the cost of being caught and the value he gets in the residual network, after the seeker's action. More links in the network and hence the higher connectivity secures a larger value after the the seeker's action *provided* he is not caught. However, a larger number of links also leads to higher exposure. Fixing the number of singleton nodes, $s$, the choice between a cycle and a core-periphery network is influenced by the change in $f$, as measured by the quantity $T(n,s)$. The probability of being caught in a cycle of size $n-s$ is $3/(n-s)$, as each node has exactly two neighbours, while only one node is lost from the cycle component if not caught. The probability of being caught in a maximal core-periphery network, on the other hand, is $2/(n-s)$ since the hider hides mixing uniformly across the periphery nodes; in the event of not being caught, two nodes are lost from the core periphery component since the seeker seeks mixing uniformly across the core nodes. If the change in $f$ between $n-s-2$ and $n-s-1$ is sufficiently high, so that $T(n,s) > \beta$ then the marginal loss from an additional node being removed from a component is high, as compared to the penalty for being caught, and, therefore, a cycle is preferred over the core-periphery network. If the change in $f$ is not sufficiently high, on the other hand, the marginal loss from an additional node being removed from a component is not sufficiently high and the hider prefers to opt for the safer, core-periphery, network.

The proof of the theorem is long and we provide a brief description of the general technique before giving the details.

We start by constructing a feasible strategy of the seeker that, for each network over the set of nodes $V$, provides a (mixed) seeking strategy on that network. This strategy determines the payoffs the seeker can secure for each possible network over $V$. Since the game is zero-sum, minus these payoffs provide an upper bound on the payoff the hider can get for each network. Next for each $s \in \{0, \ldots, n-4, n\}$, we construct a network that is optimal for the hider across all possible networks with exactly $s$ singleton nodes. In the case of $T(n,s) \geq \beta$, as well as in the case of $s$ being even, these networks yield payoffs to the hider that meet the upper bound determined in the first part of the proof. In the case of $T(n,s) < \beta$ and odd $s$, the upper bound from the first part of the proof is not exact. Therefore in this step we establish both, the optimal networks and the exact upper bound on the hider's payoff.

We will use a series of lemmas to prove the theorem. We first introduce auxiliary notions and notation. In particular, we introduce a partition of nodes into a number of different sets that will play a crucial role in further construction.

Given a (possibly disconnected) network $G$ over the set of nodes $V$, node $i \in V$ is a *singleton node* if $|N_G(i)| = 0$. The set of singleton nodes of $G$ is denoted by $S(G)$. Node $i \in V$ is a *leaf* if $|N_G(i)| = 1$. The set of leaves of $G$ is denoted by $L(G)$. Given node $i \in V$, let $l_i(G) = |N_G(i) \cap L(G)|$ denote the number of leaf-neighbours of $i$.

Let

$$M(G) = \{i \in V : l_i(G) = 1\}$$

be the set of nodes which are connected to exactly one leaf in $G$ and let

$$SL(G) = \{i \in L(G) : N_G(i) \cap M(G) \neq \varnothing\}$$

be the set of leaves connected to an element of $M(G)$. Such leaves are called *singleton leaves*. Let $R(G) = V \setminus (S(G) \cup SL(G) \cup M(G))$ be the set of nodes in $G$ which are neither a singleton, nor a singleton leaf, nor a neighbour of a singleton leaf.

We start with the construction of a strategy of the seeker that secures a certain payoff for him on each network. Take any network $H$ over $V$ and let $s = |S(H)|$ and $m = |M(H)|$. Moreover, let $HR = H[R(H)]$ be the subnetwork of $H$ generated by the set of nodes $R(H)$. In particular, when $R(H) = \varnothing$, $HR$ is the empty network with empty sets of nodes and links. Let $D(HR)$ be the set of nodes in $R(H)$ that belong to two-element subsets of $R(H)$.

Consider a mixed strategy of player $\mathsf{S}$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, of the following form

$$\boldsymbol{\sigma} = \lambda_{\mathrm{S}} \boldsymbol{\sigma}^{\mathrm{S}} + (1 - \lambda_{\mathrm{S}}) \left( \lambda_{\mathrm{R}} \boldsymbol{\sigma}^{\mathrm{R}} + (1 - \lambda_{\mathrm{R}}) \boldsymbol{\sigma}^{\mathrm{M}} \right) \tag{3}$$

where $\lambda_{\mathrm{R}}, \lambda_{\mathrm{S}} \in [0,1]$, and

$$\sigma_i^{\mathrm{S}} = \begin{cases} \frac{1}{s}, & \text{if } i \in S(H), \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_i^{\mathrm{M}} = \begin{cases} \frac{1}{m}, & \text{if } i \in M(H), \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_i^{\mathrm{R}} = \begin{cases} \frac{l_i(HR)+1}{n-s-2m}, & \text{if } i \in R(H) \setminus (L(HR), \\ \frac{1}{n-s-2m}, & \text{if } i \in D(HR), \\ 0, & \text{otherwise,} \end{cases}$$

**Lemma 2.** *$\sigma$ is a feasible strategy for the seeker $S$.*

*Proof.* Clearly, $\boldsymbol{\sigma}^{\mathrm{S}}$ is a valid probability distribution as long as $S(H) \neq \varnothing$, that is $s > 0$. Similarly, $\boldsymbol{\sigma}^{\mathrm{M}}$ is a valid probability distribution as long as $M(H) \neq \varnothing$, that is $m \geq 1$. It is also easy to see that $\boldsymbol{\sigma}^{\mathrm{R}}$ is a valid probability distribution as long as $R(H) \neq \varnothing$. To see this, notice that $R(H)$ contains exactly $n - s - 2m$ nodes and $\boldsymbol{\sigma}^{\mathrm{R}}$ can be obtained from a uniform distribution on $R(H)$ by moving the probability mass assigned to leaves in $HR \setminus D(HR)$ to their neighbours. Lastly, notice that if $S(H) \neq \varnothing$, then either all the non-singleton nodes in $H$ have degree 1, in which case $M(H) \neq \varnothing$, or there exists a node in $H$ of degree 2 or more, in which case either $M(H) \neq \varnothing$ or $R(H) \neq \varnothing$. Hence if $S(H) \neq \varnothing$, then either $\boldsymbol{\sigma}^{\mathrm{M}}$ or $\boldsymbol{\sigma}^{\mathrm{R}}$ is a valid probability distribution. By these observations, $\boldsymbol{\sigma}$ is a valid probability distribution as long as $\lambda_{\mathrm{S}} = 1$, if $s = n$, $\lambda_{\mathrm{S}} = 0$, if $s = 0$, $\lambda_{\mathrm{R}} = 0$, if $R(H) = \varnothing$, and $\lambda_{\mathrm{R}} = 1$, if $m = 0$.

So, the lemma is true. □

The idea behind the strategy $\boldsymbol{\sigma}$ is as follows. With probability $\lambda_{\mathrm{S}}$, player S seeks in the set of singleton nodes, $S(H)$, and with probability $(1 - \lambda_{\mathrm{S}})$ he seeks outside this set. Conditional on seeking outside $S(H)$, with probability $\lambda_{\mathrm{R}}$ player S seeks in the set of nodes $R(H)$ and with probability $(1 - \lambda_{\mathrm{R}})$ he seeks in the set $SL(H) \cup M(H)$.

When seeking in $S(H)$, S mixes uniformly across all the singleton nodes. When seeking in $SL(H) \cup M(H)$, S mixes uniformly across all the nodes neighbouring a singleton leaf, that is all the nodes in $M(H)$. Lastly, when seeking in the set of nodes $R(H)$, S mixes using strategy $\boldsymbol{\sigma}^{\mathrm{R}}$.

**Lemma 3.** *The probability of capture of player* H *is at least* $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n-s-2m)$*, if* H *hides in* $R(H) \setminus (S(HR) \cup SL(HR) \cup D(HR))$.

*Proof.* Take any node $i \in R(H) \setminus (S(HR) \cup SL(HR) \cup D(HR))$. Suppose, first, that $i$ is not a leaf in $HR$, i.e. $i \in R(H) \setminus L(HR)$. Then $i$ has at least two neighbours in $R(H)$ and the probability that seeker seeks at $i$ or at one of $i$'s neighbours is at least $(1 - \lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n - s - 2m)$.

Suppose, next, that $i \in L(HR) \setminus (SL(HR) \cup D(HR))$. Then $i$ has a neighbour $j \in R(H)$ that has at least one more leaf neighbour in $HR$. Since $\sigma_j = (1 - \lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n - s - 2m)$, the lemma is true. □

We now narrow down the possible strategies, $\boldsymbol{\sigma}$, by setting the value of $\lambda_{\mathrm{R}}$. This is done under the assumption that $S(H) \neq V$, that is $s \leq n - 4$ and there exist non-singleton nodes in $H$. Let

$$\begin{aligned}\rho &= \frac{(n-s-2m)(f(n-s-2)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)} \\ &= 1 - \frac{3m(f(n-s-1)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\end{aligned}$$

and

$$\lambda_{\mathrm{R}} = \begin{cases} 0, & \text{if } R(H) = \varnothing, \\ \rho, & \text{otherwise.} \end{cases} \tag{4}$$

Clearly $\rho \in [0, 1]$ and $\lambda_{\mathrm{R}} \in [0, 1]$.

**Lemma 4.** *The probability of capture of player* H *is at least* $(1-\lambda_S)\lambda_R 3/(n-s-2m)$, *if* H *hides in* $S(HR) \cup SL(HR) \cup D(HR)$.

*Proof.* In this case, $i$ must have a neighbour, $j$, in $M(H)$. For otherwise $i$ would be a singleton node in $H$ or a singleton leaf in $H$ and so $i$ would belong to $S(H)\cup M(H)$ and not to $R(H)$. Now,

$$
\begin{aligned}
& \sigma_j \\
= \;& (1-\lambda_S)(1-\lambda_R)\left(\frac{1}{m}\right) \\
\geq \;& (1-\lambda_S)\min\left(1, \frac{3m(f(n-s-1)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\right)\left(\frac{1}{m}\right) \\
= \;& (1-\lambda_S)\left(\frac{3(f(n-s-1)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\right) \\
> \;& (1-\lambda_S)\left(\frac{3(f(n-s-2)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\right) \\
= \;& (1-\lambda_S)\lambda_R\left(\frac{3}{n-s-2m}\right).
\end{aligned}
$$

Thus $i$ is caught with probability at least $(1-\lambda_S)\lambda_R 3/(n-s-2m)$. ☐

**Lemma 5.** *Conditional on* H *hiding in a node of* $R(H)$ *and* S *using* $\sigma$, *the expected payoff of* S *is at least*

$$
\begin{aligned}
L^{R}(n,m,s) = (1-\lambda_S)\Bigg(\lambda_R\bigg(\left(\frac{3}{n-s-2m}\right)\beta - \left(1-\frac{3}{n-s-2m}\right)f(n-s-1)\bigg) \\
-(1-\lambda_R)\,f(n-s-2)\Bigg) - \lambda_S f(n-s) \quad (5)
\end{aligned}
$$

*Proof.* Suppose H hides in $R(H)$. From lemmas 3 and 4, H is captured with probability at least $(1-\lambda_S)\lambda_R 3/(n-s-2m)$ when S chooses $\sigma$. If not captured, only one node is removed when S searches in $R(H)$. With probability $(1-\lambda_S)((1-\lambda_R)$, S searches in $M(H)$ and removes two nodes. Finally, with probability $\lambda_S$, S searches in $S(H)$, and does not catch H. Then, her payoff is at least $-f(n-s)$ - this happens if $H$ is connected over $n-s$ nodes. ☐

**Lemma 6.** *Conditional on* H *hiding in a node of* $M(H) \cup SL(H)$, *player* S *by choosing* $\sigma$ *obtains a payoff of at least*

$$
\begin{aligned}
L^{M}(n,m,s) = (1-\lambda_S)\Bigg((1-\lambda_R)\bigg(\left(\frac{1}{m}\right)\beta - \\
\left(1-\frac{1}{m}\right)f(n-s-2)\bigg) - \lambda_R f(n-s-1)\Bigg) - \lambda_S f(n-s),
\end{aligned}
$$

*Proof.* The probability of capture of H is at least $(1-\lambda_S)(1-\lambda_R)1/m$. If H is not captured, S guarantees that the component of the hider has size at most $n-s-2$

with probability $(1-\lambda_{\mathrm{S}})(1-\lambda_{\mathrm{R}})$ when the attack is in $M(H)$. Also, at least one node is removed with probability $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}$ when the attack is in $R(H)$. Finally, the component containing $\mathsf{H}$ has size at most $n-s$ when the attack is in $S(H)$, and this happens with probability $\lambda_{\mathrm{S}}$. □

It is straightforward to verify that the chosen value of $\lambda_{\mathrm{R}}$ ensures that $L^{\mathrm{R}}(n,m,s) = L^{\mathrm{M}}(n,m,s)$, for any $s \in \{0,\dots,n-4\}$.

Hence the lower bound on the payoff of player $\mathsf{S}$ in $H$ when $\mathsf{H}$ hides outside singleton nodes is

$$L(n,m,s) = L^{\mathrm{R}}(n,m,s) = L^{\mathrm{M}}(n,m,s) = (1-\lambda_{\mathrm{S}})\,A(n,m,s) - \lambda_{\mathrm{S}} f(n-s) \quad (6)$$

where

$$A(n,m,s) = \begin{cases} \frac{\beta}{m} - \left(\frac{m-1}{m}\right) f(n-s-2), \text{ if } R(H)=\varnothing, \\ \left(\frac{D(n,s)D(n-1,s)}{3D(n,s)-2D(n-1,s)}\right)\left(\frac{3(\beta-T(n,s))}{m(3D(n,s)-2D(n-1,s))+(n-s)D(n-1,s)} - 1\right) + \beta, \\ \text{otherwise} \end{cases}$$

with

$$D(n,s) = f(n-s-1) + \beta$$

and

$$T(n,s) = (n-s-3)D(n,s) - (n-s-2)D(n-1,s) + \beta$$

In particular, the derivation above is valid for the extreme cases of $m=0$ and $m=(n-s)/2)$. Notice that $A(n,m,s)$ is strictly increasing in $m$ if $T(n,s)>\beta$, is strictly decreasing in $m$ if $T(n,s)<\beta$, and is constant if $T(n,s)=\beta$.

To complete the definition of strategy $\boldsymbol{\sigma}$ we establish the value of $\lambda_{\mathrm{S}}$. Conditional on $\mathsf{H}$ hiding in a node of $S(H)$, using any of the strategies $\boldsymbol{\sigma}$ defined above, player $\mathsf{S}$ obtains payoff of at least $L^{\mathrm{S}}(n,m,s) = \lambda_{\mathrm{S}} B(s) - (1-\lambda_{\mathrm{S}})\,f(1)$, regardless of the strategy of the hider, as the probability of capture is $\lambda_{\mathrm{S}}/s$ and, in the case of not capturing the hider, $\mathsf{S}$ gets payoff $-f(1)$. Let

$$\lambda_{\mathrm{S}} = \begin{cases} 1, & \text{if } s=n, \\ \frac{A(n,m,s)+f(1)}{A(n,m,s)+B(s)+f(1)+f(n-s)}, & \text{if } s\neq n \text{ and } A(n,m,s) > -f(1), \\ 0, & \text{otherwise.} \end{cases}$$

To see that $\lambda_{\mathrm{S}} \in [0,1]$, notice that $B(s) > -f(1) \geq -f(n-s)$, for any $\beta \geq 0$ and $0 \leq s \leq n-4$.

It is straightforward to verify the following for any $s \in \{0,\dots,n-4\}$:

(i) if $A(n,m,s) > -f(1)$, then $L^{\mathrm{s}}(n,m,s) = L(n,m,s)$.

(ii) if $A(n,m,s) \leq -f(1)$ then $L^{\mathrm{s}}(n,m,s) \geq L(n,m,s)$.

So, if $s \leq n-4$, the lower bound on the payoff of player $\mathsf{S}$ in $H$ is

$$Q(n,m,s) = (1-\lambda_{\mathrm{S}})A(n,m,s) - \lambda_{\mathrm{S}} f(n-s),$$

Of course, if $s = n$, $\boldsymbol{\sigma}$ mixes uniformly across the singletons with $\lambda_{\mathsf{S}} = 1$.

Thus the lower bound on the payoff of $\mathsf{S}$ in $H$, secured by strategy $\boldsymbol{\sigma}$, is

$$Q(n,m,s) = \begin{cases} B(n), & s = n \\ \frac{A(n,m,s)B(s) - f(1)f(n-s)}{A(n,m,s)+B(s)+f(1)+f(n-s)}, & \text{if } s \leq n-4 \text{ and } A(n,m,s) > -f(1), \\ A(n,m,s), & \text{otherwise,} \end{cases} \tag{7}$$

Recall that $A(n,m,s)$ is increasing in $m$ when $T(n,s) > \beta$, decreasing in $m$ when $T(n,s) < \beta$, and constant in $m$ when $T(n,s) = \beta$.

This, together with Lemma 9 in the appendix implies that when $n \leq n-4$, $Q(n,m,s)$ is decreasing in $m$ when $T(n,s) < \beta$, increasing in $m$ when $T(n,s) > \beta$, and is constant in $m$ when $T(n,s) = \beta$. So for all $s \in \{0,\ldots,n-4\}$, $Q(n,m,s)$ is minimised at $m = (n-s)/2$, when $T(n,s) < \beta$, and is minimised at $m = 0$, when $T(n,s) > \beta$.

In the last step of the proof we turn to construction of networks that are optimal for the hider.

First, we construct optimal networks for the hider, given the number of singleton nodes in the network, $s \leq n-4$.

Define a new function $\bar{Q}(n,s)$ as follows

$$\bar{Q}(n,s) = \begin{cases} Q(n,0,s), & \text{if } 0 \leq s \leq n-4 \text{ and } T(n,s) \geq \beta, \\ Q(n,(n-s)/2,s), & \text{if } 0 \leq s \leq n-4,\ T(n,s) < \beta \text{ and } n-s \text{ is even,} \\ Q(n,(n-s-3)/2,s), & \text{if } 0 \leq s \leq n-4,\ T(n,s) < \beta \text{ and } n-s \text{ is odd.} \end{cases}$$

Consider the case of $n-s$ being even first.

**Lemma 7.** *Suppose $H$ builds a network with $s$ singleton nodes such that $n-s$ is even. Then, the optimal strategy for $H$ provides $H$ with payoff $-\bar{Q}(n,s)$.*

*Proof.* Fix $s$ such that $n-s$ is even. Let

$$\bar{A}(n,s) = \begin{cases} A(n,(n-s)/2,s), & \text{if } T(n,s) < \beta, \\ A(n,0,s), & \text{if } T(n,s) \geq \beta. \end{cases}$$

and let

$$\kappa = \begin{cases} \frac{B(s)+f(1)}{\bar{A}(n,s)+B(s)+f(n-s)+f(1)} & \text{if } \bar{A}(n,s) > -f(1), \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

Let $H$ choose a network $G$ such that :

(i) $G$ has exactly $s$ singletons

(ii) $G$ is a maximum core periphery on $n-s$ nodes if $T(n,s) < \beta$

(iii) $G$ is a cycle on $n-s$ nodes if $T(n,s) \geq \beta$.

Moreover, the hider hides in the component of size $n-s$ with probability $\kappa$, mixing uniformly on the periphery nodes in the case of the component being a core-periphery network, and mixing uniformly over all its nodes in the case of the component being a cycle. Also, she hides in the singleton nodes with probability $1-\kappa$, mixing uniformly on them.

By similar arguments to those used for $\lambda_{\mathrm{S}}$ above, $\kappa \in [0,1]$ and so the strategy is valid.

If the seeker seeks in the singleton nodes, this yields payoff of at least $\kappa f(n-s) - (1-\kappa)B(s)$ to the hider. Similarly, if the seeker seeks in the core-periphery component, this yields payoff of at least $-\kappa\bar{A}(n,s)+(1-\kappa)f(1)$ to the hider. With the value of $\kappa$, above, both these guarantees are equal, in the case of $\bar{A}(n,s) > -f(1)$, and the latter is greater, otherwise.

Hence, the strategy guarantees payoff $-\kappa\bar{A}(n,s) + (1-\kappa)f(1)$ to the hider. Note that

$$-\kappa\bar{A}(n,s) + (1-\kappa)f(1) = -\bar{Q}(n,s)$$

However, we have shown that $\bar{Q}(n,s)$ is the minimal payoff the seeker can get on any network with exactly $s$ singleton nodes. Since the game is zero-sum, $-\bar{Q}(n,s)$ is the maximal payoff the hider can get on any network with exactly $s$ singleton nodes and hence the network constructed above as well as the hiding strategy must be optimal for the hider. □

Next, consider the case of $n-s$ being odd.

**Lemma 8.** *Suppose that $n-s$ is odd. Then, the optimal strategy for $H$ gives him a payoff of $-Q(n,(n-s-3)/2,s)$.*

*Proof.* Let

$$\bar{A}(n,s) = \begin{cases} A(n,(n-s-3)/2,s), & \text{if } T(n,s) < \beta, \\ A(n,0,s), & \text{if } T(n,s) \geq \beta. \end{cases}$$

and let $\kappa$ be defined as in (8). If $T(n,s) \geq \beta$ than choosing a cycle over $n-s$ nodes and using the same hiding strategy as in the case of $n-s$ being even, the hider secures the highest possible payoff on a network with exactly $s$ singleton nodes.

Suppose that $T(n,s) < \beta$. Since $(n-s)/2$ is not an integer, the hider cannot attain the upper bound on his payoff determined by the lower bound on the payoff to the seeker, $\bar{Q}(n,s)$. Recall that if $T(n,s) < \beta$ then for any $0 \leq s \leq n-4$, $Q(n,m,s)$ is decreasing in $m$. We show below for any $0 \leq s \leq n-4$, the hider can attain payoff $-Q(n,(n-s-3)/2,s)$, and that this is the maximal payoff he can secure when $n-s$ is odd.

Suppose that the hider chooses a core-periphery network with three orphaned nodes over $n-s$ nodes (c.f. Figure 4).

Consider a strategy of the hider

$$\boldsymbol{\eta} = \kappa(\mu\boldsymbol{\eta}^{\mathrm{M}} + (1-\mu)\boldsymbol{\eta}^{\mathrm{R}}) + (1-\kappa)\boldsymbol{\eta}^{\mathrm{S}},$$

where

$$\eta_i^{\mathrm{M}} = \begin{cases} \frac{1}{m}, & \text{if } i \in SL(G), \\ 0, & \text{otherwise,} \end{cases}$$

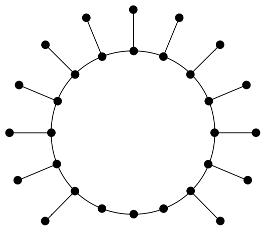


Figure 4: A core-periphery network over 23 nodes with 3 orphaned nodes.

(i.e. $\boldsymbol{\eta}^{\mathrm{M}}$ mixes uniformly on the periphery nodes of $G$),

$$\eta_i^{\mathrm{R}} = \begin{cases} 1, & \text{if } i \text{ is the middle orphaned node in } G, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta_i^{\mathrm{S}} = \begin{cases} \frac{1}{s}, & \text{if } i \in S(G), \\ 0, & \text{otherwise.} \end{cases}$$

(i.e. $\boldsymbol{\eta}^{\mathrm{S}}$ mixes uniformly on the singleton nodes of $G$), and

$$\mu = \frac{(n-s-3)f(n-s-2) + (n-s-3)\beta}{(n-s-3)f(n-s-1) + 2f(n-s-2) + (n-s-1)\beta}.$$

It is immediate to see that $\mu \in [0,1]$ and so the hiding strategy is valid. If the seeker seeks in the orphaned nodes of the core-periphery component, this yields payoff of at least $\kappa(\mu f(n-s-1) - (1-\mu)\beta) + (1-\kappa)f(1)$ to the hider and, since the game is zero-sum, of at most minus this value to the seeker. Similarly, if the seeker seeks in periphery nodes or their neighbours in the core-periphery component, this yields payoff of at least $\kappa(\mu(-2\beta/(n-s-3) + (1-2/(n-s-3))f(n-s-2)) + (1-\mu)f(n-s-2)) + (1-\kappa)f(1)$ to the hider and of at most minus this value to the seeker. With the value of $\mu$, above, both these guarantees are equal.

It is straightforward to verify that

$$\begin{aligned} \kappa(\mu f(n-s-1) - (1-\mu)\beta) + (1-\kappa)f(1) &= -\kappa A(n,(n-s-3)/2,s) + (1-\kappa)f(1) \\ &= -Q(n,(n-s-3)/2,s). \end{aligned}$$

Since $Q(n,(n-s-3)/2,s)$ is a lower bound on the payoff that the seeker can secure in a network with exactly $s$ singleton nodes and at most $(n-s-3)/2$ singleton leaves, minus this value is the highest payoff that the hider can secure in a network with exactly $s$ singleton nodes and at most $(n-s-3)/2$ singleton leaves.

The only networks that could yield a higher payoff to the seeker are networks with exactly $s$ singleton nodes and $(n-s-1)/2$ singleton leaves. In any such network, $H$, the set $R(H)$ consist of exactly one node and this node is connected to at least two nodes in $M(H)$. It cannot be connected to one node in $M(H)$, because in this case its neighbour would have two leaf-neighbours and could not be a member of $M(H)$.

Let $\tilde{\boldsymbol{\sigma}} = \lambda \boldsymbol{\sigma}^{\mathrm{S}} + (1-\lambda)\boldsymbol{\sigma}^{\mathrm{M}}$, where $\boldsymbol{\sigma}^{\mathrm{M}}$ and $\boldsymbol{\sigma}^{\mathrm{S}}$ are the mixed strategies of the seeker, defined earlier in the proof,

$$\lambda = \begin{cases} \frac{X(n,s)+f(1)}{B(s)+X(n,s)+f(1)+f(n-s)}, & \text{if } X(n,s) > -f(1), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$X(n,s) = \frac{2\beta}{n-s-1} - \left(1 - \frac{2}{n-s-1}\right) f(n-s-2).$$

Using this strategy, with probability $\lambda$, $\mathsf{S}$ mixes uniformly on the nodes in $M(H)$ and with probability $(1-\lambda)$, $\mathsf{S}$ mixes uniformly on the singleton nodes of $H$. Payoff to $\mathsf{S}$ conditioned on $\mathsf{H}$ hiding in a singleton node is at least $\lambda B(s) - (1-\lambda)f(1)$ and payoff to $\mathsf{S}$ conditioned on $\mathsf{H}$ hiding outside singleton nodes is at least $(1-\lambda)X(n,s) - \lambda f(n-s)$. It is easy to verify that the value of $\lambda$ is such that both these payoffs are equal (in the case of $X(n,s) > -f(1)$) or the latter is higher, for any value of $\lambda$. Therefore payoff to $\mathsf{S}$ from using $\tilde{\boldsymbol{\sigma}}$ against any strategy of $\mathsf{H}$ is at least

$$Y(n,s) = \begin{cases} \frac{B(s)X(n,s)-f(1)f(n-s)}{B(s)+X(n,s)+f(1)+f(n-s)}, & \text{if } X(n,s) > -f(1), \\ X(n,s), & \text{otherwise,} \end{cases}$$

and so the upper bound on the payoff to the hider on any network with $s$ singleton nodes and $(n-s-1)/2$ singleton leaves is at most $-Y(n,s)$. To see that $-Q(n,(n-s-3)/s,s) > -Y(n,s)$ notice that

$$X(n,s) - A(n,(n-s-3)/2,s) = \frac{2(f(n-s-1)-f(n-s-2))(f(n-s-2)+\beta)(n-s-3)}{(n-s-1)(f(n-s-1)(n-s-3)+2f(n-s-2)+\beta(n-s-1))} > 0$$

and so $X(n,s) > A(n,(n-s-3)/2,s)$. This, together with Lemma 9 (in the Appendix), implies that $Y(n,s) > Q(n,(n-s-3)/2,s)$. Hence $-Q(n,(n-s-3)/2,s)$ is the exact upper bound on the payoff that a hider can secure by choosing a network with exactly $s$ singleton nodes when $n-s$ is odd. $\square$

Since the game is zero-sum, the hider maximises his payoff when the seeker's payoff is minimised. Therefore, an optimal network has $s \in S^*(n)$ singleton nodes, where

$$S^*(n) = \arg\min_{s\in\{0,\ldots,n\}} \bar{Q}(n,s).$$

Lemmas 7 and 8 have therefore proved our main result.

**Theorem 1.** *For any number of nodes, $n \geq 1$, and any $\beta \geq 0$ there exists an equilibrium of the game, $((G,h),s)$ such that*

- *$G$ has exactly $s \in S^*(n)$ singleton nodes and either $s \leq n-4$ or $s = n$.*
- *If $T(n,s) \geq \beta$ and $n-s \geq 4$ then $G$ has a cycle component over the remaining $n-s$ nodes.*

- *If $T(n,s) < \beta$, $n-s \geq 4$, and $n-s$ is even then $G$ has a maximal core-periphery component over $n-s$ nodes.*
- *If $T(n,s) < \beta$, $n-s \geq 4$, and $n-s$ is odd then $G$ has a core-periphery component with three orphaned nodes over $n-s$ nodes.*
- *The hider mixes between hiding in the singleton nodes and in the connected component. When hiding in the singleton nodes, he mixes uniformly across all these nodes. When hiding in the connected component, he mixes uniformly across all the nodes (when it is a cycle), mixes uniformly across the periphery nodes (when it is a maximal core-periphery network), and mixes between hiding in periphery nodes, mixing uniformly across them, and the middle orphaned node (otherwise).*
- *The seeker mixes between seeking in the singleton nodes and in the connected component. When seeking in the singleton nodes, he mixes uniformly across all these nodes. When seeking in the connected component, he mixes uniformly across all the nodes (when it is a cycle), mixes uniformly across the core nodes (when it is a maximal core-periphery network), and mixes between seeking in the neighbours of periphery nodes, mixing uniformly across them, and the middle orphaned node (otherwise).*

*Equilibrium payoff to the hider is $-\bar{Q}(n,s)$.*

We have shown in the proof of Theorem 1, that equilibrium payoff to the seeker in an optimal network with at least one singleton node is a convex combination of $B(s)$ (which, as we show, is greater than $-f(1)$) and $-f(1)$ and so it is at least $-f(1)$. Hence the payoff that the hider can secure in such a network is at most $f(1)$. Thus if the payoff the seeker can secure in a connected component of size $n$, $\bar{A}(n,0) < -f(1)$, then the payoff the hider can secure in such a component is $-\bar{A}(n,0) > f(1)$. Therefore it is optimal for the hider to choose a connected network without singleton nodes in such a case.

If, on the other hand, the cost of being caught, $\beta$, is sufficiently high then $\bar{A}(n,0) > -f(1)$ and the payoff the hider can secure in a connected network, $-\bar{A}(n,0)$, is less than the payoff he gets if he is not caught in a singleton node. This motivates the hider to construct a network with a smaller component and $s \geq 1$ singleton nodes. If the cost of being caught is sufficiently high, it is optimal for the hider to choose a disconnected network with $s = n$ singleton nodes.

Theorem 1 provides characterization of an optimal network for the hider in terms of $T(n,s)$. We now use the theorem to provide a (partial) characterization in terms of the function $f$, which is of course an exogenous parameter of the model.

**Theorem 2.** *Suppose that either*

*(i) $f$ is concave, or*

*(ii) $f$ is convex and for all $x \geq 2$*

$$f(x+1) < \frac{x}{x-1} f(x)$$

*Then, for all $n \geq 1$, and any $\beta \geq 0$ there exists an equilibrium of the game, $((G,h),s)$ such that*

- *$G$ has exactly $s^*$ singleton nodes with $s^* \in \{0, 1, n\}$.*
- *If $n - s^* \geq 4$ and is even then $G$ has a maximal core-periphery component over $n - s^*$ nodes*
- *If $n - s^* \geq 4$ and is odd then $G$ has a core-periphery component with three orphaned nodes over $n - s^*$ nodes.*

*Proof.* Notice that

$$T(n, s) = (n - s - 3)\Delta f(n - s - 2) - f(n - s - 2)$$

and

$$T(n, s + 1) = (n - s - 3)\Delta f(n - s - 3) - f(n - s - 2)$$

Hence,

$$\begin{aligned} T(n, s + 1) - T(n, s) &= -(n - s - 3)(\Delta f(n - s - 2) - \Delta f(n - s - 3)) \\ &= -(n - s - 3)\Delta^2 f(n - s - 3). \end{aligned}$$

where $\Delta f(x) = f(x+1) - f(x)$ is the first-order (forward) difference of $f$ at $x$ and $\Delta^2 f(x) = \Delta f(x + 1) - \Delta f(x)$ is the second-order (forward) difference of $f$ at $x$. Hence, if $f$ is concave, then $\Delta^2 f(n - s - 3) \leq 0$, and so

$$T(n, s + 1) - T(n, s) \geq 0 \text{ for all } s \leq n - 4$$

In addition $T(n, n-4) = f(3) - 2f(2)$ which is negative if $f$ is concave and strictly increasing. Thus for all $n \geq 4$ and $s \leq n - 4$, $T(n, s) < 0 \leq \beta$.

From Theorem 1, the core-periphery component over $n - s$ nodes is strictly better for the hider than the cycle component over $n - s$ nodes.

If $f$ is convex then $\Delta^2 f(n - s - 3) \geq 0$ and $T(n, s + 1) - T(n, s) \leq 0$, for all $s \leq n - 4$. Thus $T(n, s)$ is decreasing in $s$ on $[0, n - 4]$, for all $n \geq 4$.

Suppose $f(x + 1) < x/(x - 1)f(x)$ for all $x \geq 2$.[6] Then $T(n, 0) = (n - 3)f(n - 1) - (n - 2)f(n - 2) < 0$ and so

$$T(n, s) \leq T(n, 0) < \beta, \text{ for all } s \in [0, n - 4].$$

Again, this shows that the core periphery is better for the hider than the cycle over $n - s$ nodes.

Next, note that if $n \leq 5$, then Lemma 1 shows that $s^* \leq 1$.

Suppose that $n \geq 6$. We show in the Appendix (Lemma 10) that if $n \geq 6$, then $\bar{Q}(n, s)$ is either minimised at $s = 0$ or it is minimised at $s = n$.

This shows that $s^* \in \{0, 1, n\}$ and completes the proof of the theorem. □

**Remark 2.** *So, the theorem shows that if $f$ is concave or even convex but grows slowly, then the core periphery is better than the cycle.*

[6] An example of a strictly increasing convex functions that satisfy this property are $f(x) = x^\gamma/(x + 1)^{\gamma-1}$ with $\gamma > 1$.

Consider convex $f$ with $(n-3)f(n-1)-(n-2)f(n-2) > 0$. If $\beta > (n-3)f(n-1)-(n-2)f(n-2)$ then $T(n,s) \leq T(n,0) < \beta$ for all $s \in [0, n-4]$ and core-periphery component over $n-s$ nodes is strictly better for the hider than the cycle component over $n-s$ nodes. If $f(3)-2f(2) < \beta < (n-3)f(n-1)-(n-2)f(n-2)$ then there exists $z \in [0, n-s]$ such that for all $s \in [0, z)$, $T(n,s) > \beta$ and cycle component is better than core-periphery component for the hider, and for all $s \in (z, n-4]$, $T(n,s) < \beta$ and core-periphery component is better than cycle component for the hider. If $f(3)-2f(2) > \beta$, then for all $n \geq 4$ and $s \leq n-4$, $T(n,s) \geq T(n, n-4) > \beta$ and cycle component over $n-s$ nodes is better for the hider than the core-periphery component over $n-s$ nodes.

Notice that

$$T(n+1,s) - T(n,s) = (n-s-2)\Delta^2 f(n-s-2).$$

Hence $T(n,s)$ is increasing in $n$ for all $s \in [0, n-4]$ when $f$ is convex. If $(n-s-2)\Delta^2 f(n-s-2)$ is unbounded (which holds when the second derivative of $f$ is increasing) then, with sufficiently large $n$, $T(n,s) > \beta$ and cycle component is better than core-periphery component for the hider. Intuitively, with larger $n$ and fixed $s$, $n-s$ is larger and so, eventually, the marginal loss from loosing a single node from a component becomes large enough to make the hider prefer to construct a better connected cycle instead of a core-periphery component, and risk being caught with higher probability.

# 5 Extension to $k > 1$

In this section, we allow for $S$ to have more seeking units. That is, $k > 1$. Throughout this section, we assume that $f$ is linear, and $n$ is even.

Consider the class of core periphery networks where nodes in the core periphery form a *clique*.

We also restrict attention to the class of networks such that $\min_{i \in N} d_i(g) \geq 3$. That is, every node has at least 3 neighbours.[7]

Assume that $S$ searches *uniformly* over all $k$-element subsets of $V$. We are aware that this is an inefficient strategy, but have not been able to develop a better strategy so far.

We first compute the probability of $H$ being caught under this uniform strategy.

$H$ is not captured if neither $H$ nor any neighbour of $H$ is inspected.

The probability that any node $i$ is not captured equals

$$\frac{\binom{n-d-1}{k}}{\binom{n}{k}} = \frac{(n-d-1)!(n-k)!}{n!(n-d-1-k)!}$$

where $d$ is degree of $i$.

The idea here is as follows: there are $n-d-1$ nodes in $V$ that are neither $i$ nor a neighbour of $i$ and $i$ is not captured iff we pick a $k$-element subset of these nodes.

[7]We conjecture that we can soon dispense with this restriction. We can also show that the core periphery is strictly better than the circle.

So, if $d \geq 3$, the probability that $h$ is not captured is at most

$$\frac{(n-d-1)!(n-k)!}{n!(n-4-k)!} = \frac{(n-k)(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)}$$

Conversely, the probability that $i$ is captured is *at least*

$$P(k) = 1 - \frac{(n-k)(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)}.$$

The expected payoff to $S$ on core periphery is

$$E(S, cp) = \left(\frac{2k}{n}\right)\beta - \frac{(n-2k)^2}{n}. \tag{9}$$

The expected payoff to $S$ on any network where the seeker captures the Hider with probability $p$ or more is at least

$$\underline{S}(p) = p\beta - (1-p)(n-k) \tag{10}$$

Substitute $p = P(k)$ , and assume minimal disruption so that $H$ is in a component of size $n-k$. Clearly, the resulting payoff to $S$ is a lower bound on the expected payoff to $S$ on any network that is not the core periphery. For any $\beta \geq 0$ and $n \geq 4k$, this lower bound is greater than $E(S, cp)$.

Hence, the core periphery is the unique optimal network in the class of connected networks.

# 6 Conclusions

We proposed and studied a strategic model network design and hiding in the network facing a hostile authority that attempts to disrupt the network and capture the hider. We characterized optimal networks for the hider as well as optimal hiding and seeking strategies in these networks. Our results suggests that the hider chooses networks that allow him to be anonymous and peripheral in the network. We also developed a technique for solving such models in the setup of zero-sum games.

There are at least two avenues for future research. Firstly, different forms of benefits from the network could be considered. For example, the utility of the hider could dependent not only on the size of his component but also on his distance to the nodes in the component. Given our results, we conjecture that this would make the core periphery components with better connected core more attractive. But answering this problem precisely requires formal analysis. Secondly, the seeker could be endowed with more than one seeking unit and the units could be used either simultaneously or sequentially. Our initial investigation suggests that solving such an extension might be an ambitious task.

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# Appendix

# A Proofs

**Lemma 9.** *Function*

$$\varphi(Z) = \begin{cases} \frac{B(s)Z - f(1)f(n-s)}{Z + B(s) + f(n-s) + f(1)}, & \text{if } Z > -f(1), \\ Z, & \text{otherwise,} \end{cases}$$

*is strictly increasing in $Z$.*

*Proof.* Notice that $\varphi(-f(1)) = -f(1)$ when $Z = -f(1)$. Moreover, $\varphi$ is increasing in $Z$ if $Z < -f(1)$. Let $Z > -f(1)$. Taking the derivative of $\varphi$ with respect to $Z$ we get

$$\varphi'(Z) = \frac{(B(s) + f(1))(B(s) + f(n-s))}{(Z + B(s) + f(n-s) + f(1))^2}$$

and it is immediate to see that $\varphi'(Z) > 0$ and $\varphi$ increases in $Z$ when $B(s) > -f(1)$ and $B(s) \geq -f(n-s)$. Notice that $B(s) = (\beta + f(1))/s - f(1) > -f(1)$ for any $\beta \geq 0$ and $s > 0$. Also $f(n-s) \geq f(1)$ for all $s \in [0, n-1]$. Thus, by the observation on function $\varphi$, above, $\varphi(Z)$ increases when $Z$ increases. $\square$

**Lemma 10.** *Let $f(x) = x$, for all $x \in \mathbb{R}_{\geq 0}$. For any natural $n \geq 6$, $t \in \{0, 1\}$ and any $s \in \{t+1, \dots, n\}$, $\bar{Q}(n, s) > \min(\bar{Q}(n, n), \bar{Q}(n, t))$*

*Proof.* When $f(x) = x$,

$$\bar{A}(n, s) = 2\left(\frac{\beta - 2}{n - s}\right) + 4 - (n - s), \text{ for } 0 \leq s \leq n - 2,$$

and

$$\bar{Q}(n, s) = \begin{cases} \bar{A}(n, s), & \text{if } \bar{A}(n, s) \leq -1 \text{ or } s = 0, \\ AB(n, s), & \text{if } 1 \leq s \leq n - 2 \text{ and } \bar{A}(n, s) > -1 \\ B(n), & \text{otherwise,} \end{cases} \tag{11}$$

with

$$AB(n, s) = (1 - \rho)\bar{A}(n, s) - \rho(n - s)$$

where $\rho$ solves

$$(1 - \rho)\bar{A}(n, s) + \rho(s - n) = \rho B(s) - (1 - \rho). \tag{12}$$

Notice that $\bar{A}(n, s)$ is increasing in $s$ on $[0, n-2]$ and it is equal to $\beta$ at $s = n-2$. Thus there exists a unique $\tilde{s} \in [0, n-2]$ such that $A(n, \tilde{s}) = -1$. Solving (12) we get

$$\rho = \frac{s(2(\beta - 2) - (n - s)(n - s - 5))}{s(2(\beta - 2) - (n - s)(n - s - 5)) + (n - s)(s(n - s - 1) + \beta + 1)}.$$

Notice that $2(\beta - 2) - (n - s)(n - s - 5) > 0$ if and only if $\bar{A}(n, s) > -1$, and $(n - s)(s(n - s - 1) + \beta + 1) > 0$ for $s \leq n - 1$. Thus if $\bar{A}(n, s) > -1$ then $\rho \in (0, 1)$. In addition $B(s) > -1$, for all $s > 0$, so if $\rho \in (0, 1)$ then $AB(n, s) > -1$.

By the observations above, if $\bar{A}(n,1) \le -1$ then $\bar{Q}(n,0) < \bar{Q}(n,1) < \bar{Q}(n,s)$ for all $s \in \{2,\dots,n\}$ and the claim of the lemma holds.

For the remaining part of the proof suppose that $A(n,1) > -1$. This implies $2(\beta-2) > (n-1)(n-6)$ and, consequently, $\beta > 2$ if $n \ge 6$. We will show that $\bar{Q}(n,s)$ is either decreasing or first increasing and then decreasing on $[0,n-1]$. On $[0,\tilde{s}]$, $\bar{Q}(n,s) = \bar{A}(n,s)$ and, as we argued above, $\bar{Q}(n,s)$ is increasing. Consider the interval $[\tilde{s},n-1]$. Notice that since $B(s) > -1 \ge n-s$, for $0 < s \le n-1$, and $\bar{A}(n,\tilde{s}) = -1$ so $AB(n,\tilde{s}) = -1$. In addition, $AB(n,n) = B(n)$. We will show that $AB(n,s)$ is either decreasing or first increasing and then decreasing on $[0,n]$. Inserting $\rho$ into (11) we get

$$\bar{Q}(n,s) = \frac{(n^2(\beta+1) - 2n(s(\beta-1)+2(\beta+1)) + s^2(\beta-3) + 6s\beta - 2(\beta+1)(\beta-2))}{s(4s-\beta+5) - n(4s+\beta+1)}.$$

Notice that $\bar{Q}(n,\tilde{s}) = A(n,\tilde{s}) = -1$. Taking the derivative of $\bar{Q}$ with respect to $s$ we get

$$\frac{\partial \bar{Q}}{\partial s} = \frac{(\beta+1)W(s)}{(s(4s-\beta+5) - n(4s+\beta+1))^2},$$

where

$$W(s) = Xs^2 - 2Ys + \left(n + \frac{\beta-2}{2}\right)Y - \left(\frac{\beta-2}{2}\right)(n-4)(\beta+1),$$

with $X = 4n - \beta - 15$ and $Y = 4n^2 + n(\beta-19) - 8(\beta-2)$.

The sign of $\partial\bar{Q}/\partial s$ is the same as the sign of $W(s)$. Notice that $W(n) = -2(\beta-2)(n+\beta-5) < 0$, as $n \ge 6$ and $\beta > 2$. When $X > 0$, then $W(s)$ is an $\bigcup$-shaped parabola and, since $W(n) \le 0$, either $W$ is negative or $W$ is first positive and the negative on $[0,n]$. Thus in this case $\bar{Q}$ is either increasing or first increasing and then decreasing on $[0,n]$. Similar observation holds when $X = 0$. Suppose that $X < 0$. In this case $W(s)$ is an $\bigcap$-shaped parabola and it has a maximum at $s^* = Y/X$. Suppose that $s^* \in (0,n-2)$. Since $X < 0$ so $Y < 0$. Moreover, for $n \ge 6$, $X < 0$ implies $\beta > 5$ and, consequently, Moreover,

$$\begin{aligned} W(s^*) &= -Ys^* + \left(n + \frac{\beta-2}{2}\right)Y - \left(\frac{\beta-2}{2}\right)(n-4)(\beta+1) \\ &= \left(n - s^* + \frac{\beta-2}{2}\right)Y - \left(\frac{\beta-2}{2}\right)(n-4)(\beta+1) < 0. \end{aligned}$$

Thus $W$ is either negative or first positive then negative on $[0,n]$, for any natural $n \ge 5$. Hence $\bar{Q}$ is either decreasing or first increasing and then decreasing on $[0,n]$, for any natural $n \ge 6$.

By the analysis above, when $A(n,1) > -1$ then $AB(n,s)$ is either decreasing or first increasing and then decreasing in $s$ on $[0,n]$ and $AB(n,n) = B(n)$. Hence, by the definition of $\bar{Q}(n,s)$, the claim of the lemma follows immediately. $\square$