Asset Price Beliefs
And Optimal Monetary Policy

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Abstract

We characterize optimal monetary policy when agents learn about endogenous asset prices. Learning leads to inefficient asset price fluctuations and distortions in consumption and investment decisions. We find that the policy-relevant natural real interest rate increases with subjective asset price beliefs. Optimal monetary policy raises interest rates when expected capital gains are high, but does not eliminate deviations of asset prices from their fundamental value. When the asset is not in fixed supply, optimal policy also "leans against the wind". In a simple calibration of the model, a positive response to capital gains in simple interest rate rules is beneficial.

Keywords: Optimal Monetary Policy, Asset Prices, Natural Real Interest Rate, Learning, Leaning Against The Wind

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The question of how, if at all, monetary policy should react to asset prices remains controversial. Some argue that asset price misalignments can pose significant risks to macroeconomic and financial stability, and that monetary policy should raise interest rates when asset prices are high; others argue that monetary policy should not pay attention at all to asset prices, or at most in order to improve forecasts of inflation and economic activity.[1]

Any answer to this question depends on what is assumed about the sources of asset price fluctuations. Are financial markets pricing assets efficiently, and if not, what is the nature of price misalignments? Standard macroeconomic models, including workhorse New-Keynesian models used for monetary policy analysis, embody the efficient market hypothesis that rules out asset price misalignments by design. Gali (2014, 2017) has added rational bubbles to these models, arguing for a negative reaction of interest rates to asset prices because rational bubbles grow more slowly at lower interest rates. Yet rational bubbles are not the only way through which asset prices can deviate from their "fundamental value". An alternative narrative holds that investors' expectations suffer from extrapolative bias and can suffer from bouts of over- and under-confidence which affect prices. This narrative can be formalized through models of learning, which are a plausible explanation for many well-known asset price characteristics (Fuster *et al.* 2012; Collin-Dufresne *et al.* 2013; Adam *et al.* 2015; Barberis *et al.* 2015 for stock prices; Adam *et al.* (2012); Caines (2016); Glaeser and Nathanson (2017) for house prices). Importantly, these models can explain the systematic bias in return expectations observed in survey data, which is inconsistent with any rational expectations model (Greenwood and Shleifer, 2014). However, the effect of asset price learning on the conduct of optimal monetary policy has not been studied previously.

In this paper, we analytically solve for optimal monetary policy in a model with subjective, extrapolative beliefs about endogenous asset prices. The model is a simple New-Keynesian model, to which we add a long-term asset and learning about the equilibrium asset price. The learning process implies extrapolative expectations and endogenous boom-bust cycles in equilibrium price dynamics.

[1] See, for example, Bernanke and Gertler (2001), Gilchrist and Leahy (2002), Christiano *et al.* (2010), Filardo and Rungcharoenkitkul (2016) and Svensson (2017).

We keep expectations close to rational by restricting them to be model-consistent conditional on subjective asset price beliefs (Winkler, forthcoming).[2] Agents remain forward-looking and correctly forecast the fundamental state of the economy. They also understand the monetary transmission mechanism and the policy strategy followed by the central bank, except for its effects on asset prices. Despite this restriction, fluctuations in subjective asset price expectations have real effects because they distort intertemporal choices of consumption and investment: Optimistic expectations create excess aggregate demand by increasing subjective wealth, and create excess asset production by increasing subjective returns on investment.

We show that the policy-relevant natural real rate of interest is not simply a function of technology and preferences, but depends positively on subjective asset price beliefs. The intuition for this result is simple: When agents expect larger capital gains on the asset, then the return on bonds must also rise for the bond market to clear, even if the capital gains expectations aren't rational. Because realized asset prices in our model are also increasing in subjective beliefs, the model gives rise to a positive relationship between the level of asset prices and the natural real interest rate. In order to follow this natural rate, the central bank needs to raise interest rates when asset prices are high.

In terms of target criteria, flexible inflation targeting remains optimal under learning when the asset is in fixed supply. When we allow for production of the asset, the optimal policy instead "leans against the wind": The central bank is willing to tolerate low inflation and output when agents are overly optimistic about asset prices, so as to mitigate investment distortions arising from subjective belief fluctuations. By contrast, flexible inflation targeting remains optimal under rational expectations.

Finally, we numerically evaluate simple Taylor-type interest rate rules with a reaction to asset prices. We find that a positive reaction to capital gains mitigates the distortions from non-rational beliefs and stabilizes asset price fluctuations. Raising interest rates when subjective beliefs are overly optimistic reduces

[2]Conditionally model-consistent expectations require that beliefs be consistent with all equilibrium conditions other than asset market clearing. The concept has also been applied in Caines (2016) and Gandré (2017).

the price of our long-term asset, thereby correcting expectations downwards. This finding is in contrast to the rational bubble literature (Gali, 2014; Dong *et al.*, 2017), where raising rates when bubbles are large makes them larger still.

Our baseline model assumes a particular process of expectation formation in which agents think that asset prices are a random walk with a small time-varying drift. This process has been shown to fit the data well (Adam *et al.*, 2017), but our results are robust to alternative choices. We show how our results carry over to a more general class of subjective beliefs, including "natural expectations" (Fuster *et al.*, 2012) and "diagnostic expectations" (Bordalo *et al.*, 2018). Another assumption we make is that expectations are model-consistent conditionally on asset price beliefs. This restriction greatly reduces the degrees of freedom for boundedly rational expectations. However, we do not mean to imply that belief distortions or learning about other aspects of the economy, such as inflation or even monetary policy itself, do not matter. Rather, we see this assumption as a modeling device that allows us to isolate the effects of asset price learning from other such distortions.

When we analyze optimal policy, we assume that the central bank has complete information and maximizes welfare under the equilibrium probability distribution of our model, which is distinct from the subjective distribution of boundedly rational agents. Effectively, the central bank makes its choices given its own view of the economy, not that of the private sector. In practice, central banks' views on asset prices do indeed diverge at least sometimes from those of financial markets, as exemplified in Alan Greenspan's "irrational exuberance" comments (Greenspan, 1996). If the central bank instead had the same beliefs as the private sector, the policy problem in our model would revert to that of a relatively standard, rational expectations New-Keynesian model, which has been extensively studied in the literature.

In our model, subjective asset price beliefs have real effects because they affect consumption and investment through changes in subjective wealth and expected returns. We abstract from more complex transmission channels of asset prices, such as credit constraints and balance sheet effects. The advantage of this simplification is that we are able to obtain closed form solutions for optimal monetary policy in the presence of learning.

Previous studies have argued for a monetary policy reaction to asset prices in environments without rational expectations, for example Dupor (2005) and Mertens (2011). In these studies, beliefs about the exogenous fundamentals of the economy are distorted, and affect welfare through an investment channel. To our knowledge, our analysis is the first in which beliefs about the asset price itself—an endogenous variable—are distorted. We find departures from the optimal policy under rational expectations even when there is no investment channel.

Our analysis is related to Adam and Woodford (2018), who study "robustly optimal policy" in a New-Keynesian model with housing quite similar to ours.[3] Like the papers previously mentioned, Adam and Woodford study distortions to beliefs about the exogenous fundamentals of the economy, while our agents learn about the endogenous asset price. But they also set themselves a different policy problem, in which the class of possible belief distortions is large and the policymaker does not know which of these distortions is realized, thus limiting the policymaker's ability to exploit distorted expectations to its advantage. In our paper, the central bank is certain of the belief distortions in the private sector. We also limit the degree to which it can exploit them, by ruling out that the central bank pursues a different policy targeting rule from what the private sector believes the targeting rule to be. Another difference is that Adam and Woodford find benefits of leaning against the wind only if the steady state is distorted in a particular direction. In contrast, our results are obtained around the fully efficient steady state.

We also complement a growing literature studying monetary policy prescriptions in models with learning. Fully optimal policy has recently been studied in a two-equation model with learning by Molnar and Santoro (2014) and Eusepi and Preston (2018). Eusepi *et al.* (2018) introduce drift in long-run expectations to a New Keynesian model and show that such beliefs introduce a policy tradeoff between stabilizing current inflation and anchoring long-horizon beliefs. In these papers, asset prices are absent and it is mostly learning about the inflation process

[3] One important difference is that utility is linear in asset holdings in their model. This effectively implies that wealth does not enter the model as a state variable. In our learning model, changes in subjective wealth are the key distortion induced by non-rational asset price expectations, and hence we cannot make the same assumption. Instead, we solve for optimal policy carrying endogenous state variables.

that drives the dynamics of the model.[4] Instead, we focus on non-rational asset price beliefs, while endowing agents with conditionally consistent beliefs about the rest of the economy.

Our analysis shares some common ground with previous studies on asset prices and monetary policy that stay within the paradigm of rational expectations. Christiano *et al.* (2010) study the optimal policy reaction to news shocks about future productivity. Good news raises both asset prices and the natural real rate of interest, so that monetary policy should optimally respond by raising interest rates. Our natural real rate is similarly increasing in subjective beliefs, but these beliefs evolve endogenously and do not rely on exogenous news shocks. Gilchrist and Saito (2009) analyze simple interest rate rules in a credit friction model in which the private sector has limited information about the trend growth rate of technology. They find that a reaction to the growth rate of asset prices in the policy rule is beneficial, while a reaction to the level of asset prices is not. Even though we abstract from credit frictions, the same result obtains in our model, too. The reason is that a reaction to the growth rate of asset prices approximates return expectations that enter the natural real rate of interest.

The remainder of this paper is structured as follows. We begin by describing the baseline model in Section 1, and our notion of a learning equilibrium in Section 2. We characterize the linearized equilibrium under rational expectations and learning in Section 3. Optimal policy is analyzed in Section 4, while Section 5 discusses how well certain simple interest rate rules approximate the optimal policy. Sections 6 and 7 discuss extensions to more general beliefs and asset production, respectively. Section 8 concludes.

1 Model description

Our model is an otherwise standard New-Keynesian model in which the representative household holds a stock of a long-term asset that yields utility. In the baseline version of the model, the supply of the asset is fixed. One can think of

[4] Airaudo (2016) augments the standard New Keynesian model with a stock market and infinite-horizon learning as in Preston (2006) to study conditions under which the rational expectations equilibrium is learnable, but stops short of characterizing optimal policy.

this asset as a stock of housing, but we will refer to it generically as a long-term asset.

A representative household supplies labor and owns firms. It can also hold nominal bonds promising a nominal return i_t. In addition, the household owns the long-term asset. The household's problem is

$$\mathbb{E}^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} + \chi \frac{H_t^{1-\theta}}{1-\theta} \right)$$

$$\text{s.t. } C_t = W_t N_t + \Pi_t + T_t - Q_t \left(H_t - H_{t-1} \right) + B_t - \frac{1+i_{t-1}}{1+\pi_t} B_{t-1}.$$

Here, C_t is the household's utility from consuming final consumption goods, N_t is the household's labor supply, and T_t are lump-sum taxes. Π_t are the profits received from firms. The quantity of the asset owned by the household is denoted H_t and trades at the price Q_t. B_t are government bonds which are in zero net supply. The price level is P_t and $\pi_t = P_t/P_{t-1} - 1$ is the inflation rate. The expectational operator $\mathbb{E}^{\mathcal{P}}$ has a superscript indicating that agents' expectations are evaluated under a subjective probability measure $\mathcal{P}$.

The first order conditions are:

$$W_t = C_t^{\gamma} N_t^{\phi}$$

$$1 = \beta \mathbb{E}_t^{\mathcal{P}} \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \frac{1+i_t}{1+\pi_{t+1}}$$

$$Q_t = \chi \frac{C_t^{\gamma}}{H_t^{\theta}} + \beta \mathbb{E}_t^{\mathcal{P}} \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} Q_{t+1}.$$

On the production side, a representative intermediate goods producer transforms household labor into intermediate goods using the decreasing returns to scale technology

$$Y_t = A_t N_t^{\alpha}.$$

It has to hire labor at the real wage rate w_t and sells its goods at the real price M_t. Its first-order condition is

$$W_t = \alpha M_t A_t N_t^{\alpha-1}.$$

Intermediate goods are bought by wholesale firms indexed by $i \in [0,1]$, who transform them into differentiated wholesale goods using a one-for-one technology. They face a standard Dixit-Stiglitz demand function and a Calvo price setting friction. When producer i is able to set a price P_{it} for its output Y_{it}, it solves:

$$\max_{P_{it}} \mathbb{E}_t^{\mathcal{P}} \sum_{s=0}^{\infty} \left(\prod_{\tau=1}^{s} \xi \Lambda_{t,t+\tau} \right) \left((1+\tau_t) P_{it} - M_{t+s} P_{t+s} \right) Y_{it+s}$$

$$\text{s.t. } Y_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\sigma} \tilde{Y}_t,$$

where σ is the demand elasticity of substitution between varieties, $\Lambda_{t,t+\tau} = \beta^{\tau} C_t^{\gamma} C_{+\tau}^{-\gamma}$ is the household discount factor between times t and $t+\tau$, and ξ is the probability of not being able to adjust the price in the future. Any profits are distributed to households. The first-order conditions are standard and give rise to the New-Keynesian Phillips curve.

The term τ_t is a government subsidy to revenue. Its steady state value $\bar{\tau} = (\sigma - 1)/\sigma$ is set so as to induce a zero steady-state markup, thus rendering the model's steady state fully efficient. Time-varying shocks to the subsidy τ_t act as cost-push shocks that affect markups and inflation but leave the first-best allocation unchanged.

A representative retailer buys differentiated wholesale goods at prices $(P_{it})_{i \in [0,1]}$ and transforms them back into a homogeneous final consumption good. The final good sells at price P_t and is produced according to the technology

$$\tilde{Y}_t = \left(\int_0^1 (Y_{it})^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}.$$

The first order condition gives rise to the CES demand function above. The price level can be expressed as $P_t = \int_0^1 P_{it} Y_{it} / \tilde{Y}_t$.

The government transfers a lump sum real amount to households

$$T_t = \tau_t \int_0^1 P_{it} Y_{it} di$$

to finance the subsidies to final goods producers and offset the tax on stock holdings. Profits and government transfers sum up to $\Pi_t + T_t = Y_t - W_t N_t$. Finally, the central bank sets the nominal interest rate, to be specified later.[5]

[5] Throughout the paper, we will assume that the central bank specifies monetary policy to guarantee uniqueness and determinacy of the equilibrium. We also abstract from the zero lower bound on nominal interest rates.

Aggregate fluctuations are caused by productivity and, potentially, cost-push shocks, which follow first-order autoregressive processes:

$$\begin{aligned}\log A_t &= (1-\rho_a)\log \bar{A} + \rho_a \log A_{t-1} + \varepsilon_{at} \\ \tau_t &= (1-\rho_\tau)\,\bar{\tau}_t + \rho_\tau \tau_{t-1} + \varepsilon_{\tau t}\end{aligned}$$

The innovations are independent white noise with variances σ_A^2 and σ_τ^2.

Market clearing in the final goods market requires $\tilde{Y}_t = C_t$. Bonds are in zero net supply and the market clearing condition is therefore $B_t = 0$. Finally, the supply of the long-term asset is fixed at unity, so that asset market clearing requires $H_t = 1$.

2 Definition of equilibrium

Let us first recall the formal definition of a rational expectations (RE) equilibrium. Let $y_t \in \mathbb{R}^N$ denote the collection of all endogenous model variables—including prices, allocations, and strategies—and by $u_t \in \mathbb{R}^M$ the collection of all exogenous model variables, i.e. the technology and the cost-push shock, which we call "fundamentals". Stochastic processes for y_t and u_t are defined on the spaces $\Omega_y = \Pi_{t=0}^{\infty}\mathbb{R}^N$ and $\Omega_u = \Pi_{t=0}^{\infty}\mathbb{R}^M$, respectively. Further, denote by $\Omega_u^{(t)}$ the set of all possible histories of exogenous variables up to period t, and its elements by $u^{(t)} \in \Omega_u^{(t)}$. Finally, let $\mathbb{P}_u$ denote the true probability measure for the exogenous variables defined on $(\Omega_u, \mathcal{S}(\Omega_u))$, where $\mathcal{S}(\cdot)$ is the Borel sigma algebra on a metric space. The topological support of $\mathbb{P}_u$ is denoted by $\operatorname{supp}(\mathbb{P}_u)$.

Definition 1. A *rational expectations equilibrium* is a sequence of mappings $g_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \ldots$ such that, for all t and $u^{(t)} \in \operatorname{supp}(\mathbb{P}_u)$:

1. the choices contained in y_t solve the time-t decision problem of each agent in the economy, conditional on decision-relevant[6] past and current outcomes

[6] A variable is *decision-relevant* if it enters the agents' decision problem, and a decision-relevant variable is *external* if its value is taken as given by the agent, while it is *internal* if the variable is part of the solution of the agents' decision problem. For example, wholesalers need to get information on current and future aggregate demand Y_t (decision-relevant and external) to set prices P_{it} (decision-relevant and internal), while they do not need to forecast wages since their only production input is the intermediate good.

contained in $u^{(t)}$ and $y^{(t)} = \left(g_0\left(u^{(0)}\right), \ldots, g_t\left(u^{(t)}\right)\right)$, and evaluating the probability of future external decision-relevant outcomes under the probability measure $\mathbb{P}_{RE}$ implied by $\mathbb{P}_u$ and the mappings $(g_t)_{t=0}^{\infty}$;

2. the allocations contained in $y_t = g_t\left(u^{(t)}\right)$ clear all markets.

Under learning, agents are not endowed with knowledge of the equilibrium asset price process, i.e. the mapping of a history of fundamentals $u^{(t)}$ to prices Q_t. Instead, they use a simple subjective model to forecast asset prices. s we show in Section 6, this subjective belief system can be made quite general, but here we confine ourselves to our preferred specification which follows Adam *et al.* (2017). Agents think that the asset price is a simple random walk model with a time-varying drift:

$$\Delta \log Q_t = \hat{\mu}_{t-1} + z_t \tag{1}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t \tag{2}$$

where $\hat{\mu}_t$ is the perceived trend price growth, g is the learning gain, and z_t is the *subjective forecast error.* Under $\mathcal{P}$, z_t is normally distributed white noise with variance σ_z^2, independent of the other exogenous shocks. The belief $\hat{\mu}_t$ is updated in the direction of the last forecast error: When agents see asset prices rising faster than they expected, they will also expect them to rise by more in the future.[7] In order to avoid complications arising from simultaneity in the determination of outcomes and beliefs, we follow Adam *et al.* (2012) and Caines (2016) and assume that in period t agents make choices conditional on $\hat{\mu}_{t-1}$, and update their beliefs according to (1) at the end of the period.

In order to determine expectations about the remaining variables of the model, we follow Winkler (forthcoming) in assuming that agents have so-called "conditionally model-consistent expectations". This is a restriction on expectations that effectively allows us to isolate the effects of asset price learning from other potential sources of learning in the economy. Let $(\Omega_z, \mathcal{S}(\Omega_z), \mathcal{P}_z)$ be the probability space that defines the subjective beliefs for z_t (i.e., that the z_t are iid normally dis-

[7] The belief system above is equivalent to a belief that asset price growth is the sum of a temporary and a permanent component, both of which are unobserved. Bayesian updating of this belief leads to the equations above, with the learning gain g representing the perceived ratio of the standard deviation of the permanent relative to the temporary component.

tributed with mean zero and variance σ_z^2). Agents' subjective beliefs depend on this perceived stochastic forecast error even though in equilibrium, model outcomes are a function only of fundamentals u_t. The subjective probability measure $\mathcal{P}$ is defined by a mapping from fundamentals u_t and the subjective forecast error z_t to model outcomes y_t.

Definition 2. *Conditionally model-consistent expectations* (CMCE) are a sequence of mappings $h_t : \Omega_u^{(t)} \times \Omega_z^{(t)} \ni \left(u^{(t)}, z^{(t)}\right) \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \ldots$ such that, for all t and $\left(u^{(t)}, z^{(t)}\right) \in \text{supp}\left(\mathcal{P}_{u,z}\right)$:

1. the choices contained in y_t solve the time-t decision problem of each agent in the economy, conditional on decision-relevant past and current outcomes contained in $u^{(t)}$ and $y^{(t)} = \left(h_0\left(u^{(0)}, z^{(0)}\right), \ldots, h_t\left(u^{(t)}, z^{(t)}\right)\right)$, and evaluating the probability of future decision-relevant outcomes under the probability measure $\mathcal{P}$ implied by $\mathbb{P}_u \otimes \mathcal{P}_z$ and the mappings $(h_t)_{t=0}^{\infty}$;

2. the allocations contained in $y_t = h_t\left(u^{(t)}, z^{(t)}\right)$ clear all markets except the markets for assets and final consumption goods;

3. asset prices under $\mathcal{P}$ follow the law of motion given by (1)–(2).

The definition of the mappings h_t is almost identical to that of a rational expectations equilibrium, except that asset market equilibrium is not part of the conditions, and instead the price Q_t evolves according to subjective beliefs.[8] Conditional model consistency restricts the subjective belief $\mathcal{P}$ to have the maximum degree of consistency with the model given agents' misspecified beliefs about asset prices. We will call the mappings h_t the subjective or perceived law of motion.

While demand for the long-term asset does not have to be equal to supply under $\mathcal{P}$, the market still has to clear in equilibrium:

Definition 3. An *equilibrium* with conditionally model-consistent expectations is a sequence of mappings $r_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto z_t \in \mathbb{R}$ and $g_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto y_t^* \in \mathbb{R}^N$, $t = 0, 1, 2, \ldots$ such that, for all t and $u^{(t)} \in \text{supp}\left(\mathbb{P}_u\right)$:

[8] In order for asset market equilibrium to not enter beliefs, Walras' law requires that at least two market clearing conditions be absent from agents' information set. Consequently, in definition 2 we impose that allocations under conditionally model-consistent beliefs do not clear the goods market. In Appendix D we discuss alternatives to these assumptions.

1. $g\left(u^{(t)}\right)=h\left(u^{(t)},\left(r_0\left(u^{(0)}\right),\ldots,r_t\left(u^{(t)}\right)\right)\right)$;

2. the allocations contained in $y_t^*=g_t\left(u^{(t)}\right)$ clear the asset market.

The probability measure implied by $\mathbb{P}_u$ and the mappings $(g_t)_{t=0}^{\infty}$ is denoted by $\mathbb{P}$.

Market clearing is brought about by finding the right value of the price Q_t that clears the asset market. We will call the mappings g_t the equilibrium or actual law of motion. To avoid confusion, we will use asterisks to denote the equilibrium stochastic processes y_t^* defined by the mappings g_t, as opposed to the perceived processes y_t defined by the mappings h_t. The equilibrium implies a particular path for the subjective asset price forecast error. In equilibrium, z_t^* is a function of the states and the shocks of the model, while under $\mathcal{P}$, z_t is perceived as an additional unforecastable exogenous disturbance. Because of this discrepancy, the subjective distribution $\mathcal{P}$ and the equilibrium distribution $\mathbb{P}$ are mutually singular ($\mathcal{P}\perp\mathbb{P}$).

Agents endowed with conditionally model-consistent expectations may not know the equilibrium mapping from fundamentals to asset prices, but their beliefs about the economy are correct conditional on their subjective view about the evolution of asset prices. This way of setting up expectations is very tractable and also allows us to transparently solve a linearized version of the model.

3 Linearized equilibrium

The analysis in this paper will focus entirely on a linearization of the model around its efficient steady-state.

3.1 Rational expectations equilibrium

Under rational expectations, a standard log-linearization of the model yields:

$$y_t=a_t+\alpha n_t \tag{3}$$

$$w_t=m_t+a_t-(1-\alpha)\,n_t \tag{4}$$

$$\pi_t=\beta\mathbb{E}_t^{\mathcal{P}}\pi_{t+1}+\frac{(1-\xi)\,(1-\beta\xi)}{\xi}m_t+\eta_t \tag{5}$$

$$w_t=\gamma c_t+\phi n_t \tag{6}$$

$$i_t = \gamma \left(\mathbb{E}_t^{\mathcal{P}} c_{t+1} - c_t \right) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} \tag{7}$$

$$q_t = \gamma c_t - (1-\beta)\, \theta h_t - \beta \gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \beta \mathbb{E}_t^{\mathcal{P}} q_{t+1} \tag{8}$$

$$c_t = y_t - \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta h_t \tag{9}$$

$$y_t = c_t \tag{10}$$

Here, lower-case variables denote log-linearizations around the (zero-inflation) steady state, except for i_t which is the difference of the nominal interest rate from its steady-state level, and $\eta_t = \frac{(1-\xi)(1-\beta\xi)}{\xi} \left(\bar{\tau} - \tau_t \right)$ is the cost-push shock process. The model still has to be closed with an equation describing monetary policy. The model is simply the textbook New-Keynesian model with an extra equation for the asset price q_t in (8). However, the asset price is redundant for the model dynamics because the asset is in fixed supply.

An important special case of the model obtains when prices are fully flexible and there are no cost-push shocks ($\xi = 0$ and $\eta_t = 0$). In this case, the allocation under rational expectations equilibrium is first-best efficient everywhere regardless of monetary policy. Output and the real interest rate are given by:

$$y_t^{n,RE} = \kappa_0 a_t \tag{11}$$

$$r_t^{n,RE} = -\gamma \kappa_0 \left(1 - \rho_a\right) a_t. \tag{12}$$

where $\kappa_0 = (1+\phi) / (1+\phi-\alpha+\alpha\gamma)$. These quantities are called the natural level of output and the natural real rate, respectively.

The equilibrium with sticky prices can be expressed in terms of the deviation from this efficient equilibrium. To this end, denote the output gap by $\hat{y}_t = y_t - y_t^{n,RE}$. The sticky price equilibrium can be summarized with a Phillips curve, an IS curve, a relation between marginal costs and the output gap, and an asset pricing equation:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)\,(1-\beta\xi)}{\xi} m_t + \eta_t \tag{13}$$

$$m_t = \frac{1+\phi-\alpha+\alpha\gamma}{\alpha} \hat{y}_t \tag{14}$$

$$\mathbb{E}_t \hat{y}_{t+1} - \hat{y}_t = \frac{1}{\gamma} \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^{n,RE} \right). \tag{15}$$

3.2 Learning equilibrium

Following Section 2, we compute the learning equilibrium in two steps. First, we solve for agents' subjective law of motion given their beliefs $\mathcal{P}$. To do this, we take the system of equations (3)–(10), but replace the goods market clearing condition (10) with the subjective law of motion for asset prices from (1)–(2):

$$\Delta q_t = \hat{\mu}_{t-1} + z_t \tag{16}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t. \tag{17}$$

This subjective law of motion is a forward-looking model that is straightforward to solve. However, there is now an additional shock, the asset price forecast error z_t, that is absent under RE. This forecast error will be predictable in equilibrium, but under $\mathcal{P}$ agents believe it to be unforecastable. Just as under RE, one still needs to add an equation describing monetary policy.

In the second step, we compute the equilibrium. To avoid confusion, we will denote with asterisks the equilibrium law of motion. We impose market clearing in the asset market:

$$h_t^* = 0.$$

This equation implicitly defines the equilibrium law of motion of the asset price q_t^*, of the forecast error z_t^*, as well as of all other equilibrium outcomes. We solve separately for the flexible- and the sticky-price equilibrium.

3.2.1 Flexible prices

We first describe the flexible price equilibrium ($\xi = 0$ and $\eta_t = 0$). We first find the subjective law of motion by solving (3)–(10). The learning model has two additional state variables compared to its RE counterpart, q_t and $\hat{\mu}_{t-1}$. We guess and verify that the asset demand function has the following form:

$$h_t^n = k_a a_t + k_h h_{t-1}^n - k_q q_t + k_\mu \hat{\mu}_{t-1} \tag{18}$$

where the coefficients satisfy $k_h \in (0, 1)$, $k_a, k_q, k_\mu > 0$. Exact expressions are in the appendix. Asset demand under learning is increasing in productivity, decreasing in the asset price, and increasing in expectations of future capital gains.

We can also solve for the values of output and the real interest rate under flexible prices. We write output and the real rate in deviation from their RE counterpart:

$$y_t^n = y_t^{n,RE} + \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha}\left(k_a a_t - (1-k_h)\, h_{t-1}^n - k_q q_t + k_\mu \hat{\mu}_{t-1}\right) \tag{19}$$

$$\begin{aligned} r_t^n &= r_t^{n,RE} + \gamma\kappa_1 \left(k_a\,(2-\rho_a-k_h)\, a_t - (1-k_h)^2\, h_{t-1}^n - k_q\,(1-k_h)\, q_t\right) \\ &\quad + \gamma\kappa_1\left((2-\rho_\mu-k_h)\, k_\mu + k_q\right)\hat{\mu}_{t-1}. \end{aligned} \tag{20}$$

where $\kappa_1 = \frac{1+\phi-\alpha}{1+\phi-\alpha(1-\gamma)}\frac{\bar{Q}\bar{H}}{\bar{Y}} > 0$. The natural rate is increasing in the price growth belief $\hat{\mu}_{t-1}$. Agents' subjective expectations about output under flexible prices are affected by the choice of asset holdings (which are not constant in agents' minds). An increase in expected capital gains will increase asset demand, and households will increase their labor supply in order to finance their purchase of the asset, thereby increasing the level of output.

The natural real rate under subjective expectations can be understood by the arbitrage relationship between the return on the long-term asset and the return on bonds. Combining the two asset pricing equations (7) and (8), we obtain:

$$r_t^n = \frac{1-\beta}{\beta}\left(\gamma c_t - \theta h_t - q_t\right) + \mathbb{E}_t^{\mathcal{P}}\Delta q_{t+1}$$

Up to first order, the expected return on the two assets has to be equal. An increase in expected capital gains $\mathbb{E}_t^{\mathcal{P}}\Delta q_{t+1} = \hat{\mu}_{t-1}$ increases the expected return to the asset, and the real interest rate on bonds therefore has to rise as well.

To find the flexible-price equilibrium under learning, i.e. the actual law of motion, one has to impose $h_t^* = 0$. From the asset demand function (18), one can then immediately solve for the equilibrium asset price and the realization of the subjective forecast error:

$$q_t^* = \frac{k_a a_t + k_\mu \hat{\mu}_{t-1}^*}{k_q}. \tag{21}$$

That is, the equilibrium asset price is increasing in both productivity and capital gains expectations. This is intuitive. The demand function (18) is downward-sloping, and so an increase in demand due to either higher productivity (i.e. higher income) or higher expected capital gains has to be met with an increase in the price to bring about equilibrium in the asset market.

Substituting the equilibrium price (21) into Equations (19) and (20), we obtain the realized level of output and the real rate under flexible prices:

$$y_t^* = y_t^{n,RE} \tag{22}$$

$$r_t^* = r_t^{n,RE} + \gamma\kappa_1 \left((1-\rho_a)\, k_a a_t + ((1-\rho_\mu)\, k_\mu + k_q)\, \hat{\mu}_{t-1}^* \right). \tag{23}$$

Under learning and flexible prices, the equilibrium level of output is the same as under RE. This coincidence arises because, under flexible prices, output is determined entirely by intratemporal conditions that are independent of expectations. Nonetheless, the real interest rate does depend on expectations, and its natural level under learning is therefore different from rational expectations. In particular, it is increasing in subjectively expected asset price price growth.

3.2.2 Sticky prices

With sticky prices, the subjective law of motion can be expressed in deviation from the flexible price allocation, just as under rational expectations. We will use tildes to denote "perceived" gaps, e.g. $\tilde{h}_t = h_t - h_t^n$ denotes the difference of asset holdings from their flexible price level under $\mathcal{P}$. The sticky price PLM can be summarized with the equations:

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} m_t + \eta_t \tag{24}$$

$$m_t = \frac{1+\phi-\alpha+\alpha\gamma}{\alpha} \left(\tilde{c}_t + \kappa_1 \Delta \tilde{h}_t \right) \tag{25}$$

$$\tilde{c}_t = \mathbb{E}_t^{\mathcal{P}} \tilde{c}_{t+1} - \frac{1}{\gamma} \left(i_t - \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} - r_t^n \right) \tag{26}$$

$$\tilde{h}_t = \frac{\gamma}{\theta(1-\beta)} \left(\tilde{c}_t - \beta \mathbb{E}_t^{\mathcal{P}} \tilde{c}_{t+1} \right). \tag{27}$$

The first equation is the familiar Phillips curve, and the second equation relates marginal costs to gaps in consumption $\tilde{c}_t$ and in asset investment $\Delta\tilde{h}_t$. The investment gap appears because, for a given level of consumption, higher asset purchases have to be financed out of additional income from production, and marginal costs are increasing in the level of production. The third equation is an IS equation written in terms of the consumption gap (which under subjective expectations does not equal the output gap). The last equation is the Euler equation for asset demand, rewritten in gap form.

Notice that the asset price q_t itself does not appear in its own Euler equation when it is written in terms of gaps, because agents perceive q_t as an exogenous process, independent of the degree of price stickiness. The asset price still implicitly enters equation (27) through the natural rate r_t^n.

To find the actual law of motion under sticky prices, one imposes $h_t^* = 0$ and solves for q_t^*. The equilibrium depends crucially on the conduct of monetary policy, which we have not yet specified.

3.3 Numerical illustration

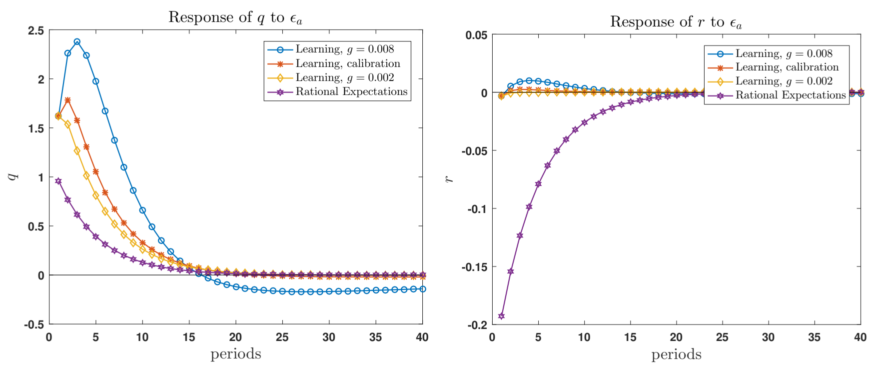
We illustrate the properties of the learning model using a simple calibration in which we interpret the long-term asset as housing. We set the labor share in output equal to $\alpha = 0.7$ and the discount factor β equal to 0.995. The coefficient of relative risk aversion is set to $\gamma = 1.39$ (Gandelman and Hernández-Murillo, 2014) and the inverse Frisch elasticity of labor supply is set to $\phi = 0.33$. The utility scaling parameter χ is set to 0.01005 in order to achieve a steady state ratio of asset wealth to output of $\bar{Q}\bar{H}/\bar{Y} = 2.01$, which corresponds to the US ratio of real estate holdings over GDP in 2016. The degree of price stickiness is set to a standard value of $\xi = 0.75$, implying an average price duration of four quarters. The elasticity of substitution σ does not affect the first-order dynamics of the model and we set it such that the relative weight on inflation in the loss function (29) below equals $\lambda_\pi = 1$. We follow Billi (2017) and set the autocorrelation of both the technology and cost-push shocks to 0.8. Finally, we calibrate the remaining four parameters $(\sigma_A, \sigma_\tau, \theta, g)$ to jointly match the volatilities of output growth $\sigma(\Delta y_t) = 0.64\%$, inflation $\sigma(\pi_t) = 0.82\%$, house price growth $\sigma(\Delta q_t) = 1.51\%$ and real wage growth $\sigma(\Delta w_t) = 0.10\%$, under the assumption that monetary policy follows the commonly used Taylor rule

$$i_t = 1.5\pi_t + 0.125\left(y_t - y_t^{n,RE}\right). \tag{28}$$

The resulting parameter values are $\sigma_A = 0.83\%$, $\sigma_p = 0.75\%$, $\theta = 0.0068$ and $g = 0.0041$.

We first document the effect of learning under flexible prices. Learning has no effect on equilibrium allocations relative to rational expectations, but manifests itself in the asset price and the natural real interest rate. Figure 1 plots the response of these two variables to a technology shock under rational expectations and learning for a range of values of the learning gain g.

Figure 1: Effect of learning under flexible prices.



Note: Response to a one standard deviation positive technology shock ε_{At}. Log percentage points. Flexible prices and zero inflation. For learning cases, outcomes are plotted for the equilibrium law of motion.

The effect of learning on q_t is typical for self-referential asset price learning models. Initially, the asset price rises on impact because higher wage income raises asset demand, as under rational expectations. But the initial increase now causes a subsequent revision in beliefs $\hat{\mu}_t$ through the learning mechanism. The household believes that the shock has some long-run impact on capital gains and responds by increasing its asset demand above the RE demand. This response drives a further increase in q_t in the next period and the shock continues to propagate through belief updating thereafter. At some point, expected price growth has risen so much that it outstrips realized price growth. At this point, beliefs $\hat{\mu}_t$ decrease, bringing about a reduction in asset demand and therefore in equilibrium asset prices, so that the process eventually reverts back to steady state. The strength of these dynamics depends crucially on the learning gain g.

The differing response of the real interest rate between the learning and RE environments directly shows the effect of expected capital gains on the natural rate of interest. Even though the impulse response of *realized* consumption is exactly identical under learning and RE, what matters for the interest rate is *expected* consumption growth. Under learning, increases in expected capital gains in the periods following the shock also increase expected consumption growth and therefore higher real interest rates, as agents anticipate selling some of their asset holdings in the future to profit from the capital gains. Optimistic capital gains expectations thus imply a higher real rate of interest than under rational expectations.

4 Optimal Policy

4.1 Welfare function

We provide second-order approximations to the expected discounted sum of utility in our model. Under learning, we will assume that the policymaker maximizes welfare under the equilibrium law of motion, applying the loss function $\mathcal{L}_t$. If, instead, welfare were maximized under the subjective law of motion of the agents, then the policy problem could be solved as a standard rational expectations problem, which has been studied extensively in the literature.

Welfare under the equilibrium law of motion is proportional to $-\sum_{t=0}^{\infty} \mathbb{E}_0 \mathcal{L}_t$ up to second order and terms independent of policy. The period loss function is given by

$$\mathcal{L}_t = \lambda_\pi \left(\pi_t^*\right)^2 + \left(\hat{y}_t^*\right)^2 . \tag{29}$$

where $\hat{y}_t = y_t - y_t^{n,RE}$ is the deviation of equilibrium output from its flexible-price level, and $\lambda_\pi > 0$ is a function of the structural model parameters (see the appendix).

This loss function is the same as in the standard, rational expectations New Keynesian model. It penalizes deviations of inflation from zero as well as deviations of output from its natural level under rational expectations (11). This natural level of output is first-best efficient.

4.2 Optimal policy without cost-push shocks

We now solve for the optimal monetary policy for the case in which there are no cost-push shocks. For exposition, we start by reviewing the optimal policy under rational expectations. As the flexible price equilibrium under RE is first-best efficient, monetary policy is optimal if it manages to replicate the flexible price allocation in the presence of nominal rigidities. This amounts to closing the output gap and completely stabilizing inflation at the same time, as can be seen from the loss function (29). Without cost-push shocks, the "divine coincidence" holds and complete stabilization is achievable: The optimal policy implements $\pi_t = 0$ and the Phillips curve (13) then immediately implies $\hat{y}_t = 0$. The optimal policy can be implemented with the following rule:

$$i_t = r_t^{n,RE} + \phi_\pi \pi_t$$

where $\phi_\pi > 1$. The interest rate has to track the natural real rate and react more than one-for-one to inflation, i.e. satisfy the Taylor principle.

Under learning, we will see that the first-best allocation is also feasible by pursuing strict inflation targeting, which is therefore the optimal policy.

Proposition 1. *It is optimal for monetary policy under learning to implement* $\pi_t = 0$*, regardless of whether welfare is evaluated under the actual or the perceived law of motion. The optimal policy can be implemented with the rule*

$$i_t = r_t^n + \phi_\pi \pi_t$$

where $\phi_\pi > 1$*.*

Proof. See the appendix. □

The intuition for this result is relatively simple: When the central bank tracks the appropriate natural real rate of interest and there are no cost-push shocks, it implements the flexible price allocation. But we have just seen that the equilibrium under flexible prices is identical to that under RE, and is therefore first-best. Moreover, the flexible-price allocation is also first-best under the subjective law of motion of agents. Therefore, for the case without cost-push shocks and a fixed asset supply, there is no tension between the central bank's optimal policy and what agents would perceive to be optimal from a subjective perspective.

It might seem at first that the optimal policy prescriptions are unchanged by the presence of learning, because the prescription of strict inflation targeting is unchanged. But the implementation of this target requires a different reaction function under learning. The nominal interest rate has to track the *subjective* natural real interest rate r_t^n, which is different from the $r_t^{n,RE}$ under rational expectations.[9]

[9] The process r_t^n is also different, from the perspective of agents, from the equilibrium realization r^* in (23). The appendix contains a discussion of this difference and how it matters for the setting of interest rates.

Whereas $r_t^{n,RE}$ is a function of productivity a_t only, r_t^n depends additionally on beliefs $\hat{\mu}_t$, prices q_t and the asset holdings h_{t-1}. In particular, the real rate rises when expected price growth $\hat{\mu}_t$ increases. In equilibrium, the asset price q_t depends positively on expected price growth, and therefore the central bank has to set higher interest rates when asset prices, or subjective capital gains expectations, are high. Under rational expectations, such a reaction is not necessary.

4.3 Optimal policy with cost-push shocks

The presence of cost-push shocks breaks the "divine coincidence" under rational expectations, so that the first-best allocation is not feasible. Under learning, there exists in principle a non-linear policy that, by exploiting agents' misperceptions, still achieves the first-best allocation:

Proposition 2. *With cost-push shocks and learning, it is possible to close the inflation and the output gaps* $(\pi_t^* = \hat{y}_t^* = 0)$ *with the non-linear targeting rule* $\pi_t = -\eta_t / (\beta \rho_\eta) + b_t z_t$*, where* b_t *is a state-dependent coefficient.*

Proof. See the appendix. □

This result relies on the manipulation of beliefs by the central bank.[10] From the Phillips curve (24), it is clear that inflation and the output gap can only be zero if expected future inflation and the cost-push shock offset each other: $\beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \eta_t = 0$. Under rational expectations, this is infeasible because zero inflation will also imply zero expected inflation. But under learning, this need not be the case. In fact, the central bank can separately control inflation and inflation expectations, if it is able to make the private sector believe that part of its actions are random when in fact they are not. This is what the policy in the proposition above accomplishes: The term b_t is set such that agents perceive it as independent from z_t, so that $b_t z_t$ appears as a random, zero-mean shock to inflation to them, while in equilibrium it will be set in a highly systematic manner so as to set equilibrium inflation to zero.

[10]The possibility of such an outcome was anticipated by Woodford (2010) who speculated that one "might even conclude that the optimal policy under learning achieves an outcome better than any possible rational-expectations equilibrium, by inducing systematic forecasting errors of a kind that happen to serve the central bank's stabilization objectives".

Because it implements the first-best allocation, this policy is obviously optimal under learning. But it uses an extreme degree of belief manipulation that is particularly vulnerable to the Lucas critique. Why should agents continue to forever expect inflation when all they ever observe is complete price stability? It is plausible that the very fact that the central bank is trying to exploit a certain expectational bias will change the nature of this bias or even eliminate it. One way out of this problem is to explicitly model interactions between the nature of agents' belief distortions and the central bank's policy, as in Woodford (2010). Instead, we will stick with the particular belief distortion we study, but limit the degree to which it can be exploited by the central bank.

We limit the class of policies considered to those in which the amount of the central bank's manipulation of beliefs is limited, in the following sense: The target criterion for inflation that the central bank pursues in equilibrium has to be the same as the criterion that agents perceive to be pursued under their subjective beliefs. Formally, we require that π_t be a function only on the fundamental shocks $u^{(t)}$, but not on the perceived asset price forecast errors $z^{(t)}$; or, equivalently:

$$\pi_t = \pi_t^* \ \mathcal{P}\text{-almost surely.} \tag{30}$$

This condition means that agents effectively have rational expectations for inflation.[11] This condition is not a restriction on the behavior of agents, who form beliefs under the same learning scheme as before, but a restriction on the set of policies that the central bank can pursue. Policies for which the private sector's beliefs of inflation are not fully model-consistent, i.e. different from the actual inflation target of the central bank, are ruled out.

The class of policies satisfying condition (30) is large, and includes all possible target criteria under rational expectations, including the RE-optimal discretionary and commitment criteria. It also includes target criteria that are made contingent on equilibrium asset price realizations q_t^*. However, policies with a target that depends on outcomes contemplated by agents, but never realized in equilibrium, are not part of this class. In particular, then, the class excludes the policy described in Proposition 2.

[11] Note that by definition of the equilibrium, it is always the case that $\pi_t = \pi_t^*$ a.s. (almost surely) under the equilibrium measure $\mathbb{P}$, but the subjective measure $\mathcal{P}$ is not absolutely continuous with respect to $\mathbb{P}$ (see Section 2).

We now solve for the optimal monetary policy within our restricted class of policies. We find that the best that monetary policy can achieve is to replicate the optimal outcomes under rational expectations:

Proposition 3. *Consider the class of policies satisfying condition* (30)*. Within this class:*

1. *The optimal target criterion under commitment (from the timeless perspective) is given by $\zeta p_t = -\Delta m_t$ for some $\zeta > 0$, where p_t is the price level. The optimal commitment policy achieves the same equilibrium allocation as the commitment solution under rational expectations.*

2. *Similarly, the optimal target criterion under discretion is given by $\zeta \pi_t = -m_t$. The optimal discretionary policy achieves the same equilibrium allocation as the discretionary solution under rational expectations.*

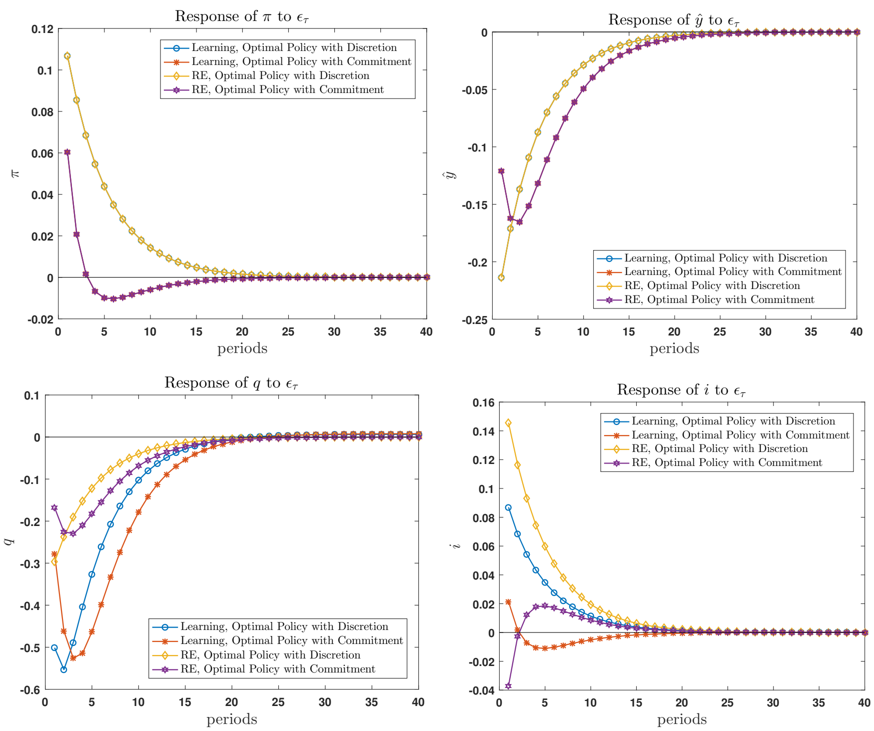
3. *In both cases above, optimal policy can be implemented with an interest rule of the form $i_t = r_t^n + A(L)\eta_t + \phi_\pi \pi_t$, where $A(L)\eta_t$ is a lag polynomial in the cost-push shock η_t and $\phi_\pi > 1$.*

Proof. See the appendix. □

In sum, monetary policy can do no better under learning than under RE, and flexible inflation targeting remains the optimal policy even when asset price expectations are not rational. However, implementing flexible inflation targeting requires the central bank to track the subjective natural real rate r_t^n (as in the previous case without cost-push shocks), which depends positively on subjective expectations of capital gains. Therefore, the optimal nominal interest rate has to react positively to asset prices.

Figure 2 compares the optimal policy response in the calibrated model to a cost-push shock under RE and learning. Comparing either optimal discretionary or commitment policies, the outcomes for inflation and the output gap under learning and under rational expectations are the same. However, the implementation of these outcomes requires different paths for the nominal interest rate. In all cases displayed, the cost push shock reduces asset prices, as lower income reduces asset demand. The drop in asset prices is magnified under learning as subjective expectations become pessimistic. The natural rate under learning falls with subjective expectations. As a consequence, the optimal discretionary nominal interest rate is lower than under rational expectations. The same holds true under commitment after the first two periods, when the difference of the asset price response becomes sufficiently large.

Figure 2: Optimal policy and alternatives after a cost-push shock.



Note: Response to a unit standard deviation positive cost-push shock $\varepsilon_{\tau t}$. Log percentage points. For learning cases, outcomes are plotted for the equilibrium law of motion. The elasticity of substitution σ is chosen such that the welfare weight on inflation in the loss function $\mathcal{L}$ equals $\lambda_\pi = 1$.

5 Simple rules

Implementing optimal policy in the learning environment requires knowledge of the natural rate of interest under learning, which depends on subjective asset price beliefs. This implies that the central bank can either directly observe these beliefs, or can infer them from the difference between the realized level and the efficient level of asset prices. Either way, measurement of the relevant quantities is fraught with difficulty, which is in fact one of the most frequent arguments made against incorporating reactions to asset prices into monetary policy.

In this section, we consider simple interest rate rules rules with a potential reaction to asset prices. These rules can be implemented without knowledge of subjective beliefs or asset price gaps, since they only depend on realized asset prices that are easy to measure and observable with high frequency. We show that for our calibrated model, incorporating a positive reaction to asset price growth is desirable in terms of welfare. This is not a straightforward consequence of the optimal policy analysis in the previous section, because simple rules can be quite far from optimal. Broadly speaking, a rule reaction to asset prices will tend to be beneficial if periods of elevated asset prices coincide with excess aggregate demand under that particular rule. For our calibrated model and Taylor-type interest rate rules, that turns out to be the case.

We re-compute the model under the assumption that the monetary authority is following a Taylor-type rule of the form:

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\phi_\pi \pi_t + \phi_y \hat{y}_t + \phi_q \sum_{s=0}^{\infty} \omega^s \Delta q_{t-s}\right). \qquad (31)$$

The rule depends on inflation and the output gap, and has an additional term for asset prices: a moving average of past price changes, with a weight on past observations that decays at the rate $\tilde{\omega} \in (0,1)$. In what follows, we keep the coefficient on inflation at $\phi_\pi = 1.5$ and find parameter combinations $(\rho_i, \phi_y, \phi_q, \tilde{\omega})$ that minimize the loss function (29).[12] We impose the constraint $0 \leq \tilde{\omega} \leq 0.999$. Table 1 summarizes the results.

The first four rows show results under rational expectations. Rows (1) and (2) show the optimal policy outcomes under discretion and commitment. Row (3) shows our baseline policy rule used to calibrate the model. Row (4) shows the optimized values of the coefficients on inertia ρ_i and the output gap ϕ_y, holding

[12] If one also optimizes over the coefficient and inflation, then the optimal policy under RE is given by $\phi_\pi \to \infty$ and $\phi_y/\phi_\pi \to \zeta > 0$ (Boehm and House, 2014). The outcomes of this limit policy are also attainable under learning with a similar policy that also responds infinitely strongly to inflation and the output gap. In this section, we rule out infinite rule coefficients by keeping the inflation coefficient fixed, and focus only on the tradeoff of reacting to the output gap and asset prices.

Table 1: Performance of optimized simple rules.

	Rational Expectations	$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
(1)	*Optimal Policy, Discretion*	0.436	0.085	1.011	0.962
(2)	*Optimal Policy, Commitment*	0.205	0.192	1.010	0.385
(3)	$i_t = 1.5\pi_t + 0.125\hat{y}_t$	0.349	0.451	0.801	1.583
(4)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(1.5\pi_t + \phi_y^* \cdot \hat{y}_t\right)$	0.362	0.221	0.776	0.838
	$\{\rho_i^*, \phi_y^*\} = \{0.681, 1.611\}$				
	Learning	$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
(5)	*Optimal Policy, Discretion*	0.436	0.085	1.724	0.962
(6)	*Optimal Policy, Commitment*	0.204	0.193	1.724	0.385
(7)	$i_t = 1.5\pi_t + 0.125\hat{y}_t$	0.148	0.428	1.503	1.000
(8)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(1.5\pi_t + \phi_y^* \hat{y}_t\right)$	0.133	0.428	1.608	0.940
	$\{\rho_i^*, \phi_y^*\} = \{0.147, 0.106\}$				
(9)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(1.5\pi_t + \phi_y^* \hat{y}_t + \phi_q^* \sum_{s=0}^{\infty} \omega^{*s} \Delta q_{t-s}\right)$	0.088	0.418	1.541	0.850
	$\{\rho_i^*, \phi_y^*, \phi_q^*, \omega^*\} = \{0.302, 0.133, 0.025, 0.999\}$				
(10)	FM (2007) w/ asset price				
	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(1.5\pi_t + \phi_y^* \hat{y}_t + \phi_q^* q_t\right)$	0.088	0.417	1.540	0.850
	$\{\rho_i^*, \phi_y^*, \phi_q^*\} = \{0.301, 0.136, 0.025\}$				
(11)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)1.5\pi_t$	0.014	0.541	1.911	1.393
	$\rho_i^* = 0$				
(12)	BG (1999) w/ asset price				
	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(1.5\pi_t + \phi_q^* q_{t-1}\right)$	0.050	0.536	1.842	1.380
	$\{\rho_i^*, \phi_q^*\} = \{0, -0.005\}$				
(13)	BG (2001) w/ asset price				
	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(r_{ss} + 1.5\pi_t + \phi_q^* q_t\right)$	0.012	0.541	1.911	1.393
	$\{\rho_i^*, \phi_q^*\} = \{0, 0.001\}$				
(14)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(r_{ss} + 1.5\pi_t + \phi_q^* \sum_{s=0}^{\infty} \omega^{*s} \Delta q_{t-s}\right)$	0.078	0.518	1.707	1.310
	$\{\rho_i^*, \phi_q^*, \omega^*\} = \{0, 0.024, 0.701\}$				

Note: Losses are evaluated as the unconditional expectation of the loss function $\mathcal{L}$. The welfare weight on inflation is set to $\lambda_\pi = 1$ and the losses are normalized to 1 for row (7).

constant the inflation coefficient ϕ_π. This rule is more aggressive than the standard Taylor rule and leads to welfare gains from output gap stabilization. Allowing for a non-zero asset price response in the optimization leads to an optimal coefficient of $\phi_q^* = 0$: There is no benefit from leaning against the wind under rational expectations.

Under learning, the picture is quite different. Rows (5) and (6) show optimal policy outcomes, which differ from their RE counterparts only by a higher asset price volatility. Row (7) shows the baseline policy rule used for the calibration. Row (8) optimizes the output gap and inertia coefficients in rule (31), while Row (9) also optimizes the asset price reaction parameters ϕ_q and ω. The optimal asset price response ϕ_q^* is positive. The optimal coefficient on the output gap is positive as well, and the optimal ω^* is set very close to one. With this value, the moving average of asset prices closely tracks the subjective belief $\hat{\mu}_t$, which itself is a moving average of past price changes. The reaction to the asset price stabilizes both inflation and the output gap, resulting in a 15 percent reduction in expected losses.

To get a better idea of the effects of monetary policy reactions to asset prices and the output gap under learning, we compute loss function values as well as the volatilities of inflation, the output gap and asset prices over a range of parameters for the rule in (31). We fix the moving average weight to $\tilde{\omega} = 0.9$, the interest rate inertia to $\rho_i = .147$ as in Row (8) of Table 1, and vary the magnitude of the response coefficients ϕ_y and ϕ_q on the output gap and inflation.[13] Figure 3 contains the results as surface plots.

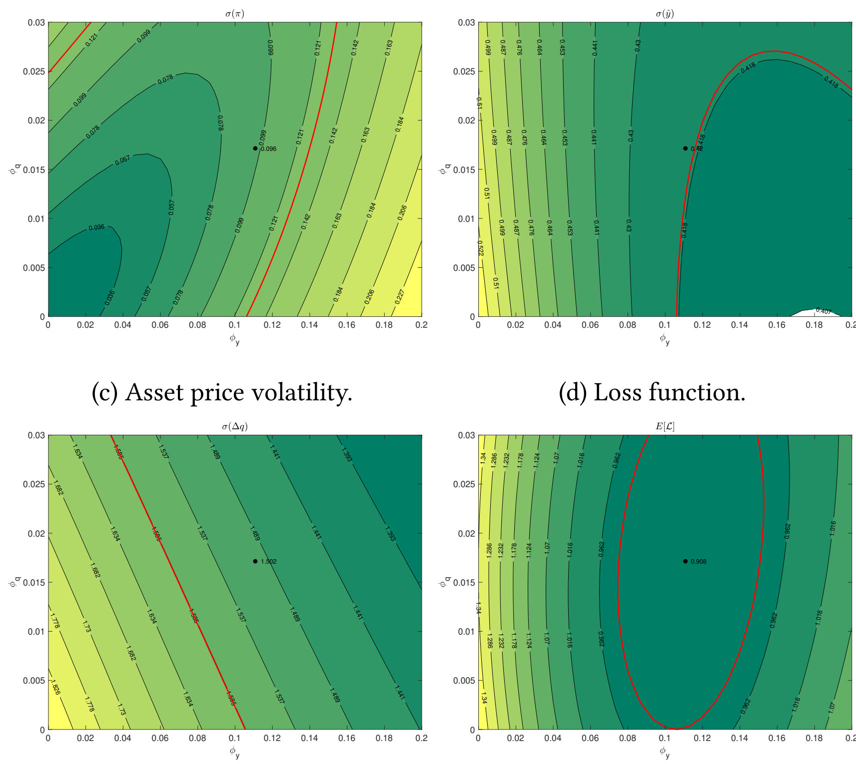
The effect of changes in the output gap coefficient are as expected: They lower the volatility of the output gap itself, but increase the volatility of inflation. This trade-off arises because the model has cost-push shocks in it. A reaction to the output gap also lowers asset price volatility in this model.

But the asset price coefficient also plays an important role. The volatility of asset prices is decreasing in the asset price response ϕ_q. The volatility of the output gap is affected little by the asset price response, but the volatility of inflation is reduced significantly with $\phi_q > 0$. Therefore, the loss function is minimized at a strictly interior point at which the central bank reacts to both the output gap and

[13] Our results are qualitatively robust to changes in the moving average weight ω. In particular, a positive reaction to asset prices $\phi_q > 0$ always reduces asset price volatility.

Figure 3: Loss values and volatilities for different output gap and asset price coefficients.

(a) Inflation volatility. (b) Output gap volatility.



(c) Asset price volatility. (d) Loss function.

Note: Unconditional standard deviation of inflation π_t^*, asset price growth Δq_t^* and output gap $\hat{y}_t^*$ under the equilibrium law of motion, and loss function $\mathcal{L}$, as a function of ϕ_y and ϕ_q. The parameter ρ_i is kept at 0.147 as in Row (8) of Table 1 and $\tilde{\omega} = 0.9$ throughout. The welfare weight on inflation is set to $\lambda_\pi = 1$ $\lambda_\pi = 1$ and the losses are normalized as in Table 1. Red lines denote contour lines at the optimal coefficient ϕ_y with $\phi_q = 0$, i.e. the rule in Row (8) of Table 1 . Black dots denote values attained under the optimal coefficients.

asset price growth.

Importantly, a reaction to asset prices always decreases asset price volatility (regardless of the value of $\tilde{\omega}$). This is in stark contrast to the rational bubbles of Gali (2014, 2017). Rational bubbles grow at the rate of interest, and so raising rates when a bubble is growing makes it grow even faster, causing more volatility. By contrast, raising rates in our learning model has the effect of lowering the asset price today: A higher real rate requires a higher expected asset return. For a given expected capital gain $\hat{\mu}_t$, a higher return needs to be brought about by a lower price today. The reduction in the asset price today then reduces optimism about future price growth.

Table 1 also evaluates different forms of asset price reactions that have been proposed in the literature. In Row (10), we evaluate a rule which reacts to the level of asset prices. Here, too, we obtain a positive optimal response coefficient on asset prices, in contrast to Faia and Monacelli (2007) who evaluated this type of rule in a rational expectations model with credit constraints. Rows (11) to (14) evaluate rules without an output gap reaction. Including a reaction to the level of asset prices, as proposed in Bernanke and Gertler (1999) and evaluated in Rows (12) and (14), does not enable the central bank to improve outcomes over a rule without such a reaction in Row (10). However, a positive reaction to a moving average of past asset price growth in Row (14) yields a small welfare gain.

6 Extension: General Asset Price Beliefs

We can show that our results derived so far are robust to a wide range of alternative specifications for agents' subjective asset price beliefs. We replace the subjective law of motion for asset prices in (1)–(2) with a general belief of the form:

$$q_t = A\left(L\right) z_t + B\left(L\right) u_t.$$

where A and B are arbitrary lag polynomials. Subjective beliefs can depend in an arbitrary way on the fundamental shocks u_t (i.e. productivity and cost-push shocks) as well as a subjective forecast error z_t. This general form of beliefs encompasses rational expectations (for which $A = 0$ and B represents the equilibrium process for asset prices), our baseline belief process (for which $B = 0$ and A represents the subjective law of motion 1–2), but also other behavioral expectations, such as the "natural expectations" of Fuster *et al.* 2012, the "diagnostic expectations" of Bordalo *et al.* (2018), general forms of extrapolation or attenuation bias, and more. The only assumptions we do retain are that i) expectations are conditionally model-consistent in the sense of Definition 2; and ii) the subjective law of motion for q_t is independent of policy. This second assumption is somewhat limiting, as it would be interesting to study how changes in policy can change the form of subjective asset price beliefs. However, we conjecture that an environment in which agents think that monetary policy is more powerful in shaping subjective asset price beliefs will provide an even stronger rationale for reacting to asset prices than the one we lay out here.

The appendix shows that with this general belief system, Propositions 1 through 3 continue to hold: The optimal policy under learning replicates the outcomes from the optimal policy under rational expectations, but has to be implemented by following the perceived natural real interest rate which is increasing in asset price expectations.

7 Extension: Asset Production

In the baseline model discussed up until now, learning causes distortions solely through wealth effects affecting aggregate consumption. Here, we extend the model to allow for the long-term asset to be produced instead of being in fixed supply. In this extension, asset price misalignments also distort investment decisions in addition to aggregate demand. This complicates the monetary policy tradeoff because learning now causes two distortions, one through aggregate demand and one through the misallocation of resources along the consumption-investment margin. We show that, in this case, the optimal policy target criterion under learning is no longer the same as under rational expectations. Instead, the optimal policy "leans against the wind" as defined by Svensson (2017): The central bank should tolerate low inflation at times when current or future expected asset prices are inefficiently high.

Relative to the baseline model, we now assume that the stock of the asset depreciates at the rate δ. The representative household owns firms that can produce an amount X_t of the asset from K_t consumption goods. Their production function has decreasing returns to scale:

$$X_t = A_h K_t^{\omega}.$$

Production takes place within one period. The profits of the investment firms (which accrue to households) are:

$$\Pi_t = Q_t X_t - K_t$$

and profit maximization leads to the first order condition:

$$I_t = A_h \left(\omega Q_t A_h\right)^{\frac{\omega}{1-\omega}}.$$

The budget constraint of the household now takes into account profits from asset producers and depreciation of the asset:

$$C_t + Q_t \left(H_t - (1-\delta) H_{t-1}\right) + \frac{1+i_{t-1}}{1+\pi_t} B_{t-1} \quad = \quad W_t N_t + \Pi_t + T_t + B_t.$$

Market clearing in the asset market now requires

$$H_t = (1-\delta)\, H_{t-1} + X_t. \tag{32}$$

The equilibrium is defined analogously to section 2. Agents do not know the market clearing condition (32), but instead hold subjective beliefs that the asset price follows equations (1)–(2). Beliefs about the hidden state μ_t are updated using the Kalman filter as before, and expectations about the remaining equilibrium objects satisfy conditional model consistency as defined in section 2.

7.1 Flexible price equilibrium

The linearization of the model as well as the derivation of the flexible-price equilibrium under learning and RE is very similar to our baseline model, which we relegate to the appendix. The only added complication is that the model now has one additional endogenous state variable, the asset quantity h_{t-1}.

The allocation under rational expectations and flexible prices is still first-best. In particular, the asset investment choice $x_t^{n,RE}$ is efficient. However, the learning equilibrium is not, because subjective asset price beliefs now distort investment decisions, so that $x_t^n \neq x_t^{n,RE}$. Consumption and output distortions under flexible prices are functions of the investment distortion:

$$\begin{aligned} c_t^n - c_t^{n,RE} &= -\kappa_1 \left(x_t^n - x_t^{n,RE}\right) \\ y_t^n - y_t^{n,RE} &= \frac{\alpha\gamma}{1+\phi-\alpha} \kappa_1 \left(x_t^n - x_t^{n,RE}\right) \end{aligned}$$

where the constant κ_1 is now defined as

$$\kappa_1 = \frac{1+\phi-\alpha}{\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right) + \alpha\gamma} \frac{\delta \bar{Q} \bar{H}}{\bar{Y}}.$$

The real interest rate under learning and flexible prices can be expressed in deviation from its counterpart under rational expectations in the form

$$r_t^n = r_t^{n,RE} - b_{h1} h_{t-1}^n - b_{h2}\left(h_{t-1}^n - h_{t-1}^{n,RE}\right) + b_a a_t - b_q q_t + b_\mu \hat{\mu}_{t-1}, \tag{33}$$

where $b_{h1}, b_{h2} b_q, b_\mu > 0$. Closed-form expressions for the coefficients are given in the appendix. Most important for our analysis is that $b_\mu > 0$: The natural real rate of interest continues to be increasing in the asset price belief $\hat{\mu}_t$.

7.2 Welfare function

To evaluate different policies, we derive a quadratic approximation of the welfare function under the equilibrium probability measure $\mathbb{P}$. Welfare is proportional (up to second order and terms independent of policy) by $-\sum_{t=0}^{\infty} \mathbb{E}_0 \mathcal{L}_t$, where the period loss function is given by

$$\mathcal{L}_t = \lambda_\pi \left(\pi_t^*\right)^2 + \left(\hat{c}_t^* + \kappa_1 \hat{x}_t^*\right)^2 + \lambda_h \left(\hat{h}_t^*\right)^2 + \lambda_x \left(\hat{x}_t^*\right)^2, \tag{34}$$

where $\lambda_\pi, \lambda_h, \lambda_x > 0$ are functions of the structural model parameters (see the appendix), and $\hat{c}_t = c_t - c_t^{n,RE}$, $\hat{h}_t = h_t - h_t^{n,RE}$, and $\hat{x}_t = x_t - x_t^{n,RE}$. As before, asterisks denote the process for a variable under the equilibrium law of motion, as opposed to the subjective law of motion under which agents make their decisions. Compared to the loss function in the baseline model, we now have to take into account variation in the asset stock h_t that the household owns, as well as variations in asset investment x_t.

7.3 Optimal policy

As in the baseline model, we can write the perceived law of motion under sticky prices in deviation from the flexible price PLM:

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} m_t + \eta_t \tag{35}$$

$$m_t = \frac{\bar{C}(1+\phi-\alpha) + \bar{Y}\alpha\gamma}{\bar{Y}\alpha}\left(\tilde{c}_t + \kappa_1 \tilde{x}_t\right) \tag{36}$$

$$i_t = \gamma\left(\mathbb{E}_t^{\mathcal{P}} \tilde{c}_{t+1} - \tilde{c}_t\right) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + r_t^n \tag{37}$$

$$\theta\left(1-\beta(1-\delta)\right)\tilde{h}_t = \gamma \tilde{c}_t - \beta(1-\delta)\gamma \mathbb{E}_t^{\mathcal{P}} \tilde{c}_{t+1}. \tag{38}$$

Note that the asset price q_t does not enter this system save through the dependence of the natural real rate r_t^n in the IS curve (37). In fact, the system has the same form as in the baseline model with slightly different coefficients. However, our results on optimal policy are changed in this version with production, due to the fact that the flexible-price allocation under learning is no longer efficient.

As before, we restrict ourselves to the class of policies for which the target criterion for inflation is robust to the Lucas critique, in the sense that beliefs about the inflation objective coincide with the central bank's actual inflation objective, $\pi_t = \pi_t^*$. Again, this class includes the optimal discretionary and commitment policies under rational expectations.

Within this class, we are able to analyze the policy problem under learning as a recursive linear-quadratic problem. We find that unlike in the baseline model, the optimal monetary policy under learning "leans against the wind" in the following sense:

Proposition 4. *Consider the class of policies for which $\pi_t = \pi_t^*$ $\mathcal{P}$-almost surely. Within this class, the optimal commitment policy takes the form*

$$\zeta p_t = -\left(m_t + f_t - (1-\delta) f_{t-1}\right)$$

$$f_t = k_h f_{t-1} + c_0 \hat{h}_{t-1}^* + \sum_{s=0}^{\infty} c_s \mathbb{E}_t \hat{q}_{t+s}^*$$

for some $\zeta > 0$, where p_t is the price level. If the gain parameter g is sufficiently small, then the coefficients c_{-1} and $(c_s)_{s=0}^{\infty}$ are all strictly positive.

Proof. See the appendix. □

The proposition establishes that the optimal commitment[14] policy "leans against the wind": Even when production is at its efficient level, i.e. the marginal cost deviation m_t is zero, and the inherited asset stock h_{t-1} is efficient, the welfare-maximizing central bank still wants to set inflation lower than its target if current or future expected asset prices are inefficiently high. Inefficiently high asset prices imply real distortions because they lead to over-investment. Low inflation mitigates over-investment because it induces lower output and higher real interest rates, both of which reduce asset demand.

[14] We can also numerically compute the optimal discretionary policy with the LQ-formulation given in the proof of the proposition, but the presence of several endogenous state variables prevents us from providing an analytical characterization.

The proposition comes with the qualification that the learning gain g must be sufficiently small. For large values of g, we cannot ensure that the central bank always wants to lean against the wind. The reason is that large values of g lead to oscillatory patterns in the response of asset prices to subjective return surprises, including those induced by monetary policy. While it will always be the case that tighter monetary policy will lead to lower asset prices and investment today, the endogenous belief dynamics of the model can lead to higher asset prices and investment in the future, rendering the effects of policy ambiguous. However, for low values of the gain g, tighter monetary policy will not increase asset prices at any future horizon.

7.4 Numerical illustration

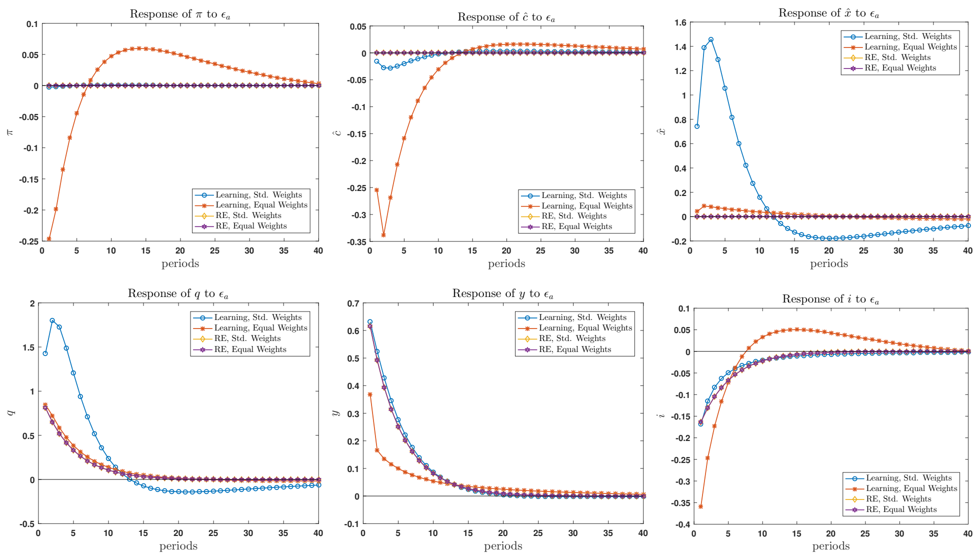
We illustrate our optimal policy results with a simple calibration. We take the same parameters as in the baseline model, and calibrate $\{\theta, g, \omega, \delta, \sigma_A, \sigma_p\}$ so as to match the volatilities of output growth, inflation, house price growth, real wage growth, residential investment, as well as the mean output share of residential investment, under the assumption that monetary policy follows the Taylor rule in (28). The resulting parameter values are $\theta = 0.081$, $g = 0.027$, $\omega = 0.547$, $\delta = 0.021$, $\sigma_A = 0.72\%$, and $\sigma_p = 0.90\%$. As in the baseline model, we set the elasticity of substitution σ such that the welfare weight on inflation in the loss function $\mathcal{L}$ equals $\lambda_\pi = 1$.

In Figure 4, we compare the optimal commitment polices under RE and learning, as derived in Proposition 4.[15] The figure shows impulse responses to a productivity shock.

The optimal policy under rational expectations is simply to fully stabilize inflation with respect to the technology shock. This policy simultaneously closes all welfare-relevant gaps. Under learning, this outcome is infeasible because eliminating nominal rigidities does not also eliminate investment distortions from asset price learning. The impulse responses under learning in Figure 4 illustrate how the central bank leans against the wind. The extent to which the central bank departs from inflation targeting depends on the parameters of the model. With

[15]The impulse response functions under discretion are qualitatively similar.

Figure 4: Optimal commitment policy with asset production.



Note: Responses to a unit standard deviation positive technology shock ε_{At} under sticky prices and with asset production. Log percentage points. In all cases, the elasticity of substitution σ is chosen such that the welfare weight on inflation in the loss function $\mathcal{L}$ equals $\lambda_\pi = 1$. For cases labeled "Equal Weights", all loss function weights are set to $\lambda_\pi = \lambda_x = \lambda_h = 1$.

our calibration and welfare-theoretic weights in the loss function, inflation barely deviates from zero in the top left panel of Figure 4. Consumption (top middle panel) is lower, and investment (top right panel) is higher than under RE because of the strong asset price response (bottom left panel) caused by subjective optimism. Higher investment also leads to higher output (bottom middle panel). The optimal nominal rate (bottom right panel) is set somewhat higher under learning than under RE, but this reaction does not eliminate the investment boom caused by optimistic asset price expectations.

When we change the weights in the loss function to equal weights for inflation, consumption, asset holdings and investment, then leaning against the wind becomes much more pronounced. Inflation drops by a sizable amount and opens up a negative consumption gap. The investment gap stays positive but is much more muted. Likewise, the asset price response gets endogenously dampened by the optimal policy response. The central bank accepts lower output in exchange for preventing an investment boom. The real interest rate (not shown) is above the level under RE throughout the time period shown, although the nominal interest

Table 2: "Leaning against the wind" with asset production, no cost-push shocks.

		$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma\left(\hat{h}_t\right)$	$\sigma(\hat{x}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
Rational Expectations	$\pi_t = 0$	0.000	0.000	0.000	0.000	0.858	0
Learning, standard weighting	$\pi_t = 0$	0.000	0.075	1.836	4.425	1.688	1.000*
	Opt. Commitment Policy	0.006	0.081	1.831	4.392	1.676	0.994*
Learning, equal weighting	$\pi_t = 0$	0.000	0.075	1.836	4.425	1.688	1.000†
	Opt. Commitment Policy	1.178	0.813	1.178	1.678	0.891	0.249†

*,† For each learning case (loss function with standard weights and loss function with equal weights), the loss is normalized to 1 for $\pi_t = 0$.

rate initially falls below the level under RE due to the fall in inflation.

In Table 2, we switch off cost-push shocks and compare strict inflation targeting to the optimal commitment policy. Under rational expectations (top panel), strict inflation targeting is optimal and all gaps are closed. Under learning, this is no longer the case. With standard loss function weights (middle panel), optimal policy reduces asset price and investment fluctuations somewhat at the expense of price stability, though the effect is quantitatively small. With equal weights (bottom panel), the central bank is willing to tolerate a much larger amount of variability in inflation in order to stabilize the economy along the asset investment margin.

In Table 3, we compare outcomes for optimal policies under RE and learning alongside simple rules. The first three rows describe the optimal discretionary and commitment policies as well as the standard Taylor rule used in our calibration. Rows (4) and (5), which describe the corresponding optimal policies under learning, confirm that the central bank cannot attain the same outcomes, as the values of the loss function are higher than under RE. This stems from the fact that the larger asset price volatility translates into inefficient investment fluctuations that cannot be undone by monetary policy without hurting inflation and aggregate demand outcomes at the same time.

We also evaluate simple interest rate rules, as in the baseline model. When we optimize coefficients on an extended interest rate rule of the form in (31), we find a positive coefficient on asset price growth in Row (7) of Table 3. All other results from Section 5 also carry over.

Table 3: Performance of optimized simple rules, with asset production.

	Rational Expectations	$\sigma(\pi_t)$	$\sigma(\hat{c}_t)$	$\sigma\left(\hat{h}_t\right)$	$\sigma(\hat{x}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
(1)	*Optimal Policy, Discretion*	0.528	0.098	0.048	0.163	0.862	1.098
(2)	*Optimal Policy, Commitment*	0.247	0.225	0.154	0.369	0.857	0.432
(3)	$i_t = 1.5\pi_t + 0.125\hat{y}_t$	0.340	0.469	0.226	0.779	0.724	1.316
	Learning	$\sigma(\pi_t)$	$\sigma(\hat{c}_t)$	$\sigma\left(\hat{h}_t\right)$	$\sigma(\hat{x}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
(4)	*Optimal Policy, Discretion*	0.525	0.126	1.798	4.432	1.680	1.148
(5)	*Optimal Policy, Commitment*	0.247	0.231	1.817	4.481	1.678	0.542
(6)	$i_t = 1.5\pi_t + 0.125\hat{y}_t$	0.154	0.472	0.662	3.069	1.515	1.000
(7)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*) \cdot ...$	0.156	0.431	0.509	2.456	1.366	0.765
	$\left(1.5\pi_t + \phi_y^* \hat{y}_t + \phi_q^* \sum_{s=0}^{\infty} \omega^{*s} \Delta q_{t-s}\right)$						
	$\{\rho_i^*, \phi_y^*, \phi_q^*, \omega^*\} = \{0.370, 0.232, 0.036, 0.999\}$						

Note: Losses are evaluated as the unconditional expectation of the loss function $\mathcal{L}$. The welfare weight on inflation is set to $\lambda_\pi = 1$ and the losses are normalized to 1 for row (6).

8 Conclusion

In this paper, we have characterized optimal monetary policy in a model in which agents are learning about asset prices. Our model is the standard New-Keynesian model with a long-term asset. Agents form expectations about asset prices in an extrapolative fashion. However, their expectations remain model-consistent, conditional on their beliefs about asset prices, which allows us to isolate the effects of learning about asset prices from the many other ways in which distorted beliefs can affect the economy. Learning amplifies asset price fluctuations in the model, and leads to perceived wealth effects that create inefficient fluctuations in consumption, saving, and investment decisions.

We have given an analytical solution to the optimal policy with learning. Our central insight is that the natural real rate of interest under learning depends positively on asset price expectations and realized asset prices. In our baseline model, flexible inflation targeting remains the optimal target criterion for monetary policy, but it requires a very different implementation: The interest rate has to increase with asset prices and subjective expectations of future capital gains. When we extend the model to allow for asset production, it becomes beneficial to "lean against the wind", i.e. tolerate low inflation when asset prices are high, in order to mitigate inefficient investment fluctuations. Our results are robust to a wide range of alternative belief specifications.

Our model is highly stylized, which allows us to derive many results analytically. However, the method for computing optimal policy that we present here is readily applicable to larger linear models with learning. Future work could evaluate our findings from a quantitative perspective.

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A Details on the derivations

A.1 Asset demand in the model with fixed asset supply

We reduce the system (3)–(10) to the equations:

$$
\begin{aligned}
q_t &= \gamma c_t - (1-\beta)\,\theta h_t - \beta\gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \beta \mathbb{E}_t^{\mathcal{P}} q_{t+1} \\
c_t &= \frac{1+\phi}{1+\phi-\alpha} a_t - \frac{\alpha\gamma}{1+\phi-\alpha} n_t - \frac{\bar{Q}\bar{H}}{\bar{Y}}\left(h_t - h_{t-1}\right) \\
q_t &= q_{t-1} + \hat{\mu}_{t-1} + z_t \\
\hat{\mu}_t &= \rho_\mu \hat{\mu}_{t-1} + g z_t
\end{aligned}
$$

and conjecture a solution for h_t of the form in (18). Substituting the guess and solving for the coefficients yields:

$$
\begin{aligned}
k_h &= \frac{1}{2\beta}\left(1+\beta+\frac{\theta}{\gamma\kappa_1}(1-\beta) - \sqrt{\left(1+\beta+\frac{\theta}{\gamma\kappa_1}(1-\beta)\right)^2 - 4\beta}\right) \in (0,1) \\
k_a &= \frac{\kappa_0\left(1-\beta\rho_a\right)}{\kappa_1 + (1-\beta)\frac{\theta}{\gamma} + \beta\kappa_1\left(1-k_h-\rho_a\right)} > 0 \\
k_q &= \frac{1}{\gamma}\frac{1-\beta}{(1-\beta)\frac{\theta}{\gamma} + \kappa_1\left(1-\beta k_h\right)} > 0 \\
k_\mu &= \frac{1}{\gamma}\frac{\beta}{(1-\beta)\frac{\theta}{\gamma} + \kappa_1\left(1-\beta\rho_\mu\right) + \kappa_1\beta\left(1-k_h\right)}\frac{(1-\beta)\frac{\theta}{\gamma} + \kappa_1\beta\left(1-k_h\right)}{(1-\beta)\frac{\theta}{\gamma} + \kappa_1\left(1-\beta k_h\right)} > 0.
\end{aligned}
$$

A.2 Asset demand with extended beliefs

We reduce the system (3)–(10) to

$$
\begin{aligned}
q_t &= \gamma c_t - (1-\beta)\,\theta h_t - \beta\gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \beta \mathbb{E}_t^{\mathcal{P}} q_{t+1} \\
c_t &= \frac{1+\phi}{1+\phi-\alpha} a_t - \frac{\alpha\gamma}{1+\phi-\alpha} n_t - \frac{\bar{Q}\bar{H}}{\bar{Y}}\left(h_t - h_{t-1}\right)
\end{aligned}
$$

and directly conjecture a solution for h_t of the form in (86). Substituting the guess and solving for the coefficients yields the same coefficients k_h, k_a, k_q as in the baseline model, and $\tilde{k}_\mu$ reads as follows:

$$\tilde{k}_\mu = \frac{\beta/\gamma}{(1-\beta)\frac{\theta}{\gamma} + \kappa_1(1+\beta-\beta k_h)} \frac{(1-\beta)\frac{\theta}{\gamma} + \kappa_1\beta(1-k_h)}{(1-\beta)\frac{\theta}{\gamma} + \kappa_1(1-\beta k_h)} > 0.$$

The natural real rate of interest under $\mathcal{P}$ is now given by

$$\frac{r_t^n - r_t^{n,RE}}{\gamma\kappa_1} = k_a(2-\rho_a-k_h)a_t - (1-k_h)^2 h_{t-1}^n - k_q(1-k_h)q_t$$
$$+ \left(\left(2-\tilde{k}_h\right)\tilde{k}_\mu + k_q\right)\mathbb{E}_t^{\mathcal{P}}\Delta q_{t+1} - \tilde{k}_\mu\left(1-\beta+\beta(1-k_h)^2\right)\sum_{s=0}^{\infty}(\beta k_h)^s\,\mathbb{E}_t^{\mathcal{P}}\Delta q_{t+s+2}. \tag{39}$$

A.3 Linearized equilibrium conditions and natural rate in the model with asset production

Under RE, the following set of equations describe the linearized equilibrium (up to a monetary policy rule):

$$y_t = a_t + \alpha n_t \tag{40}$$
$$w_t = m_t + a_t - (1-\alpha)n_t \tag{41}$$
$$\pi_t = \beta\pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}m_t + \eta_t \tag{42}$$
$$w_t = \gamma c_t + \phi n_t \tag{43}$$
$$\bar{Y}y_t = \bar{C}c_t + \bar{Q}\bar{H}\delta x_t \tag{44}$$
$$\delta x_t = h_t - (1-\delta)h_{t-1} \tag{45}$$
$$x_t = \frac{\omega}{1-\omega}q_t \tag{46}$$
$$i_t = \gamma(\mathbb{E}_t c_{t+1} - c_t) + \mathbb{E}_t\pi_{t+1} \tag{47}$$
$$q_t + \left(1-\tilde{\beta}\right)\theta h_t = \gamma c_t + \tilde{\beta}\mathbb{E}_t(q_{t+1} - \gamma c_{t+1}). \tag{48}$$

where $\tilde{\beta} = \beta(1-\delta)$. The flexible price equilibrium under RE is characterized as:

$$h_t^{n,RE} = k_h^{RE}h_{t-1}^{n,RE} + k_a^{RE}a_t,\ k_h^{RE} \in (0, 1-\delta),\ k_a^{RE} > 0 \tag{49}$$

$$x_t^{RE} = -\frac{1-\delta-k_h^{RE}}{\delta} h_{t-1}^{n,RE} + \frac{k_a^{RE}}{\delta} a_t \tag{50}$$

$$y_t^{n,RE} = \frac{\bar{C}}{\bar{Y}} \kappa_0 a_t + \frac{\alpha\gamma}{1+\phi-\alpha} \kappa_1 x_t \tag{51}$$

$$c_t^{n,RE} = \kappa_0 a_t - \kappa_1 x_t \tag{52}$$

$$\begin{aligned} r_t^{n,RE} &= \gamma \left(\frac{\kappa_1}{\delta} \left(1-\rho_a+1-\delta-k_h^{RE}\right) k_a^{RE} - \kappa_0 \left(1-\rho_a\right) \right) a_t \\ &\quad - \gamma \frac{\kappa_1}{\delta} \left(1-\delta-k_h^{RE}\right) \left(1-k_h^{RE}\right) h_{t-1}^{RE} \end{aligned} \tag{53}$$

where the coefficients k_h^{RE} and k_a^{RE} are given by:

$$\begin{aligned} k_h^{RE} &= \frac{1}{2\tilde{\beta}} \left(1+\tilde{\beta}\left(1-\delta\right) + \frac{\theta}{\gamma} \frac{\left(1-\tilde{\beta}\right)\delta}{\kappa_1 + \frac{1-\omega}{\gamma\omega}} - \sqrt{\left(1+\tilde{\beta}\left(1-\delta\right) + \frac{\theta}{\gamma} \frac{\left(1-\tilde{\beta}\right)\delta}{\kappa_1 + \frac{1-\omega}{\gamma\omega}}\right)^2 - 4\tilde{\beta}\left(1-\delta\right)} \right) \\ &\in \left(0, 1-\delta\right) \end{aligned}$$

$$k_a^{RE} = \frac{\kappa_0 \left(1-\tilde{\beta}\rho_a\right)}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma} + \delta^{-1}\left(\kappa_1 + \frac{1-\omega}{\gamma\omega}\right)\left(1+\tilde{\beta}\left(1-\delta-k_h-\rho_a\right)\right)} > 0.$$

The constant κ_0 is given by

$$\kappa_0 = \frac{1+\phi}{\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right)+\alpha\gamma}.$$

The sticky-price RE equilibrium is characterized in deviation from the flexible price equilibrium through the equations:

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{\left(1-\xi\right)\left(1-\beta\xi\right)}{\xi} m_t + \eta_t \tag{54}$$

$$m_t = \frac{\bar{C}\left(1+\phi-\alpha\right)+\bar{Y}\alpha\gamma}{\bar{Y}\alpha}\left(\tilde{c}_t + \kappa_1 \tilde{x}_t\right) \tag{55}$$

$$i_t = \gamma\left(\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t\right) + \mathbb{E}_t \pi_{t+1} + r_t^n \tag{56}$$

$$\theta\left(1-\tilde{\beta}\right)\hat{h}_t = \tilde{\beta}\mathbb{E}_t\left(\frac{1-\omega}{\omega}\hat{x}_{t+1} - \gamma\hat{c}_{t+1}\right) - \left(\frac{1-\omega}{\omega}\hat{x}_t - \gamma\hat{c}_t\right). \tag{57}$$

Under learning, we can do a similar exercise. We first tackle the PLM. All we do is to replace the market clearing condition (46) with the subjective law of motion for asset prices:

$$y_t = a_t + \alpha n_t \tag{58}$$

$$w_t = m_t + a_t - (1-\alpha)\, n_t \tag{59}$$

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} m_t + \eta_t \tag{60}$$

$$w_t = \gamma c_t + \phi n_t \tag{61}$$

$$\bar{Y} y_t = \bar{C} c_t + \bar{Q}\bar{H}\left(h_t - (1-\delta)\, h_{t-1}\right) \tag{62}$$

$$i_t = \gamma\left(\mathbb{E}_t^{\mathcal{P}} c_{t+1} - c_t\right) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} \tag{63}$$

$$q_t = \gamma c_t - \left(1-\tilde{\beta}\right)\theta h_t - \tilde{\beta}\gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \tilde{\beta}\mathbb{E}_t^{\mathcal{P}} q_{t+1} \tag{64}$$

$$q_t = q_{t-1} + \hat{\mu}_{t-1} + z_t \tag{65}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t. \tag{66}$$

Under learning and flexible prices, we can boil things down to these two equations to solve for the PLM:

$$0 = \gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha}\right)\left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}} \tilde{h}_t\right) - \frac{1+\phi}{\alpha} a_t \tag{67}$$

$$q_t = \gamma c_t - \left(1-\tilde{\beta}\right)\theta h_t - \tilde{\beta}\gamma \mathbb{E}_t c_{t+1} + \beta\delta \mathbb{E}_t^{\mathcal{P}} q_{t+1}. \tag{68}$$

Guess and verify

$$h_t^n = k_a a_t + k_h h_{t-1}^n - k_q q_t + k_\mu \hat{\mu}_{t-1} \tag{69}$$

where the coefficients are given by:

$$k_h = \frac{1}{2\tilde{\beta}}\left(1 + \tilde{\beta}(1-\delta) + \frac{\theta\delta}{\gamma\kappa_1}\left(1-\tilde{\beta}\right) - \sqrt{\left(1 + \tilde{\beta}(1-\delta) + \frac{\theta\delta}{\gamma\kappa_1}\left(1-\tilde{\beta}\right)\right)^2 - 4\tilde{\beta}(1-\delta)}\right)$$
$$\in \left(k_h^{RE}, 1-\delta\right)$$

$$k_a = \kappa_0 \frac{1-\tilde{\beta}\rho_a}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma} + \kappa_1\delta^{-1}\left(1+\tilde{\beta}(1-\delta-k_h-\rho_a)\right)} > k_a^{RE}$$

$$k_q = \frac{1}{\gamma}\frac{1-\tilde{\beta}}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma} + \kappa_1\delta^{-1}\left(1-\tilde{\beta}k_h - \tilde{\beta}\delta\right)} > 0$$

$$k_\mu = \frac{1}{\gamma}\frac{\tilde{\beta}}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma} + \kappa_1\delta^{-1}\left(1-\tilde{\beta}\rho_\mu + \tilde{\beta}(1-\delta-k_h)\right)}\frac{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma} + \kappa_1\delta^{-1}\tilde{\beta}(1-\delta-k_h)}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma} + \kappa_1\delta^{-1}\left(1-\tilde{\beta}k_h - \tilde{\beta}\delta\right)} > 0$$

We can characterize the flexible-price PLM investment, consumption, output and interest rates:

$$
\begin{aligned}
x_t^n &= x_t^{n,RE} + \frac{k_h - k_h^{RE}}{\delta} h_{t-1}^n + \frac{k_h^{RE} - 1 + \delta}{\delta} \left(h_{t-1}^n - h_{t-1}^{n,RE} \right) \\
&\quad + \frac{k_a - k_a^{RE}}{\delta} a_t - \frac{k_q}{\delta} q_t + \frac{k_\mu}{\delta} \hat{\mu}_{t-1} && (70) \\
c_t^n &= c_t^{n,RE} - \kappa_1 \left(x_t^n - x_t^{n,RE} \right) && (71) \\
y_t^n &= y_t^{n,RE} + \frac{\alpha\gamma}{1+\phi-\alpha} \kappa_1 \left(x_t^n - x_t^{n,RE} \right) && (72) \\
r_t^n &= r_t^{n,RE} - \frac{\gamma\kappa_1}{\delta} \left((1-\delta-k_h)(1-k_h) - \left(1-\delta-k_h^{RE}\right)\left(1-k_h^{RE}\right) \right) h_{t-1}^n \\
&\quad - \frac{\gamma\kappa_1}{\delta} \left(1-\delta-k_h^{RE}\right)\left(1-k_h^{RE}\right)\left(h_{t-1}^n - h_{t-1}^{n,RE} \right) - \frac{\gamma\kappa_1}{\delta} k_q (1-\delta-k_h) q_t \\
&\quad + \frac{\gamma\kappa_1}{\delta} \left((1-\rho_a+1-\delta)\left(k_a - k_a^{RE}\right) - k_h k_a + k_h^{RE} k_a^{RE} \right) a_t \\
&\quad + \frac{\gamma\kappa_1}{\delta} \left(k_\mu (1-\rho_\mu+1-\delta-k_h) + k_q \right) \hat{\mu}_{t-1}. && (73)
\end{aligned}
$$

In order to find the ALM under flexible prices, we impose market clearing for the asset and obtain:

$$
\begin{aligned}
\frac{\omega\delta}{1-\omega} q_t^* &= k_a a_t + k_h h_{t-1}^* - k_q q_t^* + k_\mu \hat{\mu}_{t-1}^* - (1-\delta) h_{t-1}^* \\
\Leftrightarrow q_t^* &= \frac{1}{\frac{\omega}{1-\omega}\delta + k_q} \left(k_a a_t + k_\mu \hat{\mu}_{t-1}^* - (1-\delta-k_h) h_{t-1}^* \right). && (74)
\end{aligned}
$$

The equilibrium price is increasing in productivity, increasing in asset price beliefs, and decreasing in the existing asset stock.

When the equilibrium asset price q_t from equation (74) is substituted into the expression for the natural rate r_t^n, the natural real rate in the ALM r_t^{n*} is increasing in asset price expectations $\hat{\mu}_{t-1}^*$, just as in the baseline model. The sign of the other coefficients are ambiguous and depend on the parameterization.

A.4 Welfare approximations

For the model with fixed supply, the approximation of welfare is standard. Following e.g. Woodford, 2003, welfare is approximated by

$$
\mathcal{L} = \sigma \frac{\xi}{(1-\xi)(1-\beta\xi)} \frac{\alpha}{1+\phi-\alpha+\alpha\gamma} \left(\pi_t^*\right)^2 + \left(\hat{y}_t^*\right)^2.
$$

Note that this loss function approximates welfare under the equilibrium law of motion $\mathbb{P}$, which takes into account that the asset supply is fixed. Welfare under the subjective law of motion $\mathcal{P}$ takes on a different form.

For the model with asset production, we can derive an approximation of the loss function under $\mathbb{P}$ as:

$$\mathcal{L}_t = \sigma \frac{\xi}{1-\xi} \frac{1}{1-\beta\xi} \left(\pi_t^*\right)^2 + \left(\gamma + \frac{1+\phi-\alpha}{\alpha} \frac{\bar{C}}{\bar{Y}}\right) \frac{\bar{C}}{\bar{Y}} \left(\hat{c}_t + \kappa_1 \hat{x}_t\right)^2$$
$$+ \left(1 - \beta\left(1-\delta\right)\right) \theta \frac{\bar{Q}\bar{H}}{\bar{Y}} \hat{h}_t^2 + \frac{\bar{Q}\bar{H}}{\bar{Y}} \left(\left(1-\omega\right)\delta^2 + \gamma\delta\kappa_1\right) \hat{x}_t^2.$$

B Proofs

Proof of Proposition 1. Suppose that the central bank implemented $\pi_t = 0$. The Phillips curve (24) then reduces to the relationship

$$\tilde{y}_t = \frac{\gamma\alpha}{1+\phi-\alpha\left(1-\gamma\right)} \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta\tilde{h}_t.$$

Substituting into the asset demand equation (27), we obtain a second-order difference equation of the form

$$\left(1-\beta\right) \frac{\theta}{\gamma} \tilde{h}_t = -\kappa_1 \left(\Delta\tilde{h}_t - \beta \mathbb{E}_t^{\mathcal{P}} \Delta\tilde{h}_{t+1}\right).$$

It is easily verified that the only solution to this equation is $\tilde{h}_t = 0$. But this implies that we implement the flexible price allocation under the subjective law of motion. From the perspective of agents, the flexible price allocation is first-best efficient, and $\mathcal{L}_t = 0$. Moreover, the actual equilibrium in this economy has $\pi_t^* = \pi_t = 0$ and $y_t^* = y_t^{n,RE}$, as was shown in the last section. This allocation is also first-best efficient under model-consistent expectations.

To show that the rule $i_t = r_t^n + \phi_\pi \pi_t$ implements $\pi_t = 0$, we first substitute it into the IS curve (26)and note that the system (24)–(27) admits the solution $\pi_t = 0$. We then need to verify that the rule also ensures determinacy of the equilibrium for $\phi_\pi > 1$. Following e.g. Lubik and Marzo (2007), we express the system (24)–(27)

in the form

$$\overbrace{\begin{pmatrix} \beta & 0 & \kappa\kappa_1 \\ 1 & \gamma & 0 \\ 0 & -\beta\gamma & -\theta\left(1-\beta\right) \end{pmatrix}}^{=:B}_{t+1} x_{t+1} + \overbrace{\begin{pmatrix} -1 & \kappa & -\kappa\kappa_1 \\ -\phi & -\gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}}^{=:C} x_t = 0$$

where $x_t = \left(\pi_t, \tilde{c}_t, \tilde{h}_{t-1}\right)'$ and

$$\kappa = \frac{\left(1-\xi\right)\left(1-\beta\xi\right)}{\xi}\frac{1+\phi-\alpha+\alpha\gamma}{\alpha}.$$

Because we have two terminal and one initial condition (i.e. two forward- and one backward-looking variable), we then need to verify that the matrix $-B^{-1}C$ has exactly two eigenvalues outside the unit circle. The characteristic polynomial $\det\left(B^{-1}C-\lambda I\right)$ is of the form $A_3\lambda^3+A_2\lambda^2+A_1\lambda+A_0=0$, where the coefficients are given by:

$$\begin{aligned}
A_3 &= \beta\gamma\left(\kappa\kappa_1+\theta\left(1-\beta\right)\right) > 0 \\
A_2 &= -\gamma\kappa\kappa_1\left(1+\beta+\beta\phi\right)-\left(1-\beta\right)\theta\left(\kappa+\gamma+\gamma\beta\right) < 0 \\
A_1 &= \gamma\kappa\kappa_1\left(1+\phi+\beta\phi\right)+\left(1-\beta\right)\theta\left(\gamma+\kappa\phi\right) > 0 \\
A_0 &= -\gamma\kappa\kappa_1\phi < 0.
\end{aligned}$$

We then verify the following sufficient conditions for two roots outside the unit circle:

$$\begin{aligned}
A_3+A_2+A_1+A_0 &> 0 \\
-A_3+A_2-A_1+A_0 &< 0 \\
A_0\left(A_0-A_2\right)+A_3\left(A_1-A_3\right) &> 0.
\end{aligned}$$

The first inequality follows from $A_3+A_2+A_1+A_0=\left(1-\beta\right)\theta\kappa\left(\phi-1\right)$, which is positive when $\phi>1$. The second inequality follows directly from the signs of the coefficients. To establish the third inequality, we collect powers of ϕ to express

$$\begin{aligned}
&A_0\left(A_0-A_2\right)+A_3\left(A_1-A_3\right) \\
&=\left(\gamma\kappa\kappa_1\right)^2\left(1-\beta\right)\phi^2 \\
&+\left[A_3\left(1-\beta\right)\theta\kappa+\gamma\kappa\kappa_1\left(\left(1+\beta\right)\left(A_3+\left(1-\beta\right)\theta\gamma+\kappa\kappa_1\right)+\left(1-\beta\right)\theta\kappa\right)\right]\phi
\end{aligned}$$

$$+A_3^2\left(\frac{1-\beta}{\beta}\right).$$

This is a quadratic polynomial in ϕ for which all coefficients are positive, implying that it takes positive values for all $\phi \geq 0$. □

Proof of Proposition 2. Substituting the policy $\pi_t = -(\beta\rho_\eta)^{-1}\eta_t + b_t z_t$ into the Phillips curve (24), we obtain:

$$\pi_t = -(\beta\rho_\eta)^{-1}\eta_t + b_t z_t = \kappa\left(\tilde{c}_t + \kappa_1\Delta\tilde{h}_t\right)$$

with κ defined as in the proof of Proposition 1. We will find b_t such that $\pi_t^* = 0\,\mathbb{P}$-a.s., i.e. under the equilibrium law of motion. In this case, $\tilde{c}_t^* + \kappa_1\Delta\tilde{h}_t^* = y_t^* - y_t^{RE} = 0$ under the ALM as well, and the first-best allocation is attained.

In order to solve for b_t, then, we need to compute the equilibrium value of z_t, which amounts to solving for the equilibrium price q_t^*. To do this, we first need to derive the demand function for the asset under the subjective law of motion. Let $x_t = \tilde{c}_t + \kappa_1\Delta\tilde{h}_t$. In analogy to the computations of the flexible-price equilibrium, we can compute the asset demand function as the solution to the following system of equations:

$$(1-\beta)\,\theta h_t = \gamma c_t - q_t - \beta\mathbb{E}_t^{\mathcal{P}}\left[\gamma c_{t+1} - q_{t+1}\right]$$

$$(1+\phi-\alpha+\alpha\gamma)\,c_t = \alpha x_t + (1-\alpha)\,a_t - (1+\phi-\alpha)\,\frac{\bar{Q}\bar{H}}{\bar{Y}}\Delta h_t.$$

With $x_t = \left(b_t z_t - (\beta\rho_\eta)^{-1}\eta_t\right)/\kappa$, we can solve:

$$h_t = k_h h_{t-1} + k_a a_t - k_q q_t + k_\mu\hat{\mu}_{t-1} + k_{bz} b_t z_t + k_\eta\eta_t.$$

Here, the coefficients k_h, k_a, k_q and k_μ are the same as in the flexible-price demand function (18). The coefficients k_η and k_{bz} are given by:

$$k_\eta = -\frac{k_h}{\beta\rho_\eta\kappa}\frac{1-\beta\rho_\eta}{1-k_h\beta\rho_\eta}\frac{\bar{Y}}{\bar{Q}\bar{H}}\frac{\alpha}{1+\phi-\alpha}$$

$$k_{z,t} = -\frac{k_h}{\kappa}\frac{\bar{Y}}{\bar{Q}\bar{H}}\frac{\alpha}{1+\phi-\alpha}$$

Now the equilibrium is found by imposing $h_t^* = 0$, and this condition leads to the following expression for z_t^*:

$$z_t^* = \frac{k_q}{k_q - k_{bz} b_t} \left(\frac{1}{k_q} \left(k_a a_t + k_\mu \hat{\mu}_{t-1}^* + k_\eta \eta_t \right) - q_{t-1}^* - \hat{\mu}_{t-1}^* \right)$$

Imposing $\beta \rho_\eta b_t z_t^* = \eta_t$ then leads to:

$$b_t = \left(\frac{k_{bz}}{k_q} + \frac{\beta \rho_\eta}{k_q} \left(k_a a_t + k_\mu \hat{\mu}_{t-1}^* + k_\eta \eta_t \right) - \beta \rho_\eta \left(q_{t-1}^* + \hat{\mu}_{t-1}^* \right) \right)^{-1} \eta_t.$$

□

Proof of Proposition 3. When $\pi_t = \pi_t^*$ $\mathbb{P}$-a.s., we can write $\mathbb{E}_t^{\mathcal{P}} \pi_t = \mathbb{E}_t \pi_{t+1}$ in the Phillips curve (24). The Phillips curve then implies that $m_t = m_t^*$ $\mathbb{P}$-a.s. as well. Furthermore, combining the static equilibrium conditions for labor supply and demand, the production function and the household budget constraint, we obtain:

$$\frac{1+\phi-\alpha+\alpha\gamma}{\alpha} y_t = m_t + \frac{1+\phi}{\alpha} a_t + \gamma \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta h_t.$$

This equation has to hold regardless of expectations. Furthermore, under the equilibrium law of motion we have $\Delta h_t^* = 0$, while under RE and flexible prices, we have $\Delta h_t^{n,RE} = 0$ and $m_t^{n,RE} = 0$. Together, these conditions imply that equilibrium output under learning equals $y_t^* = y_t^{n,RE} + \alpha / \left(1+\phi-\alpha+\alpha\gamma\right) m_t$. We can therefore write the policy problem under learning as follows:

$$\max \sum_{t=0}^{\infty} \beta^t \left(\lambda \pi_t^2 + \left(\hat{y}_t^*\right)^2 \right)$$

$$\text{s.t. } \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} m_t + \eta_t$$
$$\hat{y}_t^* = \frac{\alpha}{1+\phi-\alpha+\alpha\gamma} m_t.$$

This problem is identical to the policy problem under rational expectations with standard solutions to the commitment and discretion policies. This establishes parts (1) and (2) of the proposition.

To prove part (3), we write the IS equation (26) as

$$i_t = r_t^n + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \gamma \mathbb{E}_t^{\mathcal{P}} \Delta \tilde{c}_{t+1}.$$

Let $\phi_\pi > 1$. The optimality conditions for π_t in cases (1) and (2) combined with (24) and (27) imply that $(\pi_t)_{t=0}^{\infty}$ and $(\tilde{c}_t)_{t=0}^{\infty}$ are linear processes of the cost-push shock η_t only. Therefore, we can evaluate $\mathbb{E}_t^{\mathcal{P}}\pi_{t+1} + \gamma\mathbb{E}_t^{\mathcal{P}}\Delta\tilde{c}_{t+1} - \phi_\pi\pi_t = A(L)\eta_t$ for some lag polynomial $A(L)$. The rule satisfies determinacy by the same argument made in the proof of Proposition 1.

We start by noting from the Phillips curve that $c_t + \kappa_1 x_t = \alpha / \left(\frac{\bar{C}}{\bar{Y}}(1+\phi-\alpha) + \alpha\gamma\right) m_t$ and $m_t = 0$ under flexible prices. Therefore, we can rewrite

$$\hat{c}_t + \kappa_1\hat{x}_t = \alpha / \left(\frac{\bar{C}}{\bar{Y}}(1+\phi-\alpha) + \alpha\gamma\right) m_t^*.$$

Dividing through the (positive) coefficient on m_t^*, we can bring the loss function (34) into the form

$$\mathcal{L}_t = \ell_\pi(\pi_t^*)^2 + (m_t^*)^2 + \ell_x(\hat{x}_t^*)^2 + \ell_h\left(\hat{h}_t^*\right)^2,$$

where $\ell_\pi, \ell_x, \ell_h > 0$.

Next, we note that π_t has the same distribution under the subjective belief measure $\mathcal{P}$ as under the actual belief measure $\mathbb{P}$. We can therefore omit the asterisk notation and simply write $\pi_t^* = \pi_t$ in the planner's problem. Equations (35) and (38) then imply that the same holds true for $\tilde{c}_t$, $\tilde{h}_t$ and m_t. To evaluate the asset and investment gaps in the loss function, we note that

$$\hat{h}_t^* = \left(h_t - h_t^{n,RE}\right)^* = \tilde{h}_t + h_t^{n*} - h_t^{n,RE}. \tag{75}$$

The first equality is the definition of the asset gap, and the second equality follows from the fact that $\tilde{h}_t = \tilde{h}_t^*$ and $h_t^{n,RE}$ is independent of the learning process. To find h_t^{n*}, we can make use of the closed-form expression for h_t^n derived in (69).

The policymaker's commitment problem can then be described as follows (Lagrange multipliers to the constraints are in parentheses):

$$\max \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(\ell_\pi\pi_t^2 + m_t^2 + \ell_x(\hat{x}_t^*)^2 + \ell_h\left(\hat{h}_t^*\right)^2\right)$$

$$\text{s.t. } \pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}m_t \tag{μ_t}$$

$$\theta\left(1-\tilde{\beta}\right)\tilde{h}_t = \frac{\alpha\gamma}{\frac{\bar{C}}{\bar{Y}}(1+\phi-\alpha)+\alpha\gamma}\left(m_t - \tilde{\beta}\mathbb{E}_t m_{t+1}\right) - \gamma\kappa_1\left(\tilde{x}_t - \tilde{\beta}\mathbb{E}_t^{\mathcal{P}}\tilde{x}_{t+1}\right) \tag{χ_t}$$

$$\hat{h}_t^* = \tilde{h}_t + h_t^{n*} - h_t^{n,RE} \tag{$\hat{\psi}_t$}$$

$$h_t^{n*} = k_a a_t + k_h h_{t-1}^{n*} - k_q q_t^* + k_\mu \hat{\mu}_{t-1}^* \tag{ψ_t}$$

$$\frac{\omega}{1-\omega} q_t^* = x_t^{n*} + \tilde{x}_t \tag{Ω_{qt}}$$

$$\hat{\mu}_t^* = \rho_\mu \hat{\mu}_{t-1}^* + g\left(\Delta q_t^* - \hat{\mu}_{t-1}^*\right). \tag{$\Omega_{\mu t}$}$$

The first two constraints are the Phillips curve and the asset Euler equation in gap form (where we have substituted out $\tilde{c}_t$), which only involve variables that are measurable under $\mathbb{P}$. Next, the policymaker needs to evaluate the housing and investment gaps, for which we make use of the identity (75) and the law of motion (69) for h_t^n. We impose equilibrium through the market-clearing condition (Ω_{qt}), and finally have to take into account the effect of prices q_t on beliefs $\hat{\mu}_t$, for which we can combine equations (65)–(66).

This problem takes the form of a recursive linear-quadratic dynamic programming problem, for which commitment and discretionary solutions are well understood. The discretionary policy is difficult to evaluate analytically, but easy to compute numerically (following e.g. REF).

For the commitment solution, the first-order conditions of the planner are:

$$\ell_\pi \pi_t = \Delta \mu_t \tag{76}$$

$$\frac{(1-\xi)(1-\beta\xi)}{\xi}\mu_t = -m_t - \frac{\bar{Y}\alpha}{\bar{C}(1+\phi-\alpha)+\bar{Y}\alpha\gamma}\left(\chi_t - (1-\delta)\chi_{t-1}\right) \tag{77}$$

$$\theta\left(1-\tilde{\beta}\right)\chi_t = \gamma\frac{\kappa_1}{\delta}\left(\tilde{\beta}\mathbb{E}_t\left(\chi_{t+1} - (1-\delta)\chi_t\right) - \chi_t + (1-\delta)\chi_{t-1}\right) + \hat{\psi}_t + \frac{\Omega_{qt} - \tilde{\beta}\mathbb{E}_t\Omega_{qt+1}}{\delta} \tag{78}$$

$$\hat{\psi}_t = \ell_h \hat{h}_t^* + \ell_x \frac{\hat{x}_t^* - \tilde{\beta}\mathbb{E}_t\hat{x}_{t+1}^*}{\delta} \tag{79}$$

$$\psi_t = \hat{\psi}_t + \beta k_h \mathbb{E}_t \psi_{t+1} + \frac{\Omega_{qt} - \tilde{\beta}\mathbb{E}_t\Omega_{qt+1}}{\delta} \tag{80}$$

$$\frac{\omega}{1-\omega}\Omega_{qt} = g\left(\Omega_{\mu t} - \beta\mathbb{E}_t\Omega_{\mu t+1}\right) - k_q\psi_t \tag{81}$$

$$\Omega_{\mu t} = \beta k_\mu \mathbb{E}_t\psi_{t+1} + \beta\left(\rho_\mu - g\right)\mathbb{E}_t\Omega_{\mu t+1} \tag{82}$$

We can combine optimality conditions (76) and (77) into

$$\ell_\pi \frac{(1-\xi)(1-\beta\xi)}{\xi} p_t = -m_t - \left(f_t - (1-\delta) f_{t-1}\right). \tag{83}$$

with $f_t = \frac{\bar{Y}\alpha}{\bar{C}(1+\phi-\alpha)+\bar{Y}\alpha\gamma}\chi_t$. We conjecture the solution

$$\chi_t = k_h \chi_{t-1} + \sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{\psi}_{t+s} \tag{84}$$

where $0 < k_h < 1-\delta$, $b_0 > 0$ and $b_s > \tilde{\beta} b_{s-1}$ for all $s \geq 1$. This conjecture implies that

$$\begin{aligned}
\sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{\psi}_{t+s} &= \sum_{s=0}^{\infty} b_s \mathbb{E}_t \left(\ell_h \hat{h}^*_{t+s} + \ell_x \frac{\hat{x}^*_{t+s} - \tilde{\beta}\mathbb{E}_t \hat{x}^*_{t+s+1}}{\delta} \right) \\
&= \frac{\ell_x}{\delta} \sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{x}^*_{t+s} - \frac{\ell_x}{\delta} \tilde{\beta} \sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{x}^*_{t+s+1} + \ell_h \sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{h}^*_{t+s} \\
&= \frac{\ell_x}{\delta} b_0 \hat{x}^*_t + \frac{\ell_x}{\delta} \sum_{s=1}^{\infty} \left(b_s - \tilde{\beta} b_{s-1} \right) \mathbb{E}_t \hat{x}^*_{t+s} \\
&+ \ell_h \left(\sum_{s=0}^{\infty} b_s (1-\delta)^{s+1} \right) \hat{h}^*_{t-1} + \ell_h \sum_{s=0}^{\infty} \left(\sum_{\tau=s}^{\infty} (1-\delta)^{\tau} b_s \right) \hat{x}^*_{t+s},
\end{aligned}$$

so that the coefficients on $\hat{h}^*_{t-1}$ and $\mathbb{E}_t \hat{x}^*_{t+s}$, $s \geq 0$ are all strictly positive. Moreover, we can appeal to the market clearing condition (46) to write $\mathbb{E}_t \hat{x}^*_{t+s} = \frac{\omega}{1-\omega} \mathbb{E}_t \hat{q}^*_{t+s}$.

What is left is to verify conjecture (84). The equation (78) is a second-order difference equation in χ_t when $\hat{\psi}_t$ and Ω_{qt} are treated as given. The homogeneous form of this second-order difference equation is the same as that for the natural level of asset holdings h^n_t derived earlier. This establishes that k_h is indeed in $(0, 1-\delta)$. Furthermore, equations 80–82 form a linear forward-looking system in $(\Omega_{qt}, \Omega_{\mu t}, \psi_t)$ when $\hat{\psi}_t$ is treated as given. This implies that the solution in (84) will indeed depend only on forward-looking terms in $\hat{\psi}_t$.

Finally, we need to show that $b_s > 0$ for all $s \geq 0$. We prove this property for $g = 0$ and then make a continuity argument to extend it to g small enough. When $g = 0$, equation (81) reduces to $\Omega_{qt} = -\frac{1-\omega}{\omega} k_q \psi_t$. Substituting into (80), we obtain

$$\begin{aligned}
\psi_t &= \hat{\psi}_t + \beta k_h \mathbb{E}_t \psi_{t+1} - \frac{1-\omega}{\omega\delta} k_q \left(\psi_t - \tilde{\beta} \mathbb{E}_t \psi_{t+1} \right) \\
&= \frac{\hat{\psi}_t}{1 + \frac{1-\omega}{\omega\delta} k_q} + \tilde{\beta} \frac{\frac{k_h}{1-\delta} + \frac{1-\omega}{\omega\delta} k_q}{1 + \frac{1-\omega}{\omega\delta} k_q} \mathbb{E}_t \psi_{t+1}.
\end{aligned}$$

Now, through equation (80) we also have that

$$\hat{\psi}_t + \frac{\Omega_{qt} - \tilde{\beta}\mathbb{E}_t\Omega_{qt+1}}{\delta} = \psi_t - \beta k_h \mathbb{E}_t \psi_{t+1}$$
$$= \sum_{t=0}^{\infty} a_s \mathbb{E}_t \hat{\psi}_{t+s}. \tag{85}$$

Because $k_h < 1 - \delta$, we have

$$\tilde{\beta}\frac{\frac{k_h}{1-\delta} + \frac{1-\omega}{\omega\delta}k_q}{1 + \frac{1-\omega}{\omega\delta}k_q} = \beta k_h \frac{1 + \frac{1-\delta}{k_h}\frac{1-\omega}{\omega\delta}k_q}{1 + \frac{1-\omega}{\omega\delta}k_q} > \beta k_h$$

and therefore $a_s > 0$ for all $s \geq 0$.

We now substitute (85) and (84) into the first-order condition (78) to solve for the coefficients b_s. We obtain:

$$\left(\frac{\delta}{\gamma\kappa_1}\theta\left(1 - \tilde{\beta}\right) + 1 + \tilde{\beta}\left(1 - \delta - k_h\right)\right)\left(k_h \chi_{t-1} + \sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{\psi}_{t+s}\right)$$
$$= \tilde{\beta}\sum_{s=0}^{\infty} b_s \mathbb{E}_t \hat{\psi}_{t+s+1} + (1-\delta)\chi_{t-1} + \frac{\delta}{\gamma\kappa_1}\sum_{t=0}^{\infty} a_s \mathbb{E}_t \hat{\psi}_{t+s}.$$

Comparing coefficients, we get

$$s = 0\text{:}\ \left(\frac{\delta}{\gamma\kappa_1}\theta\left(1 - \tilde{\beta}\right) + 1 + \tilde{\beta}\left(1 - \delta - k_h\right)\right) b_0 = \frac{\delta}{\gamma\kappa_1} a_0$$

$$s \geq 1\text{:}\ \left(\frac{\delta}{\gamma\kappa_1}\theta\left(1 - \tilde{\beta}\right) + 1 + \tilde{\beta}\left(1 - \delta - k_h\right)\right) b_s = \tilde{\beta} b_{s-1} + \frac{\delta}{\gamma\kappa_1} a_s.$$

By an induction argument, these expressions establish that $b_0 > 0$ and $b_s > \tilde{\beta} b_{s-1}$ for all $s \geq 1$. □

C Extension to general beliefs

As before, we start by considering the flexible-price allocation. The appendix shows that the asset demand function in the subjective law of motion (i.e. under $\mathcal{P}$), which previously was given by (18), is replaced by:

$$h_t^n = k_a a_t + k_h h_{t-1}^n - k_q q_t + \tilde{k}_\mu \sum_{s=0}^{\infty} (\beta k_h)^s \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+s+1}. \tag{86}$$

Instead of a single state variable representing subjective beliefs, asset demand now depends on the whole time profile of subjective expected capital gains in the future, but otherwise it has the same form. The coefficients k_a, k_h and k_q are the same as in the baseline model, and $\tilde{k}_\mu > 0$.

The expression for the flexible-price real interest rate under the equilibrium measure, previously given by (23), is replaced by:

$$\begin{aligned} r_t^* &= r_t^{,RE} + \gamma\kappa_1 k_a \left(1-\rho_a\right) a_t \\ &+ \gamma\kappa_1 \tilde{k}_\mu \left(\left(1 + \frac{k_q}{\tilde{k}_\mu}\right) \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+1} - \left(1 - \beta k_h\right) \sum_{s=0}^{\infty} \left(\beta k_h\right)^s \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+s+2}. \right) \end{aligned} \tag{87}$$

The real interest rate continues to be increasing in expectations of next period's capital gains Δq_{t+1}.

The real rate also turns out to be decreasing in expectations of capital gains further in the future Δq_{t+s}, $s \geq 2$. This relationship is most easily understood for the case $\Delta q_{t+2} > 0$: Here, investors' desire to invest in the asset out of consumption is highest in period $t+1$, immediately before the realization of high expected capital gains in $t+2$. As a result, expected consumption c_{t+1} is lower than current consumption c_t, lowering the required real interest rate $r_t^n = \gamma \mathbb{E}_t^{\mathcal{P}} \Delta c_{t+1}$ today. The real rate will still positively depend on asset price expectations as long as

$$\mathbb{E}_t^{\mathcal{P}} \Delta q_{t+s} \leq \left(\frac{1}{\beta k_h}\right)^{s-1} \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+1} \text{ if } \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+1} > 0$$
$$\text{or } \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+s} \geq \left(\frac{1}{\beta k_h}\right)^{s-1} \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+1} \text{ if } \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+1} < 0.$$

As $1/\left(\beta k_h\right) > 1$, we only need that capital gains expectations do not increase too much *further* at future horizons when they are positive for the immediate future, which is likely satisfied for all but the most extreme forms of extrapolative bias.

The results on policy we derive in Section 4 continue to hold for the generalized beliefs with the modified expression for the natural real rate above. Because allocations are determined only by intratemporal labor demand and supply conditions, the level of output under flexible prices is again unaffected by the presence of learning ($y_t^* = y_t^{n,RE}$). Since the asset price q_t remains independent of policy under the PLM, it drops out of the equations describing the dynamics of the sticky price equilibrium relative to flexible prices, and so equations (24)–(27) continue to hold. Propositions 1 through 3 continue to hold: The optimal policy under learning replicates the outcomes from the optimal policy under rational expectations, but has to be implemented by following the perceived natural real interest rate which is increasing in asset price expectations.

D CMCE with alternative assumptions

In this appendix we analyze the model with alternate assumptions about the market clearing conditions that expectations are consistent with under CMCE. As noted in section 2, in order to sustain an equilibrium with CMCE, Walras' law requires that two market clearing conditions be absent from agents' information set. The first of these is of course the market clearing condition for the asset in question, but the second one is in principle a free choice. Throughout the paper, we also remove the final goods market clearing condition. But one could alternatively consider removing either (i) the bond market clearing condition or (ii) the labor market clearing condition.

To model case (i), we modify the system (3)–(10) defining the subjective law of motion, by replacing the household budget constraint (9) with

$$y_t = c_t$$
$$B_t + (1 + \bar{r})\, B_t = \bar{Q}\bar{H}\Delta h_t.$$

The first of these equations is the market clearing condition of final consumption goods, and the second equation is the modified budget constraint that takes into account that agents do not know the bond market clearing condition $B_t = 0$. It is easy to show that the resulting equilibrium is identical to the RE equilibrium regardless of price stickiness, up to the asset price q_t. Intuitively, agents now think that they can exchange the asset for bonds. Bonds are useful to agents only insofar as they can be exchanged into physical goods or labor services; but agents do not expect to be able to do so, because at any time in the future, because they understand the supply and demand in all goods and labor markets.

To model case (i), we now have to distinguish between labor services produced by agents and labor demanded by firms in the subjective law of motion, because agents think they can buy additional labor services by selling their asset holdings and sell those services to firms. Let n_t denote the labor used by firms, and n_t^s the labor produced by agents. We modify the system (3)–(10) defining the subjective law of motion by replacing the labor supply condition (6) and the budget constraint (9) with

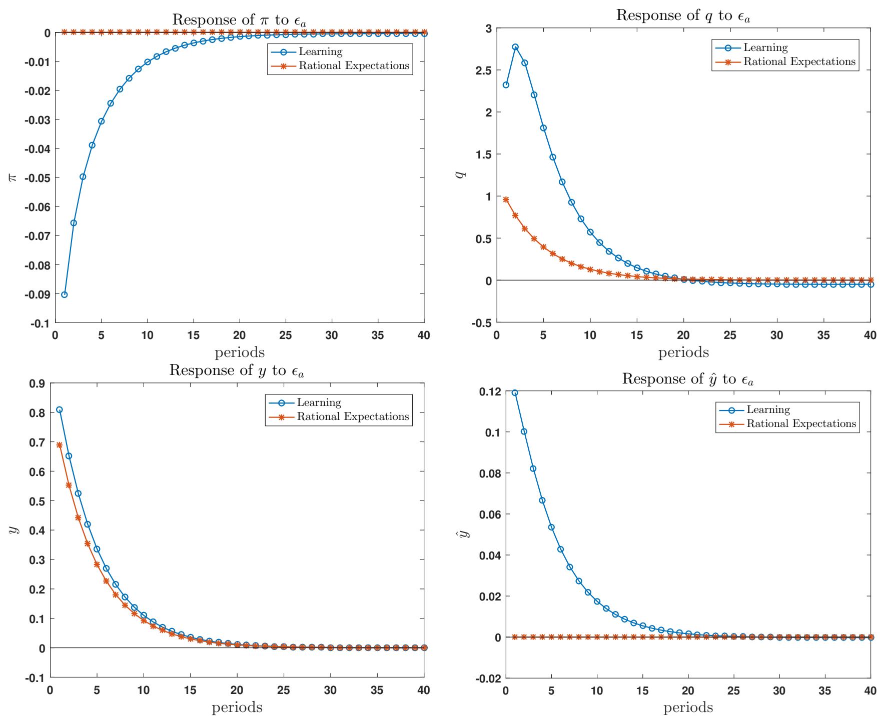
$$
\begin{aligned}
y_t &= c_t \\
w_t &= \gamma c_t + \phi n_t^s \\
y_t &= c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta h_t + \alpha \left(n_t - n_t^s \right).
\end{aligned}
$$

The first of these equations is the market clearing condition of final consumption goods, and the second equation is the modified budget constraint that takes into account that agents think they can buy labor services freely in a spot market. We can show that with this specification, the natural rate of interest in the model is still of the form in (18), with different coefficients but still respecting $k_h \in (0,1)$, $k_a, k_q, k_\mu > 0$. All our results from Sections 4 and 5 continue to hold.

As before, under flexible prices the learning model implements the first-best allocation. As a result, the new informational assumptions only affect the asset price and the natural rate of interest. Figure 5 plots impulse responses under sticky prices, assuming a Taylor-type rule. In this case a positive technology shock pushes *down* inflation. This is a consequence of the impact of the changed information assumption on expected future marginal costs. When agents' information set does not include the labor market clearing condition, then firms do not internalize labor supply effects when forecasting marginal costs. More specifically, when firms anticipate an increase in future labor demand following the shock, they fail to account for the need for future wages to increase so as to clear the labor market. This consequently pushes down the inflation response relative to baseline case.

Notwithstanding the impact on the inflation process, the learning model retains the dependence of the natural rate of interest on expected asset prices, regardless of whether the goods market or the labor market clearing condition is omitted from agents' information set. Table 4 shows optimized simple rule coefficients and their performance, as in section 5. Relative to the baseline case, the welfare gains to including asset prices in the policy reaction function are smaller. However, the optimal simple rule continues to place react positively with prices. When a weighted average of past price growth observations are included in the policy rule (Row 8), welfare is increased 1.7 percent, with the optimized coeffi-

Figure 5: IRFs to a technology shock under sticky prices, alternative CMCE assumptions.



Note: Response to a one standard deviation positive technology shock ε_{At}. Log percentage points. Model simulated under the assumption that agents' information set does not include asset market clearing or labor market clearing. Sticky prices, Taylor rule.

Table 4: Performance of optimized simple rules, alternative CMCE assumptions.

	Rational Expectations	$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
(1)	*Optimal Policy, Discretion*	0.436	0.085	1.011	0.526
(2)	*Optimal Policy, Commitment*	0.205	0.192	1.010	0.210
(3)	$i_t = 1.5\pi_t + 0.125\hat{y}_t$	0.349	0.451	0.801	0.866
	Learning	$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta q_t)$	$\mathbb{E}[\mathcal{L}]$
(4)	*Optimal Policy, Discretion*	0.436	0.085	2.134	0.527
(5)	*Optimal Policy, Commitment*	0.204	0.193	2.135	0.210
(6)	$i_t = 1.5\pi_t + 0.125\hat{y}_t$	0.195	0.580	2.702	1.000
(7)	$i_t = \rho_i^* i_{t-1} + (1-\rho_i^*)\left(1.5\pi_t + \phi_y^*\hat{y}_t + \phi_q^* \sum_{s=0}^{\infty} \omega^{*s}\Delta q_{t-s}\right)$	0.093	0.514	2.093	0.713
	$\{\rho_i^*, \phi_y^*, \phi_q^*, \omega^*\} = \{0.167, 0.019, 0.018, 0.699\}$				

Note: Model simulated under the assumption that agents' information set does not include asset market clearing or labor market clearing. Note: Losses are evaluated as the unconditional expectation of the loss function $\mathcal{L}$. The welfare weight on inflation is set to $\lambda_\pi = 1$ and the losses are normalized to 1 for row (6).

cients placing a small positive weight on the asset price term (as was the case with the baseline model in 5).

E Tracking the "right" natural real rate

The equilibrium realization of the nominal rate under the optimal policy is the expression r_t^* derived in (23). However, an instrument rule that prescribes $i_t = r_t^* + \phi_\pi \pi_t$ would fail to implement the optimal policy. The equilibrium natural rate r_t^{n*} only coincides with r_t^n when $h_t = 0$. While this must be the case in equilibrium, agents under $\mathcal{P}$ contemplate other possible realizations of the house price for which they plan on choosing $h_t \neq 0$. These off-equilibrium states of the world enter into agents' expectations of future marginal costs. Therefore, the central bank must promise to stabilize inflation even in these off-equilibrium states. Tracking only the equilibrium natural rate is insufficient: It must track the perceived natural rate.

As an illustration, Figure 6 shows impulse responses for the learning model with three interest rate equations:

$$i_t = r_t^n + 1.05\pi_t \tag{88}$$

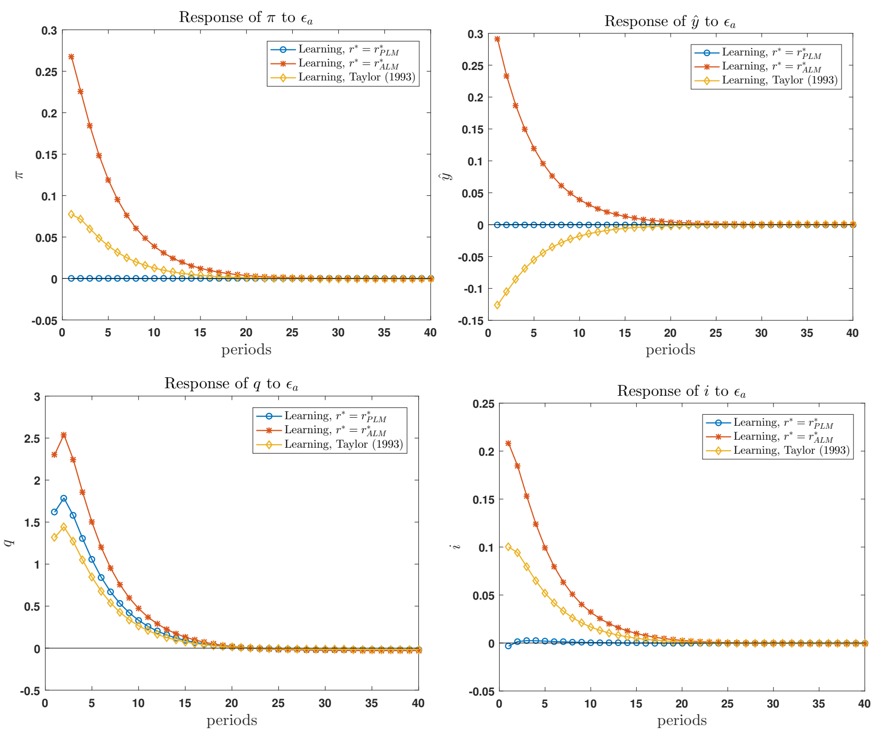
$$i_t = r_t^{n*} + 1.05\pi_t \tag{89}$$

$$i_t = r_{ss} + 1.5\pi_t + 0.125\hat{y}_t. \quad (90)$$

The first equation (88) implements strict inflation targeting as per Proposition 1. The only difference of the second equation (89) is that the monetary authority reacts to the equilibrium process of the natural rate instead of the perceived process. Figure 6 shows how using the ALM natural rate of interest in the the policy rule does not yield a zero inflation outcome. As discussed in the last section, the central bank must promise to stabilize inflation even in those states that are never reached in equilibrium—that is, when the housing market doesn't clear—but contemplated by agents under their subjective expectations. Using the ALM natural rate in the policy rule fails to do so. Due do their beliefs about the process governing Q_t, agents under the PLM do not account for the effect of the technology shock on future asset price growth. Consequently, the initial response of consumption is smaller than under rational expectations. From the standpoint of an agent under the flex price ALM on the other hand, the technology shock has an anticipated positive impact on the path of Q_t due to expected asset demand. As a result, the initial consumption response and subsequent consumption decline will be greater. The ALM natural rate of interest declines more upon the impact of the shock than does the PLM natural rate of interest. When a monetary authority uses the ALM natural rate in its policy rule as in (89), then, the nominal interest does not increase sufficiently to prevent an inflationary response.

Finally, the third equation is a standard Taylor rule. Figure 6 shows that this rule performs somewhat better in terms of outcomes, but is still far from the optimal policy. It is worth noting that the nominal interest rate is more volatile under the Taylor rule than under the optimal rule (88), which reacts to asset prices. The reason is that the stabilization benefits of reacting to asset prices make equilibrium nominal rates more stable as well.

Figure 6: Tracking the “right” natural real rate.



Note: Response to a unit standard deviation positive technology shock ε_{At} under sticky prices. Log percentage points. The interest rate rules used are given in Equations (88)–(90).