Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory

Ben Brubaker, Daniel Bump, and Solomon Friedberg
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Chapter One

Type A Weyl Group Multiple Dirichlet Series

We begin by defining the basic shape of the class of Weyl group multiple Dirichlet series. To do so, we choose the following parameters.

- $\Phi$, a reduced root system. Let $r$ denote the rank of $\Phi$.
- $n$, a positive integer.
- $F$, an algebraic number field containing the group $\mu_{2n}$ of $2n$-th roots of unity.
- $S$, a finite set of places of $F$ containing all the archimedean places, all places ramified over $\mathbb{Q}$, and large enough so that the ring
  \[ \mathfrak{o}_S = \{ x \in F \mid |x|_v \leq 1 \text{ for } v \notin S \} \]
  of $S$-integers is a principal ideal domain,
- $m = (m_1, \ldots, m_r)$, an $r$-tuple of nonzero $S$-integers.

We may embed $F$ and $\mathfrak{o}_S$ into $F_S = \prod_{v \in S} F_v$ along the diagonal. Let $(d, c)_{n,v}$ denote the $S$-Hilbert symbol, the product of local Hilbert symbols $(d, c)_{n,v} \in \mu_n$ at each place $v \in S$, defined for $c, d \in F_S^\times$. Let $\Psi : (F_S^X)^r \to \mathbb{C}$ be any function satisfying

\[ \Psi(\varepsilon_1 c_1, \ldots, \varepsilon_r c_r) = \prod_{i=1}^r (\varepsilon_i, c_i)_{n,S} \prod_{1 \leq j < k \leq r} (\varepsilon_j, c_k)_{n,S}^{-1} \Psi(c_1, \ldots, c_r) \quad (1.1) \]

for any $\varepsilon_1, \ldots, \varepsilon_r \in \mathfrak{o}_S^\times(F_S^X)^n$ and $c_1, \ldots, c_r \in F_S^X$. Here $(F_S^X)^n$ denotes the set of $n$-th powers in $F_S^X$. It is proved in [12] that the set $\mathcal{M}$ of such functions is a finite-dimensional (nonzero) vector space.

To any such function $\Psi$ and data chosen as above, Weyl group multiple Dirichlet series are functions of $r$ complex variables $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ of the form

\[ Z_{\Psi}^{(n)}(s; m; \Phi) = Z_\Phi(s; m) = \sum_{c = (c_1, \ldots, c_r) \in (\mathfrak{o}_S/\mathfrak{o}_S^\times)^r \setminus \{0\} \mid c_i \neq 0} \frac{H(c; m) \Psi(c)}{\mathcal{N}_c^{2s_1} \ldots \mathcal{N}_c^{2s_r}} \quad (1.2) \]

where $\mathcal{N}_c$ is the cardinality of $\mathfrak{o}_S/c\mathfrak{o}_S$, and it remains to define the coefficients $H(c; m)$ in the Dirichlet series. In particular, the function $\Psi$ is not independent of the choice of representatives in $\mathfrak{o}_S/\mathfrak{o}_S^\times$, so the function $H$ must possess complementary transformation properties for the sum to be well-defined.

Indeed, the function $H$ satisfies a "twisted multiplicativity" in $c$, expressed in terms of $n$-th power residue symbol and depending on the root system $\Phi$, which
Chapter Two

Crystals and Gelfand-Tsetlin Patterns

We will translate the definitions of the $\Gamma$ and $\Delta$ arrays in (1.11), and hence of the multiple Dirichlet series, into the language of crystal bases. The entries in these arrays and the accompanying boxing and circling rules will be reinterpreted in terms of the Kashiwara operators. Thus, what appeared as a pair of unmotivated functions on Gelfand-Tsetlin patterns in the previous chapter now takes on intrinsic representation theoretic meaning. Despite the conceptual importance of this reformulation, the reader can skip this chapter and the subsequent chapters devoted to crystals with no loss of continuity. For further background information on crystals, we recommend Hong and Kang [42] and Kashiwara [47] as basic references. After completing the proof of our main theorem, the subject of crystals is taken up again in Chapter 18, where we reformulate aspects of the proof in terms of crystals. Then in Chapter 20 we explain the notion of crystal bases more generally and explain why they appear naturally in the description of $p$-parts.

In the present chapter, we restrict to crystals of Cartan type $A_r$ and identify the weight lattice $\Lambda$ of $\mathfrak{gl}_{r+1}(\mathbb{C})$ with $\mathbb{Z}^{r+1}$. The weight $\lambda = (\lambda_1, \ldots, \lambda_{r+1}) \in \mathbb{Z}^{r+1}$ is dominant if $\lambda_1 \geq \lambda_2 \geq \ldots$. If $\lambda_{r+1} \geq 0$ we call the dominant weight effective. Thus an effective dominant weight is just a partition of length $\leq r + 1$. Let $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^{r+1}$ where the 1 is in the $i$-th position. Then the root system $\Phi$ of Type $A$ consists of the vectors

$$e_i - e_j \in \Lambda$$

with $i \neq j$. The positive roots $\Phi^+$ consist of the roots $e_i - e_j$ with $i < j$. We will denote by $\rho$ the Weyl vector, already defined in (1.7) to be $(r, r - 1, \ldots, 2, 1, 0)$. Regarding $\rho$ as an element of the weight lattice, it is not half the sum of the positive roots, but

$$\rho - \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

is orthogonal to the roots. This means that we may use $\rho$ and $\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ interchangeably in places such as the Weyl character formula as written in Chapter 5.

If $\lambda$ is a dominant weight then there is a crystal graph $B_\lambda$ with highest weight $\lambda$. It is equipped with a weight function $\text{wt} : B_\lambda \longrightarrow \mathbb{Z}^{r+1}$ such that if $\mu$ is any weight and $m(\mu, \lambda)$ is the multiplicity of $\mu$ in the irreducible representation of $\text{GL}_{r+1}(\mathbb{C})$ with highest weight $\lambda$ then $m(\mu, \lambda)$ is also the number of $v \in B_\lambda$ with $\text{wt}(v) = \mu$. It has operators

$$e_i, f_i : B_\lambda \longrightarrow B_\lambda \cup \{0\} \quad (1 \leq i \leq r)$$

such that:
Duality

The material contained in this chapter won’t be used again until Chapter 18 and so can be skipped without loss of continuity. Its purpose is to point out that the boxing and circling decorations of the BZL patterns that were introduced in the last chapter are in a sense dual to each other.

In Chapter 2 we used either of the notations

\[
v \left[ \begin{array}{cccc}
 b_1 & \cdots & b_N \\
 i_1 & \cdots & i_N
\end{array} \right] v' \quad \text{or} \quad v \left[ \begin{array}{cccc}
 b_1 & \cdots & b_N \\
 i_1 & \cdots & i_N
\end{array} \right]^{(f)} v'
\]

to mean that \( v' = f_{b_N}^{i_N} \cdots f_{b_1}^{i_1} v \) where each integer \( b_k \) is as large as possible in the sense that \( f_{b_k+1}^{i_k} f_{b_{k-1}}^{i_{k-1}} \cdots f_{b_1}^{i_1} v = 0 \). In this chapter, we will exclusively use the second notation - the superscript \( (f) \) will be needed to avoid confusion since we now analogously define

\[
v \left[ \begin{array}{cccc}
 b_1 & \cdots & b_N \\
 i_1 & \cdots & i_N
\end{array} \right]^{(e)} v'
\]

(3.1)

to mean that \( v' = e_{i_N}^{b_N} \cdots e_{i_1}^{b_1} v \) where \( e_{i_k}^{b_k+1} e_{i_{k-1}}^{b_{k-1}} \cdots e_{i_1}^{b_1} v = 0 \) for all \( 1 \leq k \leq N \). Thus

\[
v \left[ \begin{array}{cccc}
 b_1 & \cdots & b_N \\
 i_1 & \cdots & i_N
\end{array} \right]^{(e)} v' \quad \text{if and only if} \quad v' \left[ \begin{array}{cccc}
 b_N & \cdots & b_1 \\
 i_N & \cdots & i_1
\end{array} \right]^{(f)} v.
\]

Let us assume that (3.1) holds, where \( \Omega = (i_1, \ldots, i_N) \) and \( N = \frac{1}{2} r (r + 1) \). Assuming that \( \Omega = \Omega_\Gamma \) or \( \Omega_\Delta \), defined in (2.9) and (2.10), we may decorate the values \( b_k \) according to circling and boxing rules that are analogous to those defined in Chapter 2. Thus \( b_k \) is circled if and only if either \( b_k = 0 \) (when \( k \) is a triangular number) or \( b_k = b_{k+1} \) (when it is not). And \( b_k \) is boxed if and only if \( f_{i_k} e_{i_{k-1}}^{b_{k-1}} \cdots e_{i_1}^{b_1} v = 0 \), which is equivalent to saying that the path (3.1) includes the entire \( i_k \) string through \( e_{i_{k-1}}^{b_{k-1}} \cdots e_{i_1}^{b_1} v \). In this case, dual to Lemma 2.1, we have \( v' = v_{\text{high}} \).

The definition of the multiple Dirichlet series can be made equally well with respect to the \( e_i \). Indeed, we adapt (2.14) and define

\[
G_\Delta^{(e)}(v) = \prod_{i=1}^{\frac{1}{2} r (r+1)} \begin{cases} 
q_i^{b_i} & \text{if } b_i \text{ is circled,} \\
g(b_i) & \text{if } b_i \text{ is boxed,} \\
h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\
0 & \text{if } b_i \text{ is both circled and boxed.}
\end{cases}
\]

assuming that \( \Omega = \Omega_\Delta \); if instead \( \Omega = \Omega_\Gamma \), then \( G_\Gamma^{(e)} \) is defined by the same formula.
Weyl group multiple Dirichlet series are expected to be Whittaker coefficients of metaplectic Eisenstein series. The fact that Whittaker coefficients of Eisenstein series reduce to the crystal description that we gave in Chapter 2 is proved for Type A. In a classical setting, this was established by Brubaker, Bump, and Friedberg [13]. Alternatively, on the adele group, the corresponding local computation reduces to the evaluation of a type of $p$-adic integral. These were considered by McNamara [65], who reduced the integrals to sums over crystals by a very interesting method. A full treatment of this topic is outside the scope of this work, but we will introduce this subject by considering the case where $n = 1$.

In this chapter only, we will use $F$ to denote a nonarchimedean local field and $F$ a global field. Let $\hat{G}(\mathbb{C}) = \text{GL}_{r+1}(\mathbb{C})$, which is the $L$-group of $G = \text{GL}_{r+1}$. Let $T$ be the diagonal torus in $G$, and let $\hat{T}(\mathbb{C})$ be the diagonal torus in $\hat{G}(\mathbb{C})$. If

$$z = \text{diag}(z_1, \ldots, z_{r+1}) \in \hat{T}(\mathbb{C}),$$

then let $\chi_z$ be the unramified quasicharacter of $T(F)$ defined by

$$\chi_z(t) = \prod_i z_i^{\text{ord}(t_i)}, \quad t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1} \end{pmatrix}.$$ 

Then $z \mapsto \chi_z$ is an isomorphism of $\hat{T}(\mathbb{C})$ with the group of unramified quasicharacters of $T(F)$.

We will denote by $W$ the Weyl group of $G$ or $\hat{G}$. Each Weyl group element has a representative $w$ that is a permutation matrix, and for definiteness we will use that representative. Let $w_0$ be the long Weyl group element.

Let $B(F)$ be the Borel subgroup of upper triangular matrices, and let $U(F)$ be the subgroup of upper triangular unipotent matrices, so $B(F) = T(F)U(F)$. Given a quasicharacter $\chi$ of $T(F)$, we may extend it to a character of $B(F)$ by letting $U(F)$ be in the kernel. Let $\delta : T(F) \rightarrow \mathbb{R}^\times$ be the quasicharacter $\delta(t) = \prod_{i=1}^{r+1} |t_i|^{r+2-2i}$. Extended to $B(F)$, this is the modular character.

Let $V_\chi$ be the space of locally constant functions $f$ on $G(F)$ such that

$$f(bg) = \delta^{1/2} \chi(b) f(g), \quad b \in B(F).$$

Then $G(F)$ acts by right translation: that is, $\pi_\chi : G(F) \rightarrow \text{End}(V_\chi)$ is defined by $\pi_\chi(g)f(x) = f(xg)$. If $\chi = \chi_z$ we will also write $(\pi_z, V_\chi)$ for the representation $(\pi_\chi, V_\chi)$. The representations $\pi_z$ are called the unramified principal series. They are irreducible if $z$ is in general position.
Chapter Five

Tokuyama's Theorem

Let $z = \text{diag}(z_1, \cdots, z_{r+1})$ be an element of the group $\hat{T}(\mathbb{C})$, which is the diagonal subgroup of $GL_{r+1}(\mathbb{C})$. In the application to the Casselman-Shalika formula we will take the $z_i$ to be the Langlands parameters. (In terms of the $s_i$ the $z_i$ are determined by the conditions that $\prod z_i = 1$ and $z_i/z_{i+1} = q^{1-2s_{r+1-i}}$.)

Let us write the Weyl character formula in the form

$$\left[ \prod_{\alpha \in \Phi^+} (1 - z^{-\alpha}) \right] s_{\lambda}(z) = z^\rho \sum_{w \in W} (-1)^{l(w, w)} z^{w(\lambda + \rho)}, \quad (5.1)$$

where we recall that $s_{\lambda}(z) = s_{\lambda}(z_1, \cdots, z_{r+1})$ is the Schur polynomial. The sum over the Weyl group on the right-hand side is the numerator in the Weyl formula and the product on the left is essentially the Weyl denominator. The Weyl vector $\rho$, we recall, is $\rho = (r, r-1, \cdots, 2, 1, 0)$.

We have seen in (4.4) and (4.2) that $p$-part of the Whittaker coefficient of the Eisenstein series is

$$\left[ \prod_{\alpha \in \Phi^+} (1 - q^{-1} z^{-\alpha}) \right] s_{\lambda}(z). \quad (5.2)$$

The similarity of this expression to (5.1) indicates that the $p$-parts of Whittaker coefficients of Eisenstein series can be profitably regarded as a deformation of the numerator in the Weyl character formula.

Thus we are led to deformations of the Weyl character formula. Tokuyama [70] gave such a deformation. It is an expression of $s_{\lambda}(z)$ as a ratio of a numerator to a denominator. The denominator is a deformation of the Weyl denominator, and the numerator is a sum over Gelfand-Tsetlin patterns with top row $\lambda + \rho$. It can be rewritten as a sum over $B_{\lambda + \rho}$.

Tokuyama's formula has a parameter $t$, which can be specialized in various ways. If $t = 1$, it is the Weyl character formula. If $t = 0$, it is equivalent to the combinatorial definition of the Schur polynomial as a sum over semistandard Young tableaux, that is, over $B_\lambda$. If $t$ is specialized to $-q^{-1}$ then the denominator in Tokuyama's formula matches the product in (5.2). In this specialization, the numerator in Tokuyama's formula agrees with the $p$-parts of the Weyl group multiple Dirichlet series in the special case $n = 1$.

We will state and prove Tokuyama's formula, then translate it into the language of crystals.

If $\Xi$ is a Gelfand-Tsetlin pattern in the notation (1.5) let $s(\Xi)$ be the number of entries $a_{ij}$ with $i > 0$ such that $a_{i-1,j-1} < a_{ij} < a_{i-1,j}$. Let $l(\Xi)$ be the number
Chapter Six

Outline of the Proof

The proof of Theorem 1.1 involves many remarkable phenomena, and we wish to explain its structure in this chapter. To this end, we will give the first of a succession of statements, each of which implies the theorem. Passing from each statement to the next is a nontrivial reduction that changes the nature of the problem to be solved. We will outline the ideas of these reductions here and tackle them in detail in subsequent chapters.

Statement A. We have $H_{\Gamma} = H_{\Delta}$.

This reduction was already mentioned in the first chapter, where Statement A appeared as Theorem 1.2.

The proof that this implies Theorem 1.1 is Theorem 1 of [15]. We review the idea of the proof. To prove the functional equations that $Z_{\psi}(s; \tau)$ is to satisfy, using the method described in [20], [10], [12], and [14] based on Bochner's convexity principle [7], one must prove meromorphic continuation to a larger region and a functional equation for each generator $\sigma_1, \cdots, \sigma_r$ of the Weyl group – the simple reflections. These act on the coordinates by

$$\sigma_i(s_j) = \begin{cases} 1 - s_j & \text{if } j = i, \\ s_i + s_j - \frac{1}{2} & \text{if } j = i \pm 1, \\ s_j & \text{if } |j - i| > 1. \end{cases}$$

We proceed inductively. Taking $H = H_{\Gamma}$ as the definition of the series, all but one of these functional equations may be obtained by collecting the terms to produce a series whose terms are multiple Dirichlet series of lower rank. To see this reduction, note that we have described the $p$-part of $H$ as a sum over Gelfand-Tsetlin patterns, extended this to a definition to $H(c_1, \cdots, c_r; m_1, \cdots, m_r)$ by (twisted) multiplicativity. Equivalently, one may define $H(c_1, \cdots, c_r; m_1, \cdots, m_r)$ by specifying a Gelfand-Tsetlin pattern $\Sigma_p$ for each prime; for all but finitely many $p$ the pattern must be the minimal one

$$\left\{ \begin{array}{cccc} r & r - 1 & \cdots & 0 \\ r - 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{array} \right\}.$$ 

Summing over patterns with fixed top row (determined by the ord$_p(m_i)$) and fixed row sums (determined by ord$_p(c_i)$) gives $H(c_1, \cdots, c_r; m_1, \cdots, m_r)$. More precisely we may group the terms as follows. For each prime $p$ of $\sigma$, fix a partition $\lambda_p$ of length $r$ into unequal parts such that for almost all $p$ we have $\lambda_p =$
Chapter Seven

Statement B Implies Statement A

In this chapter we will recall the use of the Schützenberger involution on Gelfand-Tsetlin patterns in [15] to prove that Statement B implies Statement A. We will return to these statements two more times in the later chapters of the book. In Chapter 18 we will reinterpret both Statements A and B in terms of crystals, and directly prove that the reinterpreted Statement B implies the reinterpreted Statement A in Theorem 18.2. Then in Chapter 19 we will yet again reinterpret Statements A and B in a different context, and yet again directly prove that the reinterpreted Statement B implies the reinterpreted Statement A in Theorem 19.10.

We observe that the Schützenberger involution \( q_r \) can be formulated in terms of operations on short Gelfand-Tsetlin patterns. If

\[
\Xi = \begin{pmatrix}
  a_{00} & a_{01} & a_{02} & \cdots & a_{0r} \\
  a_{11} & a_{12} & \cdots & & a_{1r} \\
  \vdots & \vdots & & \ddots & \vdots \\
  & & & & a_{rr}
\end{pmatrix}
\]

is a Gelfand-Tsetlin pattern and \( 1 \leq k \leq r \), then extracting the \( r - k, r + 1 - k \) and \( r + 2 - k \) rows gives a short Gelfand-Tsetlin pattern \( t \). Replacing this with the pattern \( t' \) gives a new Gelfand-Tsetlin pattern, which is the one denoted \( t_r \Xi \) in Chapter 1. Thus

\[
t_1 \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a & b & c \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a & a + b - c \end{pmatrix}
\]

and

\[
t_2 \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a & b & c \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a' & b' & c \end{pmatrix}
\]

where \( a' = \lambda_1 + \max(\lambda_2, c) - a \) and \( b' = \lambda_3 + \min(\lambda_2, c) - b \).

We defined \( q_0 \) to be the identity map, and defined \( q_i = t_1 t_2 \cdots t_i q_{i-1} \). The \( t_i \) have order two. They do not satisfy the braid relation, so \( t_i t_{i+1} t_i \neq t_{i+1} t_i t_{i+1} \). However \( t_i t_j = t_j t_i \) if \( |i - j| > 1 \) and this implies that the \( q_i \) also have order two. We note that

\[
q_i = q_{i-1} q_{i-2} t_i q_{i-1}. \tag{7.1}
\]

Let \( A_i = \sum_j a_{i,j} \) be the sum of the \( i \)-th row of \( \Xi \). It may be checked that the row sums of \( q_r \Xi \) are (in order)

\[
A_0, A_0 - A_r, A_0 - A_{r-1}, \ldots, A_0 - A_1.
\]
Chapter Eight

Cartoons

This chapter will introduce a method of marking up a short Gelfand-Tsetlin pattern based on inequalities between its entries, that encodes the effect of the involution $t \mapsto t'$ and the boxing and circling of its accordion. This will have another benefit: it will lead to the decomposition of the pattern into pieces called episodes that will ultimately lead to the reduction to the totally resonant case.

**Proposition 8.1** (i) If $n \nmid a$ then $h(a) = 0$ and $|g(a)| = q^{a - \frac{1}{2}}$.

(ii) If $n \mid a$ then

$$h(a) = \phi(p^a) = q^{a-1}(q-1), \quad g(a) = -q^{-a-1}$$

(iii) If $n \mid a$ and $b > 0$ then

$$h(a + b) = q^a h(b), \quad g(a + b) = q^a g(b).$$

(iv) If $n \nmid a, b$ but $n \mid a + b$ then

$$g(a)g(b) = q^{a+b-1}.$$

Property (iii) means that $q^b$ and $h^b$ are periodic with period $n$.

**Proof.** This is easily checked using standard properties of Gauss sums. \qed

To define the cartoon, we will take a slightly more formal approach to the short Gelfand-Tsetlin patterns. Let

$$\Theta = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} | 0 \leq i \leq 2, 0 \leq j \leq d + 1 - i\}.$$

We call this set the substrate, and divide $\Theta$ into three rows, which are

$$\Theta_0 = \{(0, j) \in \Theta | 0 \leq j \leq d + 1\},$$

$$\Theta_1 = \{(1, j) \in \Theta | 0 \leq j \leq d\},$$

$$\Theta_2 = \{(2, j) \in \Theta | 0 \leq j \leq d - 1\},$$

Let $\Theta_B = \Theta_1 \cup \Theta_2$. Each row has an order in which $(i, j) \leq (i, j')$ if and only if $j \leq j'$.

Now we can redefine a short Gelfand-Tsetlin pattern to be an integer valued function $t$ on the substrate, subject to the conditions that we have already stated. Thus the short Gelfand-Tsetlin pattern (6.2) corresponds to the function on $\Theta$ such that $l_i = t(0, i)$, $a_i = t(1, i)$ and $b_i = t(2, i)$. The $\Gamma$ and $\Delta$ preaccordions, which are arrays having only two rows, may similarly be described as functions on $\Theta_B$ (the bottom and middle rows) in the same way. Specifying the circled and boxed elements just means specifying subsets of $\Theta_B$. 