Two-Sided Matching via Balanced Exchange *

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Abstract

We introduce a new matching model to mimic two-sided exchange programs such as tuition and worker exchanges, in which export-import balances are required for longevity of programs. These exchanges use decentralized markets, making it difficult to achieve this goal. We introduce the two-sided top-trading-cycles, the unique mechanism that is balanced-efficient, worker-strategy-proof, acceptable, individually rational, and respecting priority bylaws regarding worker eligibility. Moreover, it encourages exchange, because full participation is the dominant strategy for firms. We extend it to dynamic settings permitting tolerable yearly imbalances and demonstrate that its regular and tolerable versions perform considerably better than models of current practice.

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1 Introduction

The theory and design of two-sided matching markets, such as entry-level labor markets for young professionals, online-dating markets, or college admissions, have been one of the cornerstones of market design for more than thirty years (cf. Gale and Shapley, 1962; Roth, 1984; Roth and Peranson, 1999; Hitsch, Hortaçsu, and Ariely, 2010). Moreover, the theory of these markets has some important applications in allocation problems such as student placement and school choice (cf. Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Both sides of a given market, such as firms and workers, which have preferences over each other and participate autonomously in a decentralized or centralized market to find a partner or partners. A market equilibrium concept, such as stability, has been the key solution in theory, in the practice of decentralized markets, and in the design of centralized clearinghouses.

In this paper, we introduce and model a new class of two-sided matching markets in which there is an additional fundamental constraint. The eventual market outcome is linked to an initial status-quo matching, which may give firms and workers certain rights that constrain how future activity can play out. Eventually, a worker may end up matched with a firm different from his home institution. The most crucial property that needs to be respected in such cases is balancedness: a worker needs to be replaced with a new one at his home institution so that the market clears in a balanced manner. There are several prominent examples of such exchanges, such as national and international teacher-exchange programs, clinical-exchange programs for medical doctors, worker-exchange programs within or across firms, and student-exchange programs among colleges. The balancedness constraint is crucial in practice. For example, if a math teacher is matched with a school in a different country, the home school of this teacher may wish to hire a substitute math teacher for the year that he is away. This constraint induces preferences for firms not only over whom they get matched with (i.e., import), but also over whom they send out (i.e., export). The most basic kind of such preferences requires the firm to have a preference for balanced matchings, i.e., for import and export numbers to be equal. The first contribution of our paper is the introduction of this novel two-sided matching market model, the formulation of the balancedness condition, and the inspection of the properties and implications of balancedness in market activity. In this paper, we analyze the implications of our model in two market applications: (permanent) tuition exchange and temporary worker exchange.¹

¹See Section 2 for details of these applications.
a dependent of a faculty member at a college) can attend another institution for free, if admitted as part of a tuition-exchange program. (Tuition exchange is a part of the benefit plan of the faculty member.) Colleges have preferences over matchings. We assume only a weak structure for these preferences. Colleges’ preferences over the incoming class is assumed to be responsive to their strict preferences over individual students. Moreover, their preferences over matchings are determined through their preferences over the incoming class and how balanced the eventual matching is.\(^2\) We start by showing, through a simple example, that individual rationality and non-wastefulness, standard solution concepts in two-sided matching markets, and balancedness are in general conflicting requirements (Proposition 1). For this reason, we restrict our attention to the set of balanced–efficient mechanisms. Unfortunately, there exists no balanced–efficient and individually rational mechanism that is immune to preference manipulation for colleges (Theorem 2).

We propose a new two-sided matching mechanism that is balanced-efficient, student group–strategy-proof, acceptable, respecting internal priorities,\(^3\) individually rational, and immune to quota manipulation by colleges (Theorems 1, 3, and 4). We also show that it is the unique mechanism satisfying the first four properties (Theorem 5). To our knowledge, this is the first paper using axiomatic characterization in practical market design.

The outcome of this mechanism can be computed with a variant of David Gale’s top–trading–cycles (TTC) algorithm (Shapley and Scarf, 1974) for finding core and competitive allocations for a simple “house” (or object) exchange market without money. In the school choice problem (cf. Abdulkadiroğlu and Sönmez, 2003) and the house allocation problem with existing agents (cf. Abdulkadiroğlu and Sönmez, 1999), variants related to Gale’s TTC, have been introduced and their properties have been extensively discussed. In all of these problems, one side of the market is considered to be objects to be consumed that are not included in the welfare analysis. Schools and houses have no preferences but have “priorities.” Hence, they are not strategic agents. In two-sided matching via exchange, in contrast to school choice and house allocation, both sides of the market are strategic and must be included in the welfare analysis. As we use a variant of the Abdulkadiroğlu and Sönmez (2003) TTC algorithm to find its outcome, we refer to our mechanism as the two–sided top–trading–cycles (2S-TTC) mechanism. As far as we know, this is the first time a TTC–variant algorithm has been used to find the outcome

\(^2\)We do not rule out colleges having more complex preferences over which students they send out.

\(^3\)A mechanism respects internal priorities if, after a college increases the number of sponsored students, every student who was initially sponsored by that college and matched continues to be matched (although not necessarily with the same college).
of a two-sided matching mechanism.\footnote{Ma (1994) had previously characterized the core of a house exchange market, which can be found by Gale’s TTC algorithm, when there is a single seat at each school through Pareto efficiency, individual rationality, and strategy-proofness for students. Our characterization uses a proof technique different not only from Ma (1994), but also subsequent simpler proofs of this prior result by Sönmez (1995) and Svensson (1999). There are a few other TTC-related characterization results in the literature: Abdulkadiroğlu and Che (2010); Dur (2012); Morrill (2013) characterize school choice TTC a la Abdulkadiroğlu and Sönmez (2003); Pycia and Ünver (2016) characterize general individually rational TTC rules a la Pápai (2000) when there are more objects than agents; and Sönmez and Ünver (2010) characterize TTC rules a la Abdulkadiroğlu and Sönmez (1999) for house allocation with existing tenants. Kesten (2006) provides the necessary structure on the priority order to guarantee fairness of school choice TTC. Besides these characterizations, a related mechanism to ours was proposed by Ekici (2011) in an object allocation problem for temporary house exchanges with unit quotas.}

Although 2S-TTC is balanced–efficient, it may not match the maximum possible number of students while maintaining balancedness. We show that if the maximal-balanced solution is different from the 2S-TTC outcome even for one preference profile, it can be manipulated by students (Theorem 6).

Some tuition exchange programs require keeping a balance in a moving three-year window for their member colleges. For this reason, we extend our model to a dynamic setting, where colleges can have tolerable yearly imbalances. We propose an extension, \textit{two-sided tolerable top-trading-cycle} (or 2S-TTTC) mechanism, which allows one to keep the imbalance of each school between some upper and lower bounds, and these bounds can be adjusted over the years. Once the bound-setting and adjustment processes are externally set, we show that 2S-TTTC keeps good properties of 2S-TTC: it is student-strategy-proof, acceptable, and respecting internal priorities; moreover, no acceptable matching within the balance limits can Pareto dominate its outcome.\footnote{While 2S-TTC relies on a variant of Abdulkadiroğlu-Sönmez’s top-trading-cycles algorithm, 2S-TTTC’s algorithm is also new and original. The closest algorithm in the literature to it is the top-trading-cycles-and-chains (TTCC) algorithm proposed by Roth, Sönmez, and Ünver (2004); however, the use and facilitations of “chains” are substantially different in this algorithm than 2S-TTTC.}

As the last part of our analysis of tuition-exchange programs, we compare the performances of 2S-TTC and 2S-TTTC with that of the best-case scenario of the current practice of tuition exchange with a wide range of simulations.\footnote{We develop a model of current semi-decentralized practice in tuition exchange in Appendix A. We show that balancedness is not in general achieved through decentralized market outcomes, jeopardizing the continuation and success of such markets. We define stability for particular externalities in college preferences. We show that stable matchings exist when colleges have plausible preferences over matchings (Proposition 3). Moreover, Proposition 4 implies that stability and balancedness are incompatible. (Because of the balancedness requirement, in practice stability is not an appropriate property for our market.) We then inspect a decentralized quota reporting game with a stable market solution. We show that it is the best response for a college with a negative net balance to decrease its eligibility quota and that increasing eligibility quota is never a best response (Theorem 10). Behaving with respect to such a best response may further cause another college to have a negative net balance, as a decrease in participation...}
degrees of correlation among students’ and also colleges’ preferences, and different yearly imbalance tolerance levels, we show that 2S-TTC and its variant match considerably more students to colleges and increase students’ welfare over the naive student-proposing deferred acceptance outcome, the best case scenario for the current market.\footnote{It is the best case, since in the Appendix A, we show that under reasonable assumptions, the current market gives incentives to decrease the quotas of agents.} This is a best-case scenario for the decentralized market as it minimizes coordination failures and ignores possible college incentives to underreport their certification quotas.

We extend this model for temporary worker exchanges, such as teacher-exchange programs, the motivational examples we mentioned in the beginning. We tweak our model slightly and assume that the quotas of the firms are fixed at the number of their current employees, and, hence, firms would like to replace each agent who leaves. We also assume that firm preferences are coarser than colleges in tuition exchange due to the temporary nature of the exchanges. We assume they have weakly size-monotonic preferences over workers: larger groups of acceptable workers are weakly better than weakly smaller groups of acceptable workers when the balance of the matching with larger groups of acceptable workers is zero and the balance of the matching with smaller group of worker is non-positive.\footnote{Weakly size-monotonic preferences are weaker than dichotomous preferences (in absence of externalities), which are widely used in the literature, most notably in the kidney exchange models of Roth, Sönmez, and Ünver (2005, 2007) and Sönmez and Ünver (2014), which use it to model patient preferences over compatible kidneys. In a more abstract setting, Bogolomia and Moulin (2004) introduced dichotomous preferences in two-sided matching to model worker-firm interactions. Ekici (2011) also uses this modeling choice for “selling” preferences.} In this model, we prove that 2S-TTC not only carries all of its previous properties through but also is strategy-proof for the firms, making it a very viable candidate (Theorem 9). Our aforementioned characterization also holds in this model.

\section{Applications}

\subsection{Tuition Exchange}

Some of the best-documented matching markets with a balancedness requirement are tuition-exchange programs among US colleges. These are semi-decentralized markets, and some have failed over the years because of problems related to imbalanced matching activity.

\begin{footnotesize}
\begin{enumerate}
\item never improves the negative net balances of other colleges (Theorem 11). Hence, if we take stability as a benchmark market equilibrium concept in a decentralized market, stability discourages exchange and can prevent the market from extracting the highest gains from exchange. We use both straightforward behavior and this kind of an equilibrium in our simulations.
\end{enumerate}
\end{footnotesize}
Historically, it has been difficult for small colleges and universities to compete with bigger schools in trying to hire the best and brightest faculty. Colleges located farther away from major metropolitan areas face a similar challenge. Tuition-exchange programs play a prominent role for these colleges in attracting and retaining highly qualified faculty.9

Many colleges give qualified dependents of faculty tuition waivers. Through a tuition-exchange program, they can use these waivers at other colleges and attend these schools for free (and their home institutions do not pay tuition to the other school). The dependent must be admitted to the other college. Tuition exchange has become a desirable benefit that adds value to an attractive employment package without creating additional out-of-pocket expenses for colleges; that is, colleges do not transfer money to each other for accepting their faculty’s dependents.

One of the prominent programs is “The Tuition Exchange, Inc.” (TTEI),10 which is also the oldest and largest of its kind. TTEI is a reciprocal scholarship program for children (and other family members) of faculty and staff employed at more than 600 colleges. Member colleges are spread over 47 states and the District of Columbia. Both research universities and liberal arts colleges are members. US News and World Report lists 38 member colleges in the best 200 research universities and 46 member colleges in the best 100 liberal arts colleges. Every year an average of 20 new institutions join the program. Through TTEI, on average, 6,000 scholarships are awarded annually, with amounts averaging about $24,000. Despite TTEI’s large volume, other tuition exchange programs clear more than 50% of all exchange transactions in the US.11

Each participating college establishes its own policies and procedures for determining the eligibility of dependents for exchange and the number of scholarships it will grant each year. Each member college has agreed to maintain a balance between the number of students sponsored by that institution (“exports”) and the number of scholarships awarded to students sponsored by other member colleges (“imports”). Colleges aim to maintain a one-to-one balance between the number of exports and imports. In particular, if the number of exports exceeds the number of imports, then that college may be suspended from the tuition-exchange program. In order not to be suspended from TTEI, colleges often set the maximum number of sponsored students in a precautionary manner. Many

9“Tuition Exchange enables us to compete with the many larger institutions in our area for talented faculty and staff. The generous awards help us attract and retain employees, especially in high-demand fields like nursing and IT.” – Frank Greco, Director of Human Resources, Chatham University, from the home page of The Tuition Exchange, Inc., www.tuitionexchange.org, retrieved on 09/19/2012. Also see Footnote 14 about the results of a survey that we conducted detailing the importance of tuition-exchange programs in job choice for faculty members.


11In Online Appendix C, we describe the features of the other prominent tuition exchange programs.
colleges explicitly mention in their application documents that in order to guarantee their continuation in the program, they need to limit the number of sponsored students.\textsuperscript{12} As a result, in many cases not all qualified dependents are sponsored. Colleges often use the length of the related employee’s tenure to prioritize eligible students.

A tuition-exchange program usually functions as follows: each college determines its quotas, which are the maximum number of students it will sponsor (its “eligibility quota”) and the maximum number it will admit (its “import quota”) through the program. Then, the eligible students apply to colleges, and colleges make scholarship decisions based on preferences and quotas. A student can get multiple offers. She declines all but one, and, if possible, further scholarship offers are made in a few additional rounds. Students who are not sponsored by their home colleges cannot participate in the program, and hence do not receive a tuition-exchange scholarship. In the end, neither a student that is accepted by another college nor the home institution of the student pays tuition. The admitting institution de facto awards a tuition waiver to the dependent of the faculty of another college.

One may ask why tuition-exchange programs exist in the first place because some other colleges choose to subsidize faculty directly instead of participating in tuition-exchange programs. Although this may create flexibility for the students, any direct compensation over $5,250 is taxable income, whereas a tuition-exchange scholarship is not. Tuition exchange is not considered to be an income transfer.\textsuperscript{13,14} Moreover, tuition exchange colleges may not want to switch to such direct-compensation programs from a cost-saving perspective, regardless of the tax benefit to the faculty member. We present a simple back-of-the-envelope calculation to demonstrate these cost savings. A college typically

\textsuperscript{12}Lafayette College, Daemen College, DePaul University, and Lewis University are just a few examples. 

\textsuperscript{13}In particular, it is considered a scholarship, and it is not taxable. See http://www.irs.gov/publications/p970/ch11.html and http://www.irs.gov/publications/p970/ch01.html reached on Feb 20, 2016.

\textsuperscript{14}To collect anecdotal evidence on how much faculty members value the tuition-exchange benefit, we also conducted an IRB-approved e-mail-delivered online survey in 21 tuition-exchange schools using Qualtrics e-mail survey software. Our respondent pool is composed of 150 faculty members (with a 7.5 to 15% response rate). In this pool, there are 48, 56, and 46 assistant, associate, and full professors, respectively. 17% of the respondents have no child. In order to understand whether tuition-exchange benefits attract faculty members, we ask how important of a role their college’s membership in a tuition-exchange program played their acceptance of their offer. According to 19%/60% of the respondents, the tuition-exchange benefit was extremely important/important in their acceptance decision, respectively. Moreover, according to 23%/73% of the respondents with children, the tuition-exchange benefit played an extremely important/important role in their acceptance decision, respectively. In order to understand the value of the tuition-exchange benefit for faculty, we asked how much annual income they would give up in order to keep their tuition-exchange benefit. When we consider all respondents, the average annual value of the tuition-exchange benefit is $7,314 each year in today’s dollars per faculty member (for the ones with one or more child currently, it is only slightly higher, $8,243).
tries to keep its incoming class size almost fixed each year. The cost of admitting one more student through tuition exchange instead of a tuition-paying student is the foregone tuition payment of the student. The average tuition paid by an incoming student is on the order of half of full tuition.\textsuperscript{15} On the other hand, direct compensation for an outgoing student is on the order of full tuition (assuming that a college professor is required to pay the full tuition for his child when his home college provides direct compensation benefit). Therefore, tuition exchange saves the school about half of a full-tuition payment per student. The total surplus savings for the faculty member and the school per student is on the order of half a tuition payment and the income tax savings on the full tuition benefit (for a $40,000-a-year school, this corresponds to a $33,000 joint parent-school saving per student – assuming 1/3 of the direct compensation is paid in income tax at the margin).

### 2.2 Temporary Worker Exchanges

The balancedness requirement also matters in \textit{temporary worker-exchange programs} (such as those for teachers, students, academic staff, and medical doctors). The Commonwealth Teacher Exchange Programme (CTEP), Fulbright Teacher Exchange Program, Erasmus Student Exchange Program, and the exchange program of International Federation of Medical Students’ Associations (IFMSA) are just a few examples.\textsuperscript{16} Some of these have been running for decades,\textsuperscript{17} and thousands of participants benefit from these worker exchange programs annually. Every year more than 10,000 medical students and 200,000 college students around the world participate in IFMSA’s and Erasmus’ exchanges, respectively.\textsuperscript{18} The main difference between these programs and tuition exchange is that (1) most exchange appointments are temporary, typically lasting one year, and (2) the workers are currently employed by their associated firms, so if they cannot be exchanged, they will continue to work in their current jobs.

\textsuperscript{15}2012 Tuition Discounting Study of the National Association of College and University Business Officers report that incoming freshmen pay on average 56\% of full tuition at a private university.
\textsuperscript{16}There are also small bilateral staff-exchange programs. See Online Appendix D for details of all these programs.
\textsuperscript{17}CTEP, which allows participants to exchange teaching positions and homes with a colleague from the UK, Australia, or Canada, has been running for 100 years. See http://www.cyec.org.uk/exchanges/commonwealth-teacher-exchange.
2.3 Importance of Balancedness Requirement

Although two-sided matching via exchange induces a two–sided matching market, workers (students) cannot participate in market activity unless their home firms (colleges) sponsor them. Hence, an import/export balance emerges as an important feature of sustainable outcomes, as there are no monetary transfers between parties and there are costs for colleges associated with providing students. Balance requirements are the most important feature of these markets that distinguishes them from the previously studied matching markets. We illustrate three cases in which the absence of a balanced exchange led to the failure of the exchange program in different contexts.

The Northwest Independent Colleges Tuition Exchange program was founded in 1982 and included five members. Unlike TTEI, this program allowed children of faculty members to attend any member colleges tuition free upon admission. That is, the colleges were not able to limit their exports. Because of sizable imbalances between imports and exports, members agreed to dissolve the program, and it stopped accepting new applicants after Fall 2015 in its current form. The Jesuit universities exchange program FACHEX is another that was adversely affected. It included prominent universities such as Georgetown and Boston College. The program still does not have an embedded balancedness requirement. As a result, Georgetown abandoned the program a number of years ago, although the other universities share similar Jesuit mission statements.

The Erasmus student exchange program among universities in Europe is another example of a market in which a lack of balancedness have caused some exchange relationships to be terminated. Member colleges that want to exchange students with each other sign bilateral contracts that set the maximum number of students to be exchanged in certain years. The renewal decision of the contract depends on whether a reasonable balance is maintained between the incoming and outgoing exchange students between these colleges. In particular, if one of the colleges has more incoming students than its outgoing students, then that college does not renew the contract.

Another aspect of the requirement for balanced exchange is problems that can arise due to dynamic features of the exchange. Tuition- and worker-exchange markets are closely related to favor markets, also known as “time banks,” where time spent doing a favor or the number of favors is used as the currency of exchange. Holding of the transaction

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19 When we talk about balancedness in this paper, we are not strictly talking about zero-balance conditions where imports and exports even each other out. The idea can also be relaxed in static and dynamic manners to attain an approximate balance over time. Indeed, there could be gains for intertemporal trades and our proposals also address these issues in Section 4.2.

currency in such markets corresponds to a positive imbalance in our model. If not enough currency is injected initially to the system and there is too much uncertainty, agents may shy away from using their currency. Baby sitting co-ops are a leading example of such time banks. Such banks could be adversely affected by the lack of balanced clearing mechanisms that clear all favors in a well-defined sufficiently long time period ahead of the time.\footnote{In the mid-1970s, at the Capitol Hill Baby-Sitting Coop in Washington, DC, negative–balance aversion of families resulted in imbalances between families and decreased the number of favor exchanges between families. For details see \url{http://www.ft.com/cms/s/2/f74da156-ba70-11e1-aa8d-00144feabdc0.html} reached on Feb 20, 2016. This fits our setting perfectly: if the matches could be done in a monthly schedule using a centralized method, then balancedness requirements could be easily addressed, and such markets would not close down. Indeed, commonly used time bank softwares assign favor schedules over a time horizon. See for example \url{http://timebanks.org/get-started/community-weaver/} reached on Feb 20, 2016. We can address this issue using a many-to-many version of our proposals, although in this paper we do not formally model and analyze this but use it as a motivating example for the need for balanced exchange without uncertainty.}

3 Two–Sided Matching via Exchange: Model

In this section, we introduce our model and the desirable solution concepts.\footnote{We will keep tuition exchange in mind in naming our concepts. The minor differences in the temporary worker-exchange model will be highlighted in Section 5.} Let $C = \{c_1, ..., c_m\}$ and $S = \{s_1, s_2, ..., s_n\}$ be the set of colleges and students, respectively. The set of students is partitioned into $m$ disjoint sets, i.e. $S = \bigcup_{c \in C} S_c$ where $S_c$ is the set of students who are applying to be sponsored by college $c$. Let $q = (q_c)_{c \in C} \in \mathbb{N}^m$ be the (scholarship) admission quota vector, where $q_c$ is the maximum number of students who will be admitted by $c$ with tuition exchange scholarship, and $e = (e_c)_{c \in C} \in \mathbb{N}^m$ be the (scholarship) eligibility quota vector, where $e_c$ is the number of students in $S_c$ certified eligible by $c$. Let $\succ_C = (\succ_c)_{c \in C}$ be the list of college internal priority orders, where $\succ_c$ is a linear order over $S_c$ based on some exogenous rule.\footnote{We assume that there is no tie in the internal priorities. In practice, each college breaks any ties by using lotteries.} We denote the set of eligible students that are certified eligible by $c$ by $E_c$ where $E_c = \{s \in S_c \mid r_c(s) \leq e_c\}$ and $r_c(s)$ is the rank of $s \in S_c$ under $\succ_c$. Let $E = \bigcup_{c \in C} E_c$ be the set of all eligible students. The unassigned option, which we name the null college, is denoted by $c_\emptyset$. We set $q_{c_\emptyset} = n$.

A matching is a correspondence $\mu : C \cup S \rightarrow C \cup S \cup c_\emptyset$ such that:\footnote{We will occasionally refer to singleton $\{x\}$ as $x$ with a slight abuse of notation. The only exception will be $\{\emptyset\}$.} (i) $\mu(c) \subseteq S$ where $|\mu(c)| \leq q_c$ for all $c \in C$, (ii) $\mu(s) \subseteq C \cup c_\emptyset$ where $|\mu(s)| = 1$ for all $s \in S$, (iii) $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in C$ and $s \in S$, and (iv) $\mu(s) = c_\emptyset$ for all $s \not\in E$.\footnote{In tuition exchange, only the students who are certified eligible can be assigned to other institutions.}
Let $\mathcal{M}$ be the set of matchings. Let $X^\mu_c = \{s \in S_c \mid \mu(s) \in C \setminus c\}$ be the set of exports for $c$ under $\mu$. This is the set of certified students of $c$ who are matched\(^{26}\) with other colleges. Let $M^\mu_c = \{s \in S \setminus S_c \mid \mu(s) = c\}$ be the set of imports for $c$ under $\mu$. This is the set of students from other colleges matched with $c$. Let $b^\mu_c \in \mathbb{R}$ be the net balance of college $c$ in matching $\mu$. We set $b^\mu_c = |M^\mu_c| - |X^\mu_c|$. We say college $c$ has a zero (negative) [positive] net balance in matching $\mu$ if $b^\mu_c = 0$ ($b^\mu_c < 0$) [$b^\mu_c > 0$]. In any matching $\mu \in \mathcal{M}$ the summation of the net balances of colleges is 0, i.e., $\sum_{c \in C} b^\mu_c = 0$. Therefore, if there exists a college with a positive (negative) net balance, then there exists at least one other college with a negative (positive) net balance.

Let $\succsim = (\succsim_s, \succsim_c) = ((\succsim_s)_{s \in S}, (\succsim_c)_{c \in C})$ be the list of student and college preferences over matchings, where $\succsim_i$ is the preference relation of agent $i \in S \cup C$ on $\mathcal{M}$. We denote the strict preference of agent $i \in S \cup C$ on $\mathcal{M}$ by $\succ_i$ and her indifference relation by $\sim_i$.

Each $s \in S$ cares only about her own match in a matching and has a strict preference relation $P_s$ on $C \cup q_b$. Let $R_s$ denote the at–least–as–good–as relation associated with $P_s$ for any student $s \in S$: $cR_s c' \iff c P_s c' \ or \ c = c'$ for all $c, c' \in C \cup q_b$. Student $s$’s preferences over matchings $\succsim_s$ is defined as follows: if $\mu(s) R_s \mu'(s)$ then $\mu \succsim_s \mu'$.

On the other hand, each college potentially cares not only about its admitted class of (scholarship) students but also about its net balance. That is, colleges rank any two matchings with the same net balance by considering the set of admitted students. Colleges do not consider all students worth awarding a scholarship.\(^{27}\) For instance, a student who cannot satisfy regular admission requirements cannot be awarded a scholarship. To explain how colleges compare two matchings with the same balance, we need a ranking for each college over the sets of admitted students. The preference relation of a college $c$ over matchings, $\succsim_c$, is defined through a linear order, denoted by $P_c$, over $S \cup \{\emptyset\}$. Let $P^*_c$ be the responsive (Roth, 1985) ranking of $c$ over the subsets of students; that is, $\forall T \subseteq S$ with $|T| < q_s$ and $s, s' \in S \setminus T$: (1) $s P_c \emptyset \iff (T \cup s) \ P^*_c (T \cup s')$ and (2) $s P_c s' \iff (T \cup s) \ P^*_c (T \cup s')$. Note that $P^*_c$ is just a ranking over the sets of admitted students and is not the preference relation of $c$ over matchings. Let $R^*_c$ be the weak ranking over the subset of students induced by $P^*_c$. Throughout the paper we assume that between any two matchings in which $c$ has the same net balance, it prefers the one with the higher-ranked set of admitted students according to $R^*_c$. Formally, for any $c \in C$, college $c$’s preferences over matchings, $\succsim_c$, satisfies the following restriction: for any

\(^{26}\)When we say a student is matched to a college, we mean the student receives a tuition-exchange scholarship from that school.

\(^{27}\)We say a student $s$ is unacceptable for college $c$ if $c$ does not consider $s$ worth awarding a scholarship and $s$ is acceptable for $c$ otherwise.
\(\mu, \nu \in \mathcal{M}\), if \(b^\mu_c = b^\nu_c\) and \(\mu(c)^* R^*_c \nu(c)\) then \(\mu \succeq_c \nu\).\(^{28}\)

We now introduce the properties of desirable matchings. A matching \(\mu \in \mathcal{M}\) Pareto dominates another matching \(\nu \in \mathcal{M}\) if \(\mu \succeq_i \nu\) for all \(i \in C \cup S\) and \(\mu \succ_j \nu\) for some \(j \in C \cup S\). A matching \(\mu\) is Pareto efficient if it is not Pareto dominated by any other \(\nu \in \mathcal{M}\). A matching \(\mu \in \mathcal{M}\) is acceptable if \(s P_c \emptyset\) and \(c P_s c_\emptyset\) for all \(c \in C\) and \(s \in \mu(c)\).

A matching \(\mu \in \mathcal{M}\) is balanced if \(b^\mu_c = 0\) for all \(c \in C\).\(^{29}\) Balancedness is the key property in a tuition-exchange market. For each tuition-exchange market, the set of balanced matchings is nonempty. For instance, a matching where all students are unassigned satisfies balancedness. Moreover, there may exist multiple balanced matchings for a given market. We say a balanced matching \(\mu\) is balanced–efficient if it is not Pareto dominated by any other balanced matching.

We say \(\mu\) is blocked by a college \(c \in C\) if there exists some \(\mu' \in \mathcal{M}\) such that \(\mu' \succ_c \mu\), \(\mu'(s) = \mu(s)\) for all \(s \in S \setminus \mu(c)\), and \(\mu'(c) \subset \mu(c)\). A matching \(\mu\) is blocked by a student \(s \in S\) if \(c \emptyset P_s \mu(c)\). A matching \(\mu\) is individually rational if it is not blocked by any individual college or student. A matching \(\mu\) is nonwasteful if there does not exist a blocking pair \((c, s)\) such that \(|\mu(c)| < q_s\), \(c P_s \mu(s)\) and \(\mu'(s) = c\) and \(\mu'(s') = \mu(s')\) for all \(s' \in S \setminus s\). In Appendix A, we provide a detailed analysis of the current decentralized practice of tuition exchange and “stability”, defined there for a market with externalities.

Throughout the paper, \(C, S,\) and \(\succ_C\) are fixed; each triple of a quota vector, eligibility vector, and a preference profile defines a tuition-exchange market – or simply, a market – as \([q, e, \succeq]\).

### 3.1 Tuition-Exchange Mechanisms

The current practice of tuition exchange is implemented through indirect semi-decentralized market mechanisms. Although our new proposal can also be implemented indirectly, it will be useful to discuss it as a direct mechanism to analyze its properties. A (direct) mechanism is a systematic way of selecting a matching for each market. Let \(\varphi\) be a mechanism; then the matching selected by \(\varphi\) in market \([q, e, \succeq]\) is denoted by \(\varphi[q, e, \succeq]\), and the assignment of agent \(i \in S \cup C\) is denoted by \(\varphi[q, e, \succeq](i)\).

In a revelation game, students and colleges report their preferences; additionally, colleges report their admission quotas and the number of students they certify as eligi-

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\(^{28}\)We will introduce more structure over college preferences in Section 4 and Appendix A.

\(^{29}\)Note that \(b^\mu_c \geq 0\) for all \(c \in C\) and \(b^\nu_c \leq 0\) for all \(c \in C\) imply \(b^\nu_c = 0\) for all \(c \in C\).
We say a mechanism \( \varphi \) is **immune to preference manipulation for students** (or **colleges**) if for all \([q,e,\succ] \), there exists no \( i \in S \) (or \( i \in C \)) and \( \succ'_i \) such that \( \varphi[q,e,(\succ'_i,\succ)](i) \succ_i \varphi[q,e,\succ](i) \). A mechanism \( \varphi \) is **immune to preference manipulation** if it is immune to preference manipulation for both students and colleges. A mechanism \( \varphi \) is **immune to quota manipulation** if for all \([q,e,\succ] \), there exists no \( c \in C \) and \((q'_c, e'_c) \in \mathbb{N}^2 \) such that \( \varphi[(q'_c,q_{-c}),(e'_c,e_{-c}),\succ](c) \succ_c \varphi[q,e,\succ](c) \). A mechanism \( \varphi \) is **strategy-proof for colleges** if for all \([q,e,\succ] \), there exists no \( c \in C \) and \((q'_c, e'_c, \succ'_c) \) such that \( \varphi[(q'_c,q_{-c}),(e'_c,e_{-c}),(\succ'_c,\succ_{-c})](c) \succ_c \varphi[q,e,\succ](c) \). A mechanism is **strategy-proof for students** if it is immune to preference manipulation for students. Finally, a mechanism is **strategy-proof** if it is strategy-proof for both colleges and students.\(^{31}\)

One distinctive feature of the tuition-exchange market is the existence of internal priorities for each \( c \in C, \succ_c \). In current practice, the internal priority order is used to determine which students will be certified eligible. This priority order is usually based on the seniority of faculty members. Based on our conversations with tuition liaison officers,\(^ {32}\) when the number of scholarships is small, in general students with higher priority are awarded the scholarships. We incorporate this priority-based fairness objective into our model by introducing a new property. It is desirable that whenever a student \( s \) sponsored by \( c \) is assigned to a college in \( \varphi[q,e,\succ] \), she should also be assigned in \( \varphi[(\tilde{q}_c,q_{-c}),(\tilde{e}_c,e_{-c}),\succ] \) where \( \tilde{e}_c > e_c \) and \( \tilde{q}_c \geq q_c \). That is, the addition of students with lower internal priority should not cause \( s \) to be unassigned. Formally, a mechanism \( \varphi \) **respects internal priorities** if whenever a student \( s \in S_c \) is assigned to a college in market \([q,e,\succ] \), then \( s \) is also assigned to a college in \([\tilde{q}_c,q_{-c}),(\tilde{e}_c,e_{-c}),\succ] \) where \( \tilde{e}_c > e_c \) and \( \tilde{q}_c \geq q_c \). Given that \( s \) is assigned in \([q,e,\succ] \), then \( s \)'s internal rank should be lower than \( e_c \), \( r_c(s) \leq e_c \), and all the new students who are sponsored in \([\tilde{q}_c,q_{-c}),(\tilde{e}_c,e_{-c}),\succ] \) but not in \([q,e,\succ] \) should have a higher internal rank (and hence a lower internal priority) than \( s \).\(^ {33}\) Note that respect for internal priorities is a fairness notion rather than efficiency.

\(^{30}\)Since the internal priority order is exogenous, the set of eligible students can be determined by the number of certified students.

\(^{31}\)Since students care only about the schools they are matched with, it will be sufficient for them to report their preferences over colleges instead of over matchings. Under an additional assumption, our proposal in Section 4 can also be implemented by having colleges only report their preferences over individual students as “acceptable” or “unacceptable”.

\(^{32}\)For example, personal communication with Patricia Bussone, the director at Bradley College.

\(^{33}\)This property is used in our characterization in Section 4 where we show that this axiom does not bring additional cost to our proposed mechanism (Theorem 6).
4 Two–Sided Top Trading Cycles

In this section, we propose a mechanism that is individually rational, acceptable, balanced–efficient, and strategy-proof for students. Moreover, it respects colleges’ internal priorities. Throughout our analysis, we impose a weak restriction on college preferences. Assumption 1 below states that a college prefers a better scholarship class with zero net balance to an inferior scholarship class with a nonpositive net balance.

**Assumption 1** For any \( \mu, \nu \in \mathcal{M} \) and \( c \in C \), if \( b^c_\mu = 0 \), \( b^c_\nu \leq 0 \), and \( \mu(c)P^*_{c}\nu(c) \) then \( \mu \succ_c \nu \).

Note that, under Assumption 1, a college \( c \) may prefer a matching in which it has a negative balance to a balanced matching as long as it has a better incoming class.

We start with the following proposition, which shows the incompatibility between balancedness and individual rationality, and nonwastefulness.

**Proposition 1** Under Assumption 1, there may not exist an individually rational and nonwasteful matching that is also balanced.\(^{34}\)

Proposition 1 also shows that there exists no stable and balanced mechanism.\(^{35}\)

It will be useful to denote a matching as a directed graph, as we will find the outcome of our mechanism through an algorithm over directed graphs. In such graphs, colleges and students are nodes; a directed edge is between a college and a student, and it points to either the college or the student, but not both. Given a matching \( \mu \), let each \( s \in S \) point to \( \mu(s) \) and each \( c \in C \) point to all its matched sponsored students, i.e., those in \( S_c \setminus \mu(c_0) \); moreover, let \( c_0 \) point to students in \( \mu(c_0) \). In this graph, we define the following subgraph: A **trading cycle** consists of an ordered list of agents \( (c_1, s_1, c_2, s_2, ..., c_k, s_k) \) such that \( c_1 \) points to \( s_1 \), \( s_1 \) points to \( c_2 \), ..., \( c_k \) points to \( s_k \) and, \( s_k \) points to \( c_1 \).

In the following Remark, we state that if a matching is balanced, then we can decompose it into a finite number of disjoint trading cycles. We skip its proof for brevity.

**Remark 1** A matching \( \mu \) is balanced if and only if each student is in a trading cycle in the graph of the matching.

We are ready to propose a new two-sided matching mechanism. This is one of our main contributions in this paper. We will find its outcome using an algorithm inspired

\(^{34}\)All proofs are in Appendix B.

\(^{35}\)A matching is stable if it is individually rational and not blocked by a college-student pair. In our setting with externalities, it is formally defined in Appendix A.
by top–trading–cycles (TTC) introduced for one–sided discrete resource allocation problems, such as for school choice (by Abdulkadiroğlu and Sönmez, 2003) and dormitory room allocation (by Abdulkadiroğlu and Sönmez, 1999). These TTC algorithms were inspired by Gale’s TTC algorithm (Shapley and Scarf, 1974), which was used to find the core allocation of a simple discrete exchange economy, commonly referred to as the housing market, a subclass of one-sided matching problems. Most common mechanisms in one-sided matching problems function through algorithms that mimic agents exchanging objects that are initially allocated to them either through individual property rights or through the mechanism’s definition of the agents (see also Pápai, 2000; Pycia and Ünver, 2016). In contrast, in our market, college slots are not objects, as the colleges are active decision makers. Therefore our definition of a mechanism, and the properties of matchings and mechanisms (except strategy-proofness for students) do not have any analogous translation in such problems. However, because we use a variant of TTC algorithm to find the outcome, we refer to our mechanism as two–sided (student–pointing) top–trading–cycles (2S–TTC). Its outcome is found for any given $[q, e, \succ]$ as follows:

The Algorithm for the Two–Sided Top–Trading–Cycles Mechanism:

Round 0: Assign two counters, for admission and eligibility, for each college $c \in C$, and set them equal to $q_c$ and $e_c$, respectively.

Round $k \geq 1$: Each remaining student points to her favorite college among the remaining ones in $C \cup c_\emptyset$ that considers her acceptable, and each remaining college $c \in C$ points to the student in $S_c$ among the remaining ones who has the highest internal priority in $\succ_c$. Null college $c_\emptyset$ points to the students pointing to it. Due to the finiteness of the sets of the colleges and students, there exists at least one trading cycle. Each agent can be part of at most one cycle. Every student in each trading cycle is assigned a seat at the college she is pointing to and removed. If the cycle does not contain $c_\emptyset$, then the counters of each college in that cycle are reduced by one. If the cycle contains $c_\emptyset$, then we reduce only the eligibility counter of the college whose student is in that cycle. If any counter of a college reaches zero, then that college is removed and its remaining students are assigned to $c_\emptyset$.

The converse of this process, using an algorithm originally introduced for two-sided matching markets in one-sided matching markets, has already been utilized in market design. For certain real-life one-sided problems regarding student placement and choice, Balinski and Sönmez (1999) introduced and Abdulkadiroğlu and Sönmez (2003) advocated the Pareto–dominant fair mechanism (also commonly referred to as the student–optimal stable mechanism). This mechanism’s outcome can be found through the celebrated student–proposing deferred acceptance algorithm. This algorithm was originally introduced to find stable matchings in two-sided matching markets by Gale and Shapley (1962). Later on, many school districts in the US adopted this mechanism for public school admissions (cf. Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005 and Abdulkadiroğlu, Pathak, and Roth, 2005).
The algorithm terminates when there are no remaining students in the market.37

In the following theorem, we show that 2S-TTC is balanced–efficient, acceptable, and individually rational, and it respects internal priorities.

**Theorem 1**  Under Assumption 1, 2S-TTC is an individually rational, balanced–efficient, and acceptable mechanism that also respects internal priorities.

It should be noted that the balanced-efficiency of 2S-TTC is not an extension of the classical Pareto efficiency result of TTC in a one–sided market. Here, colleges are players who have multiple seats. Therefore, one may think that by assigning a college highly preferred students and also some unacceptable ones, an individually rational balanced matching can potentially be (weakly) improved for everyone, colleges and students alike, while violating acceptability for colleges and yet still obeying balancedness. In this theorem, through an iterative approach, we show that it is not possible to improve over 2S-TTC’s outcome in such a fashion. Moreover, 2S-TTC satisfies an even stronger version of respecting internal priorities: each eligible student is assigned the same college seat when new students are deemed eligible.

Under a centralized mechanism, incentives for participants to truthfully reveal their preferences are desirable. Unfortunately, we show that balanced–efficiency, individual rationality, and immunity to preference manipulation for colleges (hence strategy-proofness for colleges) are incompatible properties.

**Theorem 2**  There does not exist an individually rational (or acceptable) and balanced–efficient mechanism that is also immune to preference manipulation for colleges.

We prove this theorem by constructing several small markets and showing that it is not possible to satisfy all three properties in one of these markets.

Theorem 1 and 2 imply that the 2S-TTC mechanism is not strategy-proof for colleges.

Although 2S-TTC is not strategy-proof for colleges, the following theorem shows that it is group strategy-proof for students. This result is a consequence of TTC being group strategy-proof in a house allocation/exchange market (cf. Pápai, 2000).

**Theorem 3**  2S-TTC is group strategy-proof for students.

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37As this is a two-sided matching market, we could also propose the college-pointing version of the 2S-TTC mechanism in which colleges point to their most preferred students among the ones considering them acceptable and each student points to her home college in each round. This variant takes college preference intensity more seriously than students’. However, it can be easily shown that it gives incentives to both students and colleges for manipulation. On the other hand, the (student-pointing) 2S-TTC is group strategy-proof for students, as we state in Theorem 3.
As we only care whether a college finds a student acceptable or not in running 2S-TTC, it can be run as an indirect mechanism where colleges report only their acceptable incoming students. Hence, the strategy space for the colleges is very simple in using 2S-TTC in the field: their strategy is to report their admission and eligibility quotas and their sets of acceptable students based on their preferences over the matchings, set of own students, and internal priority order.

Moreover, if we focus on the game played by the tuition-exchange office of a college, when admissions preferences are fixed, truthful admission quota revelation and certification of all its own students is a (weakly) dominant strategy under 2S-TTC.\textsuperscript{38}

**Theorem 4** Under Assumption 1 and when true eligibility quotas satisfy \( e_c = |S_c| \) for all \( c \in C \), 2S-TTC is immune to quota manipulation.

The proof uses an auxiliary lemma that we state and prove in the Appendix. This lemma shows that, as the quotas of a college weakly increase, the import and export sets of this college also weakly expand under 2S-TTC.\textsuperscript{39}

Theorems 3 and 4 point out that only colleges can benefit from manipulation, and they can manipulate the 2S-TTC mechanism by misreporting their preferences. Moreover, the only way to manipulate preferences is to report an acceptable student as unacceptable. Suppose we take all the admitted students in the regular admission procedure as acceptable for a tuition-exchange scholarship. Then, to manipulate the 2S-TTC mechanism, a college needs to reject a student who satisfies the college admission requirements. Usually college admission decisions are made before the applicants are considered for scholarships. These decisions are made based on well-defined criteria. Violation of these criteria may lead to legal action. Proposition 2 below implies that colleges do not benefit from misreporting their ranking over incoming classes. Hence, in practice, colleges will be more likely to report their true preferences over matchings and true scholarship admission quota and to certify all their own students.

\textsuperscript{38}On their websites, colleges explain that the sole reason for certifying a limited number of students is maintaining a balanced exchange. 2S-TTC removes the need for this rightful caution associated with the current market practices (see Appendix A).

\textsuperscript{39}Theorem 4 is in stark contrast with similar results in the literature for Gale-Shapley-stable mechanisms. For example, it is well known that the student- and college-optimal Gale-Shapley-stable mechanisms are prone to admission quota manipulation by the colleges even under responsive preferences, regardless of imbalance aversion (cf. Sönmez, 1997); truthful revelation does not even constitute an equilibrium, and any pure strategy equilibrium, if it exists, increases all colleges' welfare above truthful revelation (cf. Konishi and Ünver, 2006). Thus, 2S-TTC presents a robust remedy for a common problem seen in centralized admissions that use the student-optimal Gale-Shapley-stable mechanism and also in tuition exchange in a decentralized market (cf. Theorems 10 and 11 in Appendix A).
Proposition 2 Under Assumption 1, colleges are indifferent among strategies that report preferences over matchings in which the same set of students is acceptable under the 2S-TTC mechanism.

We have shown that 2S-TTC has appealing properties. In the following theorem, we show that it is the unique mechanism satisfying respect for internal priorities, acceptability, balanced–efficiency, and strategy-proofness for students.

Theorem 5 Under Assumption 1, 2S-TTC is the unique student–strategy-proof, acceptable, and balanced–efficient mechanism that also respects internal priorities.

In the proof of our characterization theorem, we use a different technique from what is usually employed in elegant single quota characterization proofs such as Svensson (1999) and Sönmez (1995) for the result of Ma (1994). Our proof relies on building a contradiction with the claim that another mechanism with the four properties in the theorem’s hypothesis can exist. Suppose such a mechanism exists and finds a different matching than 2S-TTC for some market. The 2S-TTC algorithm runs in rounds in which trading cycles are constructed and removed. Suppose \( S(k) \) is the set of students removed in Round \( k \), while running the 2S-TTC algorithm in such a way that in each round only one arbitrarily chosen cycle is removed and all other cycles are kept intact. We find a Round \( k \) and construct an auxiliary market with the following three properties: (1) Eligibility quotas of home colleges of students in \( S(k) \) are set such that these are the last certified students in their respective home institutions; (2) all preferences are kept intact except those of students in \( S(k) \), whose preferences are truncated after their 2S-TTC assignments; and (3) all students in \( S(k) \) are assigned \( c_\theta \) under the alternative mechanism, while all students removed in the 2S-TTC algorithm before Round \( k \) have the same assignment under 2S-TTC and the alternative mechanism. This contradicts the balanced-efficiency of the alternative mechanism: we could give the students in \( S(k) \) their 2S-TTC assignments while keeping all other assignments intact and obtain a Pareto-dominating balanced matching. Round \( k \) and the auxiliary market are constructed in three iterative steps. In Online Appendix H, we show the independence of the axioms mentioned in Theorem 5.

Among all the axioms, only the respect for internal priorities is based on exogenous rules. One might suspect that more students will benefit from the tuition-exchange program if we allow the violation of respect for internal priorities. A natural question that arises is whether there is a balanced and individually rational mechanism that never assigns fewer students than the 2S-TTC mechanism and that selects a matching in which more students are assigned whenever there exists such an outcome. In Theorem 6, we
show that the mechanism satisfying the above conditions is not strategy-proof for students.

**Theorem 6** Any balanced and individually rational mechanism that does not assign fewer students than the 2S-TTC and selects a matching in which more students are assigned whenever such a balanced and individually rational outcome exists, is not strategy-proof for students, even under Assumption 1.

### 4.1 Market Implementation: Tuition Remission and Exchange

Incorporating tuition-remission programs by all participating colleges in tuition exchange is the best way to implement a centralized tuition clearinghouse. If parallel remission and exchange programs are run, as in current practice, a student may receive more than one scholarship offer, one from her home college and one from the tuition-exchange program. If the student accepts the home college’s offer, the net balance of the college may deteriorate.\(^{40}\)

Although the current system is inflexible in accommodating this important detail, a clearinghouse utilizing 2S-TTC can easily combine tuition exchange with remission. Indeed, in Assumption 1 for college preferences, we allowed a college to deem its own sponsored students to be acceptable. Hence, all our results in this section go through when both programs are run together.

More specifically, in the market, we propose to run an indirect version of 2S-TTC in sequential stages in a semi-decentralized fashion: first, colleges announce their tuition-exchange scholarship quotas and which of their students are eligible to be sponsored for both exchange and remission; then, eligible students apply for scholarship to the colleges they find acceptable (their home colleges or others); then colleges send out scholarship admission letters. At this stage, as students have also learned their opportunities in the parallel-running regular college admissions market, they can form better opinions about the relative ranking of the null college, i.e., their options outside the tuition-exchange market. Students submit rankings over the colleges that admitted them with a tuition-exchange scholarship and the relative ranking of their outside option. Finally, 2S-TTC is run centrally to determine the final allocation.

\(^{40}\)Indeed, Robert Lay, the Dean of Enrollment Management at Boston College told us in personal communication that a centralized tuition-exchange clearinghouse should incorporate a tuition-remission program of all participating colleges for this reason.
4.2 Allowing Tolerable Imbalances

Some tuition-exchange programs care about approximate balance over a moving time window in a dynamic setting. In this subsection, we relax the zero-balance constraint and allow each college $c \in C$ to maintain a balance within an interval $[\ell_c, u_c]$ where $\ell_c \leq 0 \leq u_c$.\(^{41}\) When either $\ell_c$ or $u_c$ equals zero for all $c \in C$, the problem turns into the case we studied in the beginning of Section 4. Let $(\ell_c, u_c)_{c \in C}$ denote the tolerance profile.

When we allow the colleges to hold a non-zero balance, then there may exist some colleges exporting (importing) more students than they import (export). In this case we cannot represent all allocations by cycles. Therefore we need to consider chains in addition to the cycles.

Formally, a chain is defined as an ordered list of college-student pairs $(c_1, s_1, c_2, s_2, ..., c_k)$ such that $c_1$ points to student $s_1$, $s_1$ points to college $c_2$, ..., $c_{k-1}$ points to student $s_{k-1}$ and $s_k$ points to college $c_k$. We refer to $c_1$ as the tail and $c_k$ as the head of the chain.

We use a mechanism similar to the 2S-TTC. We refer to it as the two-sided tolerable top-trading-cycles mechanism (2S-TTTC). For any given tuition-exchange problem and a tolerance profile $(\ell_c, u_c)_{c \in C}$, its outcome is found as follows:

**Two-Sided Tolerable Top Trading Cycles:**

*Step 0:* Fix an exogenous priority order among colleges. Assign two counters for each $c \in C$, $o^q_c$ and $o^e_c$, and set them equal to $q_c$ and $e_c$, respectively. Let $b_c$ track the current balance of $c$ in the fixed portion of the matching. Initially set $b_c = 0$ for each $c \in C$. Assign an export and an import counter for the null college and set them equal to $|S|$. All colleges are marked as importing and exporting.

*Step 1a:* If $o^q_c = 0$, and either $o^q_c = 0$ or $b_c = u_c$, then remove $c$. If $o^e_c = 0$, $o^q_c > 0$ and $b_c < u_c$, then $c$ becomes non-exporting.$^{42}$

*Step 1b:* Each student points to her favorite available importing college, which considers her acceptable, and each available exporting college $c$ points to the student $s \in S_c$ who has the highest internal priority among the available ones. The null college $c_{\emptyset}$ points to the students pointing to it.

Proceed to Step 2 if there is no cycle. Otherwise locate each cycle, and assign each student to the college that she points to. Each assigned student is removed.

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\(^{41}\)Here, $\ell_c$ and $u_c$ are integers.

\(^{42}\)I.e., a college is non-exporting if it has available quota to import but all its sponsored students are removed. Therefore, a non-exporting college cannot point to a student.
* The eligible student counter, $o_e$, of each $c$ that is in a cycle is reduced by one.
* The import counter, $o_i$, of each $c$ in a cycle is reduced by one.
* Return to Step 1a.

**Step 2:** If there are no students left, we are done. If not, then all chains end with non-exporting colleges. If $b_c = \ell_c$ for each available exporting college $c$, then remove all non-exporting colleges and go to Step 1a.\(^{43}\) Otherwise, find the chain whose tail has the highest priority among the available exporting colleges with $b_c > \ell_c$. Assign each student in that chain to the college that she points to and remove her from $S$. Denote the tail and head colleges of the chain by $c_t$ and $c_h$, respectively. Other colleges in the chain are represented by $\tilde{c}$:
* The eligible student, $o_e$, and import, $o_i$, counters of all $\tilde{c}$ are reduced by one.
* Eligible student counter $o_e$ and current balance $b_e$ are reduced by one.
* Import counter $o_i$ is reduced by one and current balance $b_i$ is increased by one.
* Return to Step 1a.

The algorithm terminates when there are no remaining eligible students left. We call each repetition of these steps a round.

The 2S-TTTC mechanism inherits the desired features of the 2S-TTC. In Theorem 7, we show that students cannot benefit from misreporting their preferences under 2S-TTTC and for any problem and tolerance profile $(\ell_c, u_c)_{c \in C}$ its outcome cannot be Pareto dominated by another acceptable matching $\nu$ such that $b_c \leq [l_c, u_c]$ for all $c \in C$.

**Theorem 7** 2S-TTTC is strategy-proof for students and for any problem $[q, e, \succeq]$ and tolerance profile $(\ell_c, u_c)_{c \in C}$, there does not exist an acceptable matching $\nu$ that Pareto dominates the outcome of 2S-TTTC and $\ell_c \leq b_c \leq u_c$ for all $c \in C$.

In the 2S-TTTC mechanism, a student starts pointing to the colleges in her preference list after all the other students with higher internal rank are assigned to a college, including the null college. Moreover, a student points to the colleges ranked over $c_0$ that consider her acceptable. As a consequence of these two features, the 2S-TTTC mechanism satisfies acceptability and respects internal ranking.

**Theorem 8** 2S-TTTC is acceptable and respects internal priorities.

Theorem 7 and 8 hold without any assumption on preferences. Under a mild assumption on college preferences, we can show that 2S-TTTC is individually rational and it is a weakly dominant strategy for colleges to certify all their students.

\(^{43}\)That is, no more chains respecting the tolerance interval can form after this point in the algorithm.
Although 2S-TTTC is defined in a static problem, we can easily extend it to the dynamic environment where the aggregate balance over years matters. In particular, for each period \( t \) and \( c \in C \), we can set counter \( b_c \) equal to college \( c \)'s aggregate balance in period \( t - 1 \) where the aggregate balance in period \( t - 1 \), is equal to the sum of balances between period 1 and \( t - 1 \). Moreover, the exogenous priority rule used in period \( t \) can be determined based on the aggregate balance colleges carry at the end of period \( t - 1 \) such that the highest priority can be given to the college with the highest aggregate balance and so on.

### 4.3 Simulations

Theoretically, 2S-TTC and the current decentralized market procedure modeled in Appendix A cannot be Pareto ranked. Moreover, when we consider the number of unassigned students, neither 2S-TTC nor the decentralized market procedure performs better than the other in every problem. In order to compare the performances of 2S-TTC and the current decentralized market procedure, we run computer simulations under various scenarios. We consider environments with 10 and 20 colleges and 5 and 10 available seats. Each student is linked to a college, and the number of students linked to a college is equal to its capacity. We construct the preference profile of each student \( s \in S_c \) by incorporating the possible correlation among students’ preferences. In particular, we calculate \( s \)'s utility from being assigned to college \( c' \in C \setminus \{c\} \) as follows:

\[
U(s, c') = \beta Z(c') + (1 - \beta)X(s, c').
\]

Here, \( Z(c') \in (0,1) \) is an i.i.d. standard uniformly distributed random variable and it represents the common tastes of students on \( c' \). \( X(s, c') \in (0,1) \) is also an i.i.d. standard uniformly distributed random variable and it represents the individual taste of \( s \) on \( c' \). The correlation in the students’ preferences is captured by \( \beta \in [0,1] \). As \( \beta \) increases, the students’ preferences over the colleges become more similar. For each student \( s \) we randomly choose a threshold utility value \( T(s) \) in order to determine the set of acceptable colleges for \( s \) where \( T(s) \in (0, 0.25) \) is an i.i.d. standard uniformly distributed random variable. We say \( c' \) is acceptable for \( s \) if \( T(s) \leq U(s, c') \). By using the utilities students get from each college and the threshold value, we construct the ordinal preferences of students over colleges.

In order to construct college preferences over students, we follow a similar method as in the student preference profile construction. In particular, we calculate \( c \)'s utility from \( s' \in S \setminus S_c \) as follows:
\[ V(c, s') = \alpha Z(s') + (1 - \alpha) Y(c, s'). \]

Here, \( Z(s') \in (0, 1) \) is an i.i.d standard uniformly distributed random variable and it represents the common tastes of colleges on \( s' \). \( Y(c, s') \in (0, 1) \) is also an i.i.d standard uniformly distributed random variable and it represents the individual taste of \( c \) on \( s \). The correlation in the college preferences is captured by \( \alpha \in [0, 1] \). Like \( \beta \), as \( \alpha \) increases the colleges’ preferences over the students become more similar. For each \( c \in C \) we randomly choose a threshold value \( T(c) \) in order to determine the set of acceptable students for \( c \) where \( T(c) \in (0, 0.5) \) is an i.i.d standard uniformly distributed random variable. We say \( s' \) is acceptable for \( c \) if \( T(c) \leq V(c, s') \). By using the utilities colleges get from each student and the threshold value, we construct the ordinal preferences of colleges over students.

Under each case, we consider a time horizon of 25 periods. In order to mimic the decentralized procedure, we use student proposing DA mechanism in each period. We consider two different strategies colleges play. Under the first strategy, each college certifies its all students as eligible in period 1. Observe that this is a naive behavior, and in a sense the best-case scenario if colleges are negative-balance averse. Under this assumption, colleges have incentives to certify fewer students than their quota (see Theorems 10 and 11 in Appendix A). For further periods, if a college \( c \) carries an aggregate negative balance of \( x \), then it certifies only \( q_c - x \) students, otherwise it certifies all its students. Under the second strategy, in each period we rerun the DA mechanism until the outcome in that period satisfies zero balance and in each run a college with negative balance excludes one student from its certified list. On the other hand, under 2S-TTC, since colleges run a zero balance, each college certifies all of its students in each period. Under each scenario, we run the DA and TTC 1,000 times by using different random draws for \( X, Y, Z, \) and \( T \) and calculate the number of students unassigned under DA and 2S-TTC, and the number of students preferring 2S-TTC over DA and vice versa. We also relax the zero-balance constraint and allow each college to run a negative balance of not more than 20% of its quota. Under the first strategy for a decentralized market, each college certifies all its students as eligible in period 1. For further periods, if a college carries an aggregate negative balance of \( x > 0.2q_c \), then it certifies only \( 1.2q_c - x \) students, otherwise it certifies all its students. Under the second strategy, we exclude students from each college only if they run a negative balance more than 20% of their quota. Similarly, since each college can run a certain amount of negative balance, we use 2S-TTTC instead of 2S-TTC with a tolerable balance interval of \([-0.2q_c, \infty)\).

In Figure 1, we illustrate the simulation results for 20 colleges and 10 seats case. The horizontal axis refers to changing levels of \( \alpha \) and \( \beta \), the preference correlation parameters.
Different values of $\beta$ are grouped together (shown in the right-bottom graph’s legend) while $\alpha$ is used as the main horizontal axis variable. The vertical axis variables in top 4 graphs demonstrate the difference of the percentage of unassigned students between the DA mechanism under the two alternative strategies of the colleges (In each row, the 1st and 3rd graphs are for straightforward behavior of DA and the 2nd and 4th graphs are for the equilibrium behavior of DA explained above) and the 2S-TTC and 2S-TTTC. In bottom 4 graphs, the vertical axes demonstrate the difference between the percentage of the students preferring the versions of 2S-TTC and the percentage of the students preferring the DA mechanism under two alternative strategies of the colleges.\textsuperscript{44}

Under all scenarios, when we compare the percentage of students preferring the versions of the 2S-TTC and the DA mechanism under two alternative strategies of the colleges, we observe that 2S-TTC and 2S-TTTC outperform both alternative strategic behaviors under DA. All results are significant at level 5%. For example, when $\alpha = 0.5$ and $\beta = 0.5$, for yearly tolerance level 0, 9.6% more of all students (i.e., the percentage of students who prefer 2S-TTC to DA minus the percentage who prefer DA to 2S-TTC) prefer 2S-TTC outcome to DA straightforward behavior outcome (while this difference increases to 22.9% for DA equilibrium simulations), as seen in the middle of the graph of the 1st (and 2nd, respectively) graph of the bottom row of Figure 1.\textsuperscript{45}

\textsuperscript{44}The results of the other cases are illustrated in Online Appendix I in Figures 3-5.
\textsuperscript{45}We do not give a separate figure for the absolute percentage of students who prefer 2S-TTC over
Except for very low correlation in both college and student preferences, we observe that the percentage of unassigned students is less under the versions of the 2S-TTC compared to the one under both alternative strategic behaviors under DA. For example, when $\alpha = 0.5$ and $\beta = 0.5$, for the yearly tolerance level 0, 2S-TTC matches 5% of all students more over the percentage matched by DA under straightforward behavior (while this difference increases to 23% over the percentage matched by DA under equilibrium behavior) as seen in the middle of the 1st graph (and 2nd graph, respectively) in the top row of Figure 1, respectively.\footnote{Although, we do not give a separate figure, it is noteworthy to mention that the absolute percentage of students unmatched under DA straightforward scenario increases from 1.4% to 40.4% of all students in both $\alpha$ and $\beta$ per period, under tolerance 0 scenario of 1. The corresponding percentages are 1.1% to 80.5% under the DA equilibrium scenario, increasing again in both $\alpha$ and $\beta$. Other treatments, including the ones reported in Appendix I, display similar pattern although percentage change interval is slightly different.}

In general, as $\alpha$, the colleges’ preference correlation parameter, increases, both welfare measures favor 2S-TTC over DA increasingly more under both tolerance level and both DA behavior scenarios. On the other hand, as $\beta$, the students’ preference correlation parameter, increases, the 2S-TTC’s dominance measures display mostly a unimodal pattern (peaking for moderate $\beta$) for any fixed $\alpha$. We conclude that 2S-TTC and 2S-TTTC approaches outperform DA methods in almost all cases.

The colleges might have negative balance over the tolerance level under the straightforward behavior of DA. We calculate the percentage of students with excess negative balance in each period and the magnitude of the excess relative to the number of available seats in each college. The case for 20 colleges and 10 seats is given in Figure 2. The average negative balance of colleges varies between 1% and 20% of the available seats at colleges and increases with $\alpha$ and $\beta$. Similarly, as $\alpha$ and $\beta$ increase the percentage of colleges with excess negative balance increases and it varies between 1% and 40%.\footnote{The results of all other cases are illustrated in Online Appendix I in Figures 6-8.}

## 5 Temporary Worker Exchanges

Many organizations have temporary worker-exchange programs that can be modeled through our balanced two-sided matching framework. The first difference between such programs and tuition exchange is that these exchanges are temporary. Hence, each firm
Figure 2: Excess balance under DA straightforward behavior simulations with 20 colleges each with 10 seats

requires a set of specific skills, e.g., a mathematics teacher to replace their own mathematics teacher. Compatibility and ability to perform the task are the main preference criterion rather than a strict preference ranking. E.g., finding a good teacher with a specific degree is the first-order requirement, rather than finer details about the rankings of all good teachers.

The second difference is that each position and each worker should be matched, unlike the tuition-exchange application. The workers are currently working for their home firms. Thus, the firms find these workers necessarily acceptable. By contrast, in tuition exchange, colleges are not required to admit all the dependents of their employees without a tuition-exchange scholarship; instead, they use blind or semi-blind criteria similar to the regular admissions process. In temporary worker exchanges, a worker who does not want to go to a different firm necessarily stays employed in his home firm.

Hence, we need to use a variant of the tuition-exchange model to facilitate balanced and efficient trade in such circumstances.

We can use the model introduced in Section 3 for the temporary worker-exchange programs, with slight changes. In Section 3, in the definition of a matching, students who are not certified as eligible are taken as assigned to \( c_{0} \). Since students who are not certified as eligible are not guaranteed to be admitted by their home colleges, defining a matching in this way is correct for tuition-exchange programs; but any worker who is
not certified as eligible continues to work at her current firm. Hence, for worker-exchange programs, the workers who are not certified as eligible in a matching are assigned to their current companies. Formally, a matching is a correspondence $\mu: C \cup S \rightarrow C \cup S$ such that, (1) $\mu(c) \subseteq S$, where $|\mu(c)| \leq q_c$ for all $c \in C$, (2) $\mu(s) \subseteq C$, where $|\mu(s)| = 1$ for all $s \in S$, (3) $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in C$ and $s \in S$, and (4) $\mu(s) = c$ for all $s \in S_c \setminus E$. Let $\mathcal{M}$ be the set of all matchings.

To capture the features of worker-exchange programs, we make assumptions about the preferences of workers and firms. Since worker $s \in S_c$ is already working at firm $c$, we assume that $s$ finds $c$ acceptable and $c$ finds $s$ acceptable, i.e., $cP_sc_0$ and $sP_c\emptyset$ for all $s \in S_c$ and $c \in C$. As discussed above, acceptable workers do not have huge differences for the firms. The compatibility assumption and Assumption 1 together imply that each firm weakly prefers a matching with zero net–balance to another matching with non-positive balance as long as it gets weakly more acceptable workers under the former one. We formally state these assumptions on preferences as follows.

**Assumption 2** (1) (Weakly size-monotonic firm preferences) For any $c \in C$ and $\mu, \nu \in \mathcal{M}$, if $b^\mu_c = 0$, $b^\nu_c \leq 0$ and $|\{s \in \mu(c) : sP_c\emptyset\}| \geq |\{s \in \nu(c) : sP_c\emptyset\}|$, then $\mu \succeq c \nu$, and

(2) (Acceptability of current match) For any $c \in C$ and any $s \in S_c$, $cP_sc_0$ and $sP_c\emptyset$.

Based on Assumption 2, a balanced mechanism that allows employees to get better firms, which consider them acceptable, improves the total welfare without hurting anyone. Hence, 2S-TTC can be applied to temporary exchange programs. In this environment, 2S-TTC inherits its desired features, i.e., it is balanced-efficient, acceptable, and individually rational, and it respects internal priorities. Moreover, it is strategy-proof. The uniqueness result presented in Section 4 also holds in the worker-exchange market.

**Theorem 9** Under Assumption 2, 2S-TTC is a balanced-efficient, individually rational, acceptable, and strategy-proof mechanism that also respects internal priorities, and it is the unique balanced-efficient, acceptable, student strategy-proof mechanism that respects internal priorities.\(^{48}\)

\(^{48}\)Moreover, 2S-TTC is stable in this domain. This result is noteworthy, because the widely-used worker-proposing deferred-acceptance mechanism with exogenous tie-breaking is not balanced-efficient, although it is stable and balanced in this special environment.

We also prove the stability of 2S-TTC in the proof of this theorem. Note that 2S-TTC is not stable with general preferences. In the current setting, even the status-quo matching is stable. Moreover, any welfare improvement cycle where workers point to the firms that are better than their home firms and find them acceptable keeps stability intact because colleges are indifferent among individual acceptable students. 2S-TTC clears the best “acceptable” improvement cycle in each step, hence, it keeps stability intact. This intuition is similar to Erdil and Ergin (2008); however, we have externalities and all “acceptable” improvement cycles are stable, unlike in their domain.
An immediate corollary of Theorem 9 is that reporting true import quota and certifying all workers is a weakly dominant strategy for firms.

**Corollary 1** Under Assumption 2, 2S-TTC is immune to quota manipulation.

## 6 Conclusions

This paper proposes a centralized market solution to overcome problems observed in decentralized exchange markets. Here, we used tuition exchange and temporary exchange programs as our leading examples, in which more than 300,000 people participate annually.

Our paper, besides introducing a new important applied problem and proposing a solution to it, has six main theoretical and conceptual contributions:

- We introduce a brand-new two-sided matching model that builds on the two mainly used matching models in the literature: discrete object allocation, including school choice, and standard many-to-one two-sided matching models, but differs in many fronts from these. This is the first time object allocation and exchange algorithms inspire the mechanism design for a two-sided matching model. This is the first time axiomatic mechanism design is directly used in practical market design to come up with the correct mechanism, as far as we know. This is one of the few instances where natural axiomatic representation is given for a TTC-based mechanism. This is one of the few instances where the stable matching theory of Gale and Shapley is extended to a setting with externalities with tractable existence, equilibrium, and comparative static results (see Appendix A).
- Finally, this is one of the few instances in which a brand-new combinatorial dynamic matching mechanism with good properties is proposed for a dynamic applied problem.

## Appendix A On Current Practice of Tuition Exchange

In this appendix, we analyze the current practice of tuition exchange. As the centralized process is loosely controlled, once each college sets its eligibility/admission quota and eligible students are determined, the market functions more like a decentralized one rather than a centralized one. Once colleges commit to the students they will sponsor, they lose their control over them. A sponsored student can sometimes get multiple offers and decide which one to accept and when to accept it. Hence, stability emerges as a relevant notion for a benchmark market-equilibrium concept when there is no other friction. To adopt stability in our model, we introduce blocking by a pair: We say \( \mu' \in \mathcal{M} \) is obtained from
μ by the **mutual deviation** of c and s if \( s \in \mu'(c) \subseteq \mu(c) \cup s \), and \( \mu'(s') = \mu(s') \) for all \( s' \in S \setminus (\mu(c) \cup s) \). A matching \( \mu \in \mathcal{M} \) is **blocked by college-student pair** \((c, s)\) if 
\[ c P_s \mu(s) \text{ and } \mu' \succ c \mu, \]
where \( \mu' \in \mathcal{M} \) is obtained from \( \mu \) by the mutual deviation of \( c \) and \( s \). As in any blocking condition in cooperative games with externalities, we need to take a stance on how other players act when a pair deviates. We assume that only one college or one student deviates at a time, and assume that the rest of the students and colleges do not make simultaneous decisions.⁴⁹ A matching \( \mu \) is **stable** if it is individually rational and not blocked by any college-student pair.

Tuition exchange market has some idiosyncratic properties different from those of previously studied two-sided matching markets.

In tuition exchange — in its current implementation — an admitted class of lower-quality students can be preferable to one with higher-quality students under two different matchings, if the latter one deteriorates the net balance of the college. The extreme version of this preference is a college being extremely averse towards negative net-balance matchings, regardless of the incoming class. Maintaining a nonnegative net balance is important for a college to continue its membership in the program. In particular, a college with negative net balance might be suspended from the program.

We will incorporate these features as two formal assumptions in this section. Assumption 3 states that a better admitted class is preferable as long as the net balance does not decrease, admission of unacceptable students deteriorates the rankings of matchings regardless of their net balances, and a college deems its own students unacceptable in tuition exchange. Assumption 4 introduces negative net-balance averse preferences. In all results in this section we will use Assumption 3, while Assumption 4 will be used in only one result. We start by stating Assumption 3.

**Assumption 3** For any \( c \in C \) and \( \mu, \nu \in \mathcal{M} \),

1. (Preference increases with a better admitted class and a non-deteriorating balance) if \( b^c \geq b^c \) and \( \mu(c)P^c_\nu(c) \), then \( \mu \succ c \nu \),

2. (Awarding unacceptable students exchange scholarships is not preferable) if there exists \( s \in \nu(c) \setminus \mu(c) \), \( \emptyset P_s s \) and \( \nu(s') = \mu(s') \) for all \( s' \in S \setminus s \), then \( \mu \succ c \nu \), and

3. (Unacceptability of the college’s own students for exchange scholarships) \( \emptyset P_c s \) for all \( s \in S_c \).

Assumption 3 implies that, if there exists \( s \in \mu(c) \) such that \( \emptyset P_s s \), then \( \mu \) is blocked by \( c \). Moreover, if \( sP_c \emptyset \) for all \( s \in \mu(c) \), then \( \mu \) is not blocked by \( c \). Hence, individual

⁴⁹See Pycia and Yenmez (2015) for more discussion of this stability concept under matching problems with externalities.

29
rationality and acceptability are equivalent under Assumption 3. Moreover, Assumption 3 implies that if \( cP_s \mu(s), sP_c \emptyset, \) and \( |\mu(c)| < q_c, \) then \((c, s)\) is a blocking pair for \( \mu. \) Similarly, if \( sP_s', sP_c \emptyset, s' \in \mu(c), \) and \( s \notin \mu(c), \) then \((c, s)\) is a blocking pair for \( \mu. \)

The existence of stable matchings has been widely studied in two-sided matching problems without externalities. For instance, in the college admission market, when the college preferences are responsive, then the set of stable matching is nonempty (cf. Gale and Shapley, 1962; Roth, 1985).\(^{50}\) We prove a similar result for our environment.

**Proposition 3** Under Assumption 3, there exists at least one stable matching in any tuition-exchange market.

We prove this proposition by constructing an associated Gale–Shapley college-admissions market in which the set of Gale–Shapley–stable matchings is identical to the set of stable tuition–exchange matchings.

We will illustrate how decentralized market forces moving toward stability can be at odds with the college’s objective of maintaining a zero net balance. We will also show that colleges always have incentives to decrease their quotas under stable outcomes. Hence, a decentralized market or “stable” centralized mechanisms discourage exchange.

In Section 4, we show the incompatibility between individual rationality, nonwastefulness, and balancedness under Assumption 1. Although Assumption 3 is stronger than Assumption 1, the incompatibility result still holds under Assumption 3.

**Proposition 4** Under Assumption 3, there may not exist an individually rational and nonwasteful matching that is also balanced.

Proposition 4 also shows that there exists no stable and balanced mechanism under Assumption 3. One can then wonder whether there exists a stable mechanism that

\(^{50}\)In the earlier two-sided matching literature, stability a la Gale and Shapley (1962) has been the central solution concept. Technically, our model is similar to a two-sided matching model with externalities, i.e., agents have preferences over allocations rather than their matches. Sasaki and Toda (1996) introduced externalities in two-sided matching markets and various stability definitions. Pycia (2010) explores existence in two-sided matching when agents have preferences over peers and matches. The first model is quite general; however, their stability notion, which guarantees existence, requires a very conservative definition of blocking. The second model, on the other hand, does not cover externalities regarding the balancedness requirement. Pycia and Yenmez (2015) also focus on the existence of stable matching in a two-sided matching problem with externalities such that preferences satisfy a substitutes condition.

However, our model has major differences from standard externality models, which generally inspect peer effects or induce different stability definitions as a solution for the decentralized market. We introduce a new stability notion for the current model.
performs better than all other stable mechanisms in terms of balancedness. We prove otherwise.\footnote{We also inspect the structure of stable matchings, as our stability concept is novel, in Online Appendix F. We show that there always exist college- and student-optimal stable matchings.}

**Proposition 5** Under Assumption 3, each college has the same net balance in all stable matchings of a given market.

We also investigate what kinds of strategic decisions a tuition-exchange office in a college would face in a quota-determination game if a stable outcome emerges in the market. Here we explicitly make the aforementioned additional assumption about negative net-balance aversion on college preferences:\footnote{This assumption is used only in Theorem 10.}

**Assumption 4** *(Negative Net-Balance Aversion)* College \( c \) prefers \( \mu \in \mathcal{M} \), such that \( b_{c}^{\mu} = 0 \) and all \( s \in \mu(c) \) are acceptable, to all \( \nu \in \mathcal{M} \) with \( b_{c}^{\nu} < 0 \).

In the quota-determination game, we fix \( C, S, \succ_{C}, \) and \( \succ \). Colleges are the players of the game and each college’s strategy is setting its admission and eligibility quotas under a simultaneous move, complete information setting. Without loss of generality, we constrain the strategy space such that a reported admission quota is not less than the reported eligibility quota. Given a true quota profile, denote the action set for \( c \) with \( A_{c} \); then, it is \( A_{c} = \{ (\hat{q}_{c}, \hat{e}_{c}) \in \mathbb{N}^{2} | \hat{q}_{c} \geq \hat{e}_{c} \geq 0 \} \). The outcome of the game is determined by a stable mechanism (solution). In Theorem 10, by using the results of Proposition 6 below, we show that in any stable solution, if a college holds a negative net balance, then the best response is only to decrease the eligibility quota. Proposition 6 also gives us a comparative result regarding how the net balances of colleges change when they certify one additional student and do not decrease their admission quotas.\footnote{Weber (1997); Engelbrecht-Wiggans and Kahn (1998); Ausubel, Cramton, Pycia, Rostek, and Weretka (2014) study demand reduction in auctions.}

**Proposition 6** Under Assumption 3, for fixed preferences \( \succ \) and for any reported quota profiles \( \hat{q} \) and \( \hat{e} \), let \( \hat{\pi} \) and \( \tilde{\pi} \) be stable matchings for the induced markets \( [\hat{q}, \hat{e}, \succ] \) and \( [(\hat{q}_{c}, \hat{q}_{-c}), (\hat{e}_{c}, \hat{e}_{-c}), \succ] \), respectively, where \( \hat{q}_{c} \geq \hat{e}_{c}, \hat{q}_{c} \geq \tilde{q}_{c} \) and \( \tilde{e}_{c} = \hat{e}_{c} + 1 \). Then \( b_{c}^{\hat{\pi}} \in \{ b_{c}^{\tilde{\pi}} - 1, b_{c}^{\tilde{\pi}} \} \) if \( b_{c}^{\tilde{\pi}} < 0 \); and \( b_{c}^{\tilde{\pi}} \in \{ b_{c}^{\tilde{\pi}} - 1, b_{c}^{\tilde{\pi}}, ..., b_{c}^{\tilde{\pi}} + \tilde{q}_{c} - \hat{q}_{c} \} \) if \( b_{c}^{\tilde{\pi}} \geq 0 \).

The proposition concludes that, when a college increases its eligibility quota by one without decreasing its admission quota, its overall net balance will decrease at most by one under any stable solution. Its net balance may increase only if it is a nonnegative net balance college to start with.\footnote{This is possible only if \( \tilde{q}_{c} > \hat{q}_{c} \).}
Theorem 10 Under Assumptions 3 and 4, for fixed preferences $\succeq$ and for any reported quota profiles $\hat{q}$ and $\hat{e}$, if $c$ has a negative net balance in a stable matching for market $[\hat{q}, \hat{e}, \succeq]$ where $\hat{q}_c \geq \hat{e}_c$, then its best response in any stable solution is to set only lower $\hat{e}_c$, but not higher; and in particular, there exist $\bar{\hat{e}}_c \leq \hat{e}_c$ such that college $c$ has a zero–balance in every stable matching of the market $[\hat{q}, (\bar{\hat{e}}_c, \hat{e}_{-c}), \succeq]$.

Theorem 10 shows that if $c$ has a negative net balance then it certifies fewer students, which will eventually increase its balance. When $c$ certifies fewer students it may cause another college $c'$ to have a negative net balance. Then $c'$ will have a negative net balance and will certify fewer students, too. In Theorem 11 below, we show this result.

Theorem 11 Under Assumption 3, for fixed preferences $\succeq$ and for any reported quota profiles $\hat{q}$ and $\hat{e}$, if a college $c$ is holding a negative net balance in a stable matching $\mu$ for market $[\hat{q}, \hat{e}, \succeq]$ where $\hat{q}_c \geq \hat{e}_c$, then $b^c_{\mu} \geq b^c_{\mu'}$ where $\mu'$ is any stable matching of market $[(q'_c, \hat{q}_{-c}), (e'_c, \hat{e}_{-c}), \succeq]$, $\hat{q}_c \geq q'_c \geq e'_c$ and $\hat{e}_c > e'_c$.

Theorems 10 and 11 do not conduct an equilibrium analysis in a quota-determination game. But they do point out that in a frictionless market, the colleges that will be likely to have a negative-balance will be conservative and will decrease their eligibility quotas for exports, which will further deteriorate the balances of other colleges.

Typically, no school fully withdraws in practice, as there is often a minimum quota of participation in place. This is instituted most likely because of the reasons outlined above. Given that continued membership is an attractive benefit, often times, smaller colleges will announce that they will import and export at this minimum quota requirement (for example, it is 1 for TTEI), and will continue to be a member of the program without fully withdrawing from the system.

We conclude that under a new design for the tuition exchange market, there should be no room for quota underreporting by the colleges due to negative net-balance aversion, if possible. A fully centralized solution disregarding decentralized market stability seems to be inevitable, as stability is at odds with balancedness and has various other shortcomings regarding other incentives.

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55 This result is in a similar vein as the results on college admissions where the DA mechanism is shown to be prone to admission quota manipulation of the colleges under responsive preferences, regardless of imbalance aversion (cf. Sönmez, 1997). However, Konishi and Ünver (2006) show that the DA mechanism would be immune to quota manipulation, if preferences of colleges over incoming students were responsive and monotonic in number. On the other hand, even under this restriction of preferences over the incoming class, our result would imply all stable mechanisms are manipulable with quota reports for colleges with negative net balances if colleges have negative net-balance aversive preferences. (See also Kojima and Pathak, 2009.)
Moreover, we deem such a stability concept inappropriate for our purpose as the rights of students to participate in market activity depends on the permission of their colleges. Thus, we claim that balanced-efficiency and individual rationality are the most important features of a tuition-exchange outcome.

Appendix B  Proofs

Proof of Propositions 1 and 4. Consider the following market. Let \( C = \{a, b\} \) and each \( c \in C \) set \( q_c = e_c = 1 \). The set of students in each college is: \( S_a = \{1\} \) and \( S_b = \{2\} \). The associated strict preference relations of students over colleges are given as \( P_1 : bP_1c_0 \) and \( P_2 : aP_2c_0 \). Student 1 is not acceptable to \( b \), i.e., \( \emptyset P_b 1 \), and \( b \) prefers any matching in which 1 is not assigned to itself over the ones in which 1 is assigned to itself. Student 2 is acceptable to \( a \) and \( a \) prefers any matching with positive balance to the ones in which no student is assigned to itself. There is one nonwasteful matching that is not individually blocked: \( \mu(1) = c_0 \) and \( \mu(2) = a \). This matching is not balanced, as college \( b \) has negative net balances under \( \mu \).

Proof of Theorem 1. Individual Rationality: Let \( \pi \) be the matching selected by 2S-TTC. Since each \( s \in S \) is assigned to a college better than \( c_0 \), \( s \) does not block \( \pi \). Since all students in \( \pi(c) \) are ranked above \( \emptyset \) in \( P_c \) for each \( c \in C \), \( \pi(c) R^*_c \tilde{S} \) for any \( \tilde{S} \subseteq \pi(c) \). In any matching \( \mu \in M \) such that \( \mu(s) = \pi(s) \) for all \( s \in S \setminus \pi(c) \) and \( \mu(c) \subseteq \pi(c), c \in C \) has a nonpositive net balance. Hence, \( \pi \) is not blocked by \( c \).

Acceptability: Students will be assigned to the null college, \( c_0 \), whenever they point to it, and, hence, they will never need to point to an unacceptable college. Moreover, a student cannot point to a college that considers her unacceptable. Therefore, the students ranked below \( \emptyset \) in \( P_c \) cannot be assigned to \( c \). Thus, the 2S-TTC is acceptable.

Respect for Internal Priorities: Suppose, contrary to the claim, that 2S-TTC does not respect internal priorities. Then there exists \( s \in S_c \) who is assigned to a college in \( [q, e, \succeq] \), but not assigned to a college in \( [\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succeq] \) where \( \tilde{q}_c \geq q_c \) and \( \tilde{e}_c > e_c \). We use a variation of the 2S-TTC in which the students with the highest priority point to a college in each round. Since only the top-priority students and students pointing to \( c_0 \) can form a cycle in each round under both versions of 2S-TTC, they will select the same outcome. Let \( S(k) \) and \( \tilde{S}(k) \) be the set of students assigned in Round \( k \) of 2S-TTC applied to the problems \( [q, e, \succeq] \) and \( [\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succeq] \), respectively. In both problems, the same set of agents will be active in the first round. Since we consider the same preference profile, \( S(1) = \tilde{S}(1) \). Then, if \( s \in S(1) \), we are done. If not, consider the second round. Since the same set of students is removed with their assignments, the set
of active students and the remaining colleges in the second round of 2S-TTC applied to
the problems will be the same. Moreover, students will be pointing to the same colleges
in both problems. Hence, $S(2) = \tilde{S}(2)$. Then, if $s \in S(2)$, we are done. If not, we can
repeat the same steps and show that $s$ will be assigned in the matching selected by the
2S-TTC in market $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succ]$.

Balanced-efficiency: Since the matching selected by the 2S-TTC consists of trading
cycles in which students and their assignments form unique cycles, its outcome is balanced
by Remark 1. Let $\pi$ be the matching selected by 2S-TTC. Let $S(k)$ be the set of students
assigned in Round $k$ of 2S-TTC. We will prove that $\pi$ is balanced-efficient in two parts.

Part I: We first prove that $\pi$ cannot be Pareto dominated by another acceptable
balanced matching. If $s \in S(1)$, then $\pi(s)$ is the highest ranked college in her preference
list that considers her acceptable. That is, no agent $s \in S(1)$ can be assigned to a better
college considering her acceptable. If there exists $\nu \in M$ such that $\nu \succ_s \pi$, then $\nu(s)$
considers $s$ unacceptable. That is, $\pi$ cannot be Pareto dominated by another $\nu$ in which
at least one student in $S(1)$ is better off under $\nu$, and all students are assigned to a college
that considers them acceptable.

If a student $s \in S(2)$ is not assigned to a more preferred $c \in C$ that considers her
acceptable, then $c$ should either fill its quota in Round 1 by another $s' \in S$, or $e_c = 1$ and
the only eligible student $\tilde{s} \in E_c$ is assigned to $c_\emptyset$. For the first case, because $s'$ is assigned
in Round 1, $\pi(s') = c$ is her favorite college among the ones considering her acceptable.
That is, in any $\nu \in M$ in which $s$ is assigned to $\pi(s')$, $s'$ will be made worse off. Hence,$\pi$ cannot be Pareto dominated by another balanced $\nu$ in which at least one student in
$S(1)$ is better off under $\nu$. For the second case, $\tilde{s}$ needs to be assigned to a college in
order for $s$ to be assigned to $c$. Otherwise, balancedness will be violated. This will violate
acceptability, since $\tilde{s}$ considers all colleges considering her acceptable as unacceptable.

We similarly show the same for all other rounds of the 2S-TTC. Thus, no student can
be assigned to a better college without harming any other student among the colleges
that consider her acceptable. Hence, no college can be made better off without harming
another college either, if we focus on matchings that are acceptable.

Part II: Next we show that there does not exist a unacceptable balanced matching that
Pareto dominates $\pi$. To the contrary of the claim, suppose there exists a unacceptable
balanced matching $\nu$ that Pareto dominates $\pi$. Then each $i \in C \cup S$ weakly prefers $\nu$ to
$\pi$, and at least one agent $j \in C \cup S$ strictly prefers $\nu$ to $\pi$. Due to the acceptability of
the 2S-TTC, every student weakly prefers her assignment in $\pi$ to $c_\emptyset$ or to being assigned
to an unacceptable college. Therefore, every assigned student in $\pi$ is also assigned to an
acceptable college under $\nu$. Thus, due to the balancedness of both $\pi$ and $\nu$, $|\nu(c)| \geq |\pi(c)|$
for all $c \in C$. As $\nu$ is unacceptable, there exists some $c_0 \in C$ such that $s_0 \in \nu(c_0)$ is unacceptable for $c_0$. As $\nu \succeq_{c_0} \pi$, there should be at least one student $s_1 \in \nu(c_0) \setminus \pi(c_0)$ such that $s_1$ is acceptable for $c_0$ by Assumption 1. We consider two cases regarding $\pi(s_1)$:

1. First, suppose $\pi(s_1) = c_0$. Denote the home college of $s_1$ by $c_1$. Hence, $|\nu(c_1)| > |\pi(c_1)|$ by balancedness of $\nu$ and $\pi$. By Assumption 1, $\nu(c_1)P_1 \pi(c_1)$, and there exists a student $s_2 \in \nu(c_1) \setminus \pi(c_1)$ such that $s_2$ is acceptable for $c_1$ and $\nu(s_2)P_2 \pi(s_2)$.

2. Next, suppose $\pi(s_1) \in C$. Denote $\pi(s_1)$ by $c_1$. As $|\nu(c_1)| \geq |\pi(c_1)|$, there exists $s_2 \in \nu(c_1) \setminus \pi(c_1)$, and $s_2$ is acceptable for $c_1$ by Assumption 1. We also have $\nu(s_2)P_2 \pi(s_2)$.

We continue with $s_2$ and $\pi(s_2)$, similarly construct $c_2$, and then $s_3$. As we continue, by finiteness, we should encounter the same student $s_k = s_\ell$ for some $k > \ell \geq 1$, that is, we’ve encountered her before in the construction. Consider the students $s_{\ell+1}, s_{\ell+2}, \ldots, s_k$. Let $s_\ell'$ be the student who is assigned in the earliest round of the 2S-TTC in this list. By definition, she points to $\pi(s_\ell')$. However, she prefers $c_\ell' - 1$ to her assignment, and she is acceptable at $c_\ell' - 1$. Moreover, we know that $c_\ell' - 1$ has not been removed yet from the algorithm, because if $c_\ell' - 1$ was constructed in Case 1 above, then $q_{\ell' - 1} > |\pi(c_\ell' - 1)|$ and $s_\ell' - 1 \in S_{\ell' - 1}$ is still not removed, and if $c_\ell' - 1$ was constructed in Case 2 above, then $s_\ell' - 1 \in \pi(c_\ell' - 1)$ is still not removed. Therefore, $s_\ell'$ should have pointed to $c_\ell' - 1$ in the 2S-TTC in that round. This is a contradiction to $\nu$ Pareto dominating $\pi$.

**Proof of Theorem 2.** Suppose that there does exist such a mechanism. Denote it by $\psi$. We use different examples to show our result.

**Case 1:** Let $C = \{a, b, c\}$ and $S_a = \{1, 2\}$, $S_b = \{3\}$, and $S_c = \{4\}$. Let $q = e = (2, 1, 1)$. Let $\succeq_S$ be the student preference profile with associated rankings over $C \cup c_0$ $P_1 : bP_1cP_1c_0$, $P_2 : cP_2c_0$, $P_3 : aP_3c_0$, and $P_4 : aP_4c_0$. Let $\succeq_C$ be the college preference profile with associated rankings over students $P_a : 3P_a4P_a\emptyset$, $P_b : 1P_b\emptyset$, and $P_c : 1P_c2P_c\emptyset$. We assume that $\succeq_C$ satisfies Assumption 1. There are two balanced–efficient and individually rational matchings: $\mu_1 = \begin{pmatrix} a & b & c \\ 4 & \emptyset & 1 \end{pmatrix}$ and $\mu_2 = \begin{pmatrix} a & b & c \\ \{3, 4\} & 1 & 2 \end{pmatrix}$.

If $\psi$ selects $\mu_1$, then $a$ can manipulate $\psi$ by submitting $\succeq_a^1$ where $P_a^1 : 3P_a^1\emptyset$ and any matching $\pi$ such that $4 \notin \pi(a)$ is preferred to $\mu_1$. Then the only individually rational and balanced–efficient matching is $\mu_3 = \begin{pmatrix} a & b & c \\ 3 & 1 & \emptyset \end{pmatrix}$. Therefore, $\psi[q, e, \succeq] = \mu_3$.

**Case 2:** We consider the same example with a slight change in $a$’s preferences. Let $\succeq_a^2$ be $a$’s preferences over the matchings with associated ranking $P_a^2 : 4P_a^23P_a^2\emptyset$. In this case, $\mu_1$ and $\mu_2$ are the only two balanced–efficient and individually rational matchings.

If $\psi$ selects $\mu_1$, then $a$ can manipulate $\psi$ by submitting $\succeq_a$. Then we will be in Case 1 and $\mu_2$ will be selected, which makes $a$ better off. Therefore, $\psi[q, e, (\succeq_a^2, \succeq_a)] = \mu_2$.

**Case 3:** Now consider the case where colleges report the following preferences $\succeq_3^3$.
where $\succeq_a^3 = \succeq_a^2 = \succeq_b$, $P_c^2 : 1P_3^3 \emptyset$ is the associated ranking with $\succeq_c^3$ and any matching $\pi$ such that $2 \notin \pi(c)$ is preferred to $\mu_2$ under $\succeq_c^3$. Then there are two individually rational and balanced–efficient matchings: $\mu_4 = \left( \begin{array}{ccc} a & b & c \\ 4 & 0 & 1 \end{array} \right)$ and $\mu_5 = \left( \begin{array}{ccc} a & b & c \\ 3 & 1 & \emptyset \end{array} \right)$.

If $\psi$ selects $\mu_4$, then in Case 2 $c$ can manipulate $\psi$ by reporting $\succeq_c^3$. Therefore, $\psi[q, e, \succeq^3] = \mu_5$.

**Case 4:** Now consider the case where colleges report the following preferences $\succeq^4$ where $\succeq_b^4 = \succeq_b$, $\succeq_a^4 = \succeq_c^3$, $P_a^1 : 4P_a^4 \emptyset$ is the associated ranking with $\succeq_a^4$ and any matching $\pi$ such that $3 \notin \pi(a)$ is preferred to $\mu_5$ under $\succeq_a^4$. There is a unique balanced–efficient and individually rational matching: $\mu_4$. In Case 3, $a$ can manipulate $\psi$ by reporting $\succeq_a^4$; then we will be in Case 4 and $a$ will be better off with respect to Case 3 preferences.

Therefore, there does not exist a balanced–efficient, individually rational mechanism that is immune to preference manipulation by colleges. ■

**Proof of Theorem 3.** Consider the preference relations of each student who ranks as acceptable only those colleges that find her acceptable. If we consider only these preferences as possible preferences to choose from for each student, we see that the 2S-TTC is group strategy-proof for students, as Pápai (cf. 2000) showed that the TTC is group strategy-proof. In the 2S-TTC, observe that students are indifferent among reporting preference relations that rank the colleges finding themselves as acceptable in the same relative order. Thus, the 2S-TTC is group strategy-proof for students. ■

The following Lemma is used in proving Theorem 4:

**Lemma 1** Let $\pi$ and $\tilde{\pi}$ be the outcome of 2S-TTC in $[q, e, \succeq]$ and $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succeq]$ where $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq e_c$ for some $c \in C$. Then, $M_c^{\tilde{\pi}} \subseteq M_c^\pi$ and $X_c^{\tilde{\pi}} \subseteq X_c^\pi$.

**Proof.** We have three cases to consider:

**Case 1:** $\tilde{q}_c = q_c$ and $\tilde{e}_c < e_c$. We consider the case in which one more student is certified by $c$, i.e., $\tilde{e}_c + 1 = e_c$. Denote the student added to the eligible set by $s$. Let $s' \in S_c$ and $r_c(s') = r_c(s) - 1$. Consider the execution of the 2S-TTC for this new market. If $c$ imports $\tilde{q}_c$ students before $s'$’s turn, then $c$ will be removed, and certifying one more student will not affect the set of students exported and imported by $c$. Now consider the case in which $c$ imports less than $\tilde{q}_c$ before $s'$’s turn. Denote the intermediate matching that we have just after $s'$ is processed by $\nu$. Since $c$ is removed just after $s'$ is processed in $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), \succeq]$, $M_c^{\tilde{\pi}} = M_c^\nu$ and $X_c^{\tilde{\pi}} = X_c^\nu$. If $s$ is assigned to a college $c' \in C \setminus c$, $c$ will import one more acceptable student. Denote that matching by $\mu$. Then, we have $M_c^{\tilde{\pi}} = M_c^\nu \subseteq M_c^\mu$ and $X_c^{\tilde{\pi}} = X_c^\nu \subseteq X_c^\mu$. If $s$ is assigned to the $c_0$ or $c$, then $c$ will have the same import and export set. If we keep certifying all $e_c - \tilde{e}_c$ students one at a time, we will have $M_c^{\tilde{\pi}} \subseteq M_c^\pi$ and $X_c^{\tilde{\pi}} \subseteq X_c^\pi$, where $\pi$ is the outcome of the 2S-TTC in $[q, e, \succeq]$. 

36
Case 2: \( \tilde{q}_c < q_c \) and \( \tilde{e}_c = e_c \). Let \( \pi \) and \( \nu \) be the outcomes of the 2S-TTC in \( [q, e, \succ] \) and \( [(\tilde{q}_c, q_{-c}), e, \succ] \), respectively. If \( |M^\nu_c| < \tilde{q}_c \) then the 2S-TTC will select \( \nu \) when \( c \) reports either \( \tilde{q}_c \) or \( q_c \). Therefore, \( M^\pi_c = M^{\nu}_c \) and \( X^\pi_c = X^\nu_c \). If \( |M^\nu_c| = \tilde{q}_c \) and if all eligible students of \( c \) are considered in \( [(\tilde{q}_c, q_{-c}), e, \succ] \) when it is removed, then it will not make a difference if \( c \) reports either \( \tilde{q}_c \) or \( q_c \). If \( |M^\nu_c| = \tilde{q}_c \) and if \( c \) is removed before all its eligible students are considered, then one more student \( s \in S_c \) might be considered when \( c \) reports \( q_c \). As in the previous case, \( c \) may import and export at least one more student. At the end, we get \( M^\nu_c \subseteq M^\pi_c \) and \( X^\nu_c \subseteq X^\pi_c \) if some of the students who are considered only when \( c \) reports \( q_c \) are assigned. Otherwise, \( M^\pi_c = M^\nu_c \) and \( X^\pi_c = X^\nu_c \).

Case 3: \( \tilde{q}_c < q_c \) and \( \tilde{e}_c < e_c \). Let \( \mu \) be the outcome of the 2S-TTC in \( [q, (\tilde{e}_c, e_{-c}), \succ] \). Then, we have \( M^\pi_c \subseteq M^\nu_c \subseteq M^\mu_c \) and \( X^\pi_c \subseteq X^\nu_c \subseteq X^\mu_c \), where the first and second subset relations come from invoking Case 1 and Case 2, respectively. ■

Proof of Theorem 4. We prove a stronger version of Theorem 4: Under the 2S-TTC, suppose that preference profiles are fixed for colleges such that no college reports an unacceptable student as acceptable in its preference report. In the induced quota-reporting game, under Assumption 1, it is a weakly dominant strategy for any \( c \in C \) to certify all its students and to reveal its true admission quota.

Take a market \( [q, e, \succ] \) and a college \( c \). Suppose that preference reports are fixed such that \( c \) does not report any unacceptable students as acceptable in these reports. We have two cases to consider for possible quota manipulations by \( c \):

Case 1: \( c \) reports \( \tilde{q}_c \leq q_c \) and \( \tilde{e}_c \leq |S_c| \): In Lemma 1 we have shown that when \( c \) reports its admission and eligibility quotas as higher, the set of students imported by \( c \) (weakly) expands. By Assumption 1, reporting \( \tilde{q}_c \leq q_c \) and \( \tilde{e}_c \leq |S_c| \) is weakly dominated by reporting the true admission quota and certifying all students.

Case 2: \( c \) reports \( \tilde{q}_c > q_c \): This strategy is weakly dominated by reporting its true admission quota \( q_c \). We prove this as follows: Let \( \nu \) and \( \mu \) be the matchings that the 2S-TTC mechanism selects when \( c \) reports \( \tilde{q}_c \) and \( q_c \), respectively. If \( |M^\nu_c| \leq q_c \), then \( M^\nu_c = M^\mu_c \) and \( X^\nu_c = X^\mu_c \); thus, it is indifferent between the two matchings. However, if \( |M^\nu_c| > q_c \), among the balanced matchings by Assumption 1, colleges’ preferences depend on the preferences on admitted students, which are only responsive up to the true admission quota, and \( \mu \) is individually rational, so it prefers \( \mu \) to \( \nu \). ■

Proof of Proposition 2. The 2S-TTC mechanism takes into account only the set of acceptable students based on the submitted preferences of colleges over the matchings. Hence, for any two different preference profiles with the same set of acceptable students, the 2S-TTC selects the same outcome. ■

Proof of Theorem 5. We use a variant of the 2S-TTC in which we select only one
cycle in one round. If there is more than one cycle, then the selection is done randomly. Let \( S(k) \) be the set of students removed in Round \( k \). Suppose the theorem does not hold. Let \( \psi \) be the mechanism satisfying all four axioms, and select a different matching for \([q, e, \succsim] \). Denote the outcome of 2S-TTC for \([q, e, \succsim] \) by \( \mu \). In the rest of the proof, we work on students’ preferences over colleges, \( P_s \), instead of matchings, \( \succsim_s \).

We first prove the following claim:

**Claim:** If there exists a student in \( S(k) \) who prefers her assignment in \( \psi[q, e, \succsim] \) to the one in \( \mu \), then there exists another student in \( \bigcup_{k'=1}^{k-1} S(k) \) who prefers her assignment in \( \mu \) to the one in \( \psi[q, e, \succsim] \). \(^{56}\)

**Proof of Claim:** We use induction in our proof. Consider the students in \( S(1) \). If \( S(1) \) is a singleton, then the student in \( S(1) \) is assigned to \( c_0 \). Any college that she prefers to \( c_0 \) considers her unacceptable. If she prefers her assignment under \( \psi \) to \( c_0 \), then she is assigned to a college that considers her unacceptable by \( \psi \). Therefore, \( \psi \) is not acceptable. If she prefers \( c_0 \) to her assignment under \( \psi \), then \( \psi \) is not acceptable. Then any acceptable mechanism will assign her to \( c_0 \). If \( S(1) \) is not a singleton, then all students in \( S(1) \) are assigned to the best colleges that consider them as acceptable, and they prefer their assignment in \( \mu \) to \( c_0 \). If \( s \in S(1) \) prefers her assignment in \( \psi[q, e, \succsim] \) to \( \mu(s) \), then \( \psi \) is not acceptable. Hence, all students in \( S(1) \) weakly prefer their assignment in \( \mu \).

In the inductive step, assume that for all Rounds 1, ..., \( k - 1 \), for some \( k > 1 \), the claim is correct. Consider Round \( k \). If there exists a student \( s \in S(k) \) such that \( c = \psi[q, e, \succsim](s)P_s\mu(s) \), then either \( c \) considers \( s \) acceptable and its seats are filled in Rounds 1, ..., \( k - 1 \) of 2S-TTC, or \( s \) is unacceptable for \( \mu \). In the latter case, \( \psi \) is not acceptable. Consider the former case. Let \( s' \) be assigned to \( c \) under \( \mu \) in Round \( k' \leq k - 1 \) but not under \( \psi[q, e, \succsim] \), as \( s \) is assigned instead of her. If she prefers \( c \) to \( \psi[q, e, \succsim](s') \), then we are done. If she does not, \( k' > 1 \), and by the inductive step, there exists a student \( s'' \in S(k'') \) for some \( k'' < k' \leq k - 1 \) who prefers \( \mu(s'') \) to \( \psi[q, e, \succsim](s'') \).

Now we are ready to prove the theorem. By the Claim and the observation above, as \( \mu \neq \psi[q, e, \succsim] \), there exists a student \( s \) and some round \( k \) such that \( s \in S(k) \) prefers \( \mu(s) \) to \( \psi[q, e, \succsim](s) \), and \( \mu(s') = \psi[q, e, \succsim](s') \) for all \( s' \in \bigcup_{k'=1}^{k-1} S(k') \).

We will construct our proof in three steps. Assign to each round of the 2S-TTC mechanism a counter and set it as \( \text{Counter}(k') = |S(k')| - 1 \) for all rounds \( k' \).

**Step 1:** Construct a preference profile \( \tilde{\succsim} \) with associated ranking \( \tilde{P} \) as follows: Let student \( s \in S_c \) rank only \( \mu(s) \) as acceptable in \( \tilde{P}_s \) and \( \tilde{\succsim}_{-s} = \succsim_{-s} \). The 2S-TTC will select \( \mu \) for \([q, e, \tilde{\succsim}] \). Since \( \psi \) is student strategy-proof and acceptable, \( \psi[q, e, \tilde{\succsim}](s) = c_0 \). \(^{56}\)

\(^{56}\)We take \( \bigcup_{k'=1}^{0} S(k') = \emptyset \).
Then, we check whether the assignments of students in $\cup_{k'=1}^{k-1} S(k')$ are the same in $\psi[q,e,\tilde{\nu}]$ and $\mu$. If not, then for some $\tilde{k} < k$, there exists a student $\tilde{s} \in S(\tilde{k})$ preferring $\mu(\tilde{s})$ to $\psi[q,e,\tilde{\nu}](\tilde{s})$ and each student in $\cup_{k'=1}^{k-1} S(k')$ gets the same college in $\mu$ and $\psi[q,e,\tilde{\nu}]$. Then we repeat Step 1 by taking $\zeta_0 := \zeta_n$, $s := \tilde{s}$, and $k := \tilde{k}$.

This repetition will end by the finiteness of rounds and the fact that $\cup_{k'=1}^{k-1} S(k') = \emptyset$. When all students in $\cup_{k'=1}^{k-1} S(k')$ get the same college in $\mu$ and $\psi[q,e,\tilde{\nu}]$, then we proceed to Step 2.

**Step 2:** In Step 1, we have shown that $s$ prefers $\mu(s)$ to $\psi[q,e,\tilde{\nu}](s) = \mu_0$. Suppose $c$ is the home college of $s$. Set a new eligibility quota $\tilde{e}_c$ equal to the rank of student $s$ in $c$'s internal priority order, that is, $\tilde{e}_c = r_c(s)$, and let $\tilde{e}_c = e_c$. In $[q,e,\tilde{\nu}]$, the 2S-TTC assigns all students in $\cup_{k'=1}^{k-1} S(k')$ to the same college as in $\mu$. $\psi[q,e,\tilde{\nu}](s) = \mu_0$ since $\psi$ respects internal priorities and we weakly decreased $c$'s eligibility quotas. We check whether the assignments of students in $\cup_{k'=1}^{k-1} S(k')$ are the same in both $\psi[q,e,\tilde{\nu}]$ and $\mu$. If not, then by the Claim, there should exist $\tilde{s} \in S(\tilde{k})$ preferring $\mu(\tilde{s})$ to $\psi[q,e,\tilde{\nu}](\tilde{s})$, and each student in $\cup_{k'=1}^{k-1} S(k')$ gets the same college in $\mu$ and $\psi[q,e,\tilde{\nu}]$ where $\tilde{k} < k$; then we restart from Step 1 by taking $\zeta_0 := \zeta_n$, $s := \tilde{s}$, $k := \tilde{k}$, and $e := \tilde{e}$.

Eventually, by the finiteness of the rounds of the 2S-TTC and as we reduce the round $k$ in each iteration, we reach the point in our proof such that students in $\cup_{k'=1}^{k-1} S(k')$ get the same college in $\mu$ and $\psi[q,e,\tilde{\nu}]$.

Observe that $s$ is the last remaining eligible student of $c$ in Round $k$ of 2S-TTC for $[q,e,\tilde{\nu}]$ by the choice of $\tilde{e}_c = r_c(s)$. Since for all $s'' \in \cup_{k'=1}^{k-1} S(k')$, $\mu(s'') = \psi[q,e,\tilde{\nu}](s'')$ and $\mu(s) \tilde{P}_s \psi[q,e,\tilde{\nu}](s) = \mu_0$, $s' \in S(k) \cap \mu(c)$ will be assigned to a different college in $\psi[q,e,\tilde{\nu}]$ than $c$. Otherwise, $\psi$ is not balanced. As for all $s'' \in \cup_{k'=1}^{k-1} S(k')$, $\mu(s'') = \psi[q,e,\tilde{\nu}](s'')$, and $s'$ points to the best available college that finds her acceptable in Round $k$, $c = \mu(s') \tilde{P}_s \psi[q,e,\tilde{\nu}](s')$. We decrease Counter($k$) by 1. If Counter($k$) > 0 then we turn back to Step 1 by taking $\zeta_0 := \zeta_n$ and $s := s'$; otherwise we continue with Step 3. Note that eventually we will find a Step $K$ such that Counter($K$) = 0, because we weakly decrease all counters and decrease one counter by 1 in each iteration of Step 2.

**Step 3:** By the construction above, each $\tilde{s} \in S(k)$ ranks only $\mu(\tilde{s})$ as acceptable in $\tilde{P}_s$ and she is the last certified student by her home college in $[q,e,\tilde{\nu}]$. In Step 2, we showed that there exist at least 2 students $s_1 (=s$ in Step 2) and $s_2 (=s$ in Step 2) in $S(k)$ who are not assigned to $\mu(s_1)$ and $\mu(s_2) = c_1 (= c$ in Step 2), respectively, in $\psi[q,e,\tilde{\nu}]$, where $c_1$ is the home college of $s_1$. Then, they are assigned to $c_0$ in $\psi[q,e,\tilde{\nu}]$, by the individual rationality of $\psi$. Recall that in the 2S-TTC for $[q,e,\tilde{\nu}]$, each student certified by the home colleges of $s_1$ and $s_2$ — colleges $c_1$ and $c_2$, respectively — other than $s_1$ and $s_2$ is removed in a round earlier than $k$. Suppose for $s_3 \in S(k)$, $\mu(s_3) = c_2$. Since
\(\psi[q, e, \tilde{c}](s_2) = c_0\), for all \(s \in S(k)\), \(\psi[q, e, \tilde{c}](s) = \mu(s)\) (by Step 2), and \(s_3\) is balanced, \(s_3\) cannot be assigned to \(c_2\) in \(\psi[q, e, \tilde{c}]\), and hence, \(\psi[q, e, \tilde{c}](s_3) = c_0\). We continue similarly with \(s_3\) and the home college of \(s_3\), say college \(c_3\), eventually showing that for all \(s \in S(k)\), \(\psi[q, e, \tilde{c}](s) = c_0\). Recall that students in \(S(k)\) formed a trading cycle in which each agent in the cycle was assigned in \(\mu\) the home college of the next student in the cycle. Thus, \(\psi[q, e, \tilde{c}]\) is Pareto dominated by the balanced matching \(\nu\) obtained as \(\nu(s) = \psi[q, e, \tilde{c}](s)\) for all \(s \in S(k)\) and \(\nu(s) = \mu(s)\) for all \(s \in S(k)\); that is, \(\nu\) is obtained from \(\psi[q, e, \tilde{c}]\) by students in \(S(k)\) trading their assignments with each other to get their assignments in \(\mu\). This is because each college in the cycle of Round \(k\) gets one acceptable student more and each student weakly prefers \(\mu\) to \(\psi[q, e, \tilde{c}]\). This contradicts the balanced-efficiency of \(\psi\). Hence, \(\psi[q, e, \tilde{c}] = \mu\), i.e., \(\psi\) is equivalent to 2S-TTC. ■

**Proof of Theorem 6.** Let \(\psi\) satisfy all conditions and be strategy-proof for students. Then, consider the following example. There are 3 colleges \(C = \{a, b, c\}\) with \(q = e = (2, 1, 1)\). Let \(S_a = \{1, 2\}, S_b = \{3\}, S_c = \{4\}\), and each student be acceptable to each college and the college preference profile satisfy Assumption 1. The internal priorities and student preference profiles are given as: \(1 \succ_a 2, bP_1c_3, cP_2\emptyset, aP_3\emptyset, bP_4aP_4c_3\).

The 2S-TTC selects \(\mu = \begin{pmatrix} a & b & c \\ \{3, 4\} & 1 & 2 \end{pmatrix}\). \(\psi\) will also select \(\mu\), since any other matching in which all students are assigned is individually irrational (and unacceptable).

If student 4 reports \(\succ_4'\) with associated ranking \(P_4' : bP_4'c_3P_4'a\) then 2S-TTC will select \(\mu' = \begin{pmatrix} a & b & c \\ 3 & 1 & \emptyset \end{pmatrix}\). The only balanced and individually rational (acceptable) matching in which more than two students are assigned is \(\mu'' = \begin{pmatrix} a & b & c \\ 3 & 4 & 2 \end{pmatrix}\). Therefore, the outcome of \(\psi\) when 4 reports \(\succ_4'\) is \(\mu''\). Hence, 4 can manipulate \(\psi\). ■

**Proof of Theorem 7.** We first prove the strategy-proofness of 2S-TTTC for students. Consider a tuition exchange problem \([q, e, \tilde{c}]\) and tolerance profile \(t = (\ell_c, u_c)_{c \in C}\). We use a variation of 2S-TTTC in which only the student with the highest internal priority at some college points to a college in each round. Let \(\mu\) be the matching selected by 2S-TTTC under truth-telling. Let \(k > 0\) be the first round that we cannot locate a cycle. Student \(s\) assigned in Round \(k' < k\) (under truth-telling) cannot affect the assignments done in earlier rounds. Before Round \(k'\), all the colleges considering \(s\) acceptable that \(s\) prefers to \(\mu(s)\) should have been removed since each student points to the most preferred college among the remaining ones. If \(s\) forms a cycle by misreporting in Round \(k'' < k'\), then she should have pointed to a worse college than \(\mu(s)\). Therefore, student \(s\) cannot get a better college by misreporting.
Now consider Round $k$. First assume that we have a chain respecting tolerance interval. As we mentioned above, any active student in Round $k$ cannot affect the assignments done in earlier rounds. Then consider the student pointed to by the tail college of the chain. This student will be assigned in this round no matter which college she points to. Therefore she will point to the most preferred college considering her acceptable among the remaining ones. The next students in the chain will do so as well. The other active students in this round cannot affect the assignment of students in the chain without hurting themselves.

Now consider the case where we don’t have a chain respecting tolerance interval. That is, each exporting college $c$ has a balance of $\ell_c$. Then we will remove all the non-exporting colleges and 2S-TTTC reduces to the 2S-TTC mechanism. It is easy to see that we will not have chains in the future rounds also.

Let $k'' > k$ be the first round after Round $k$ that we cannot locate a cycle. Then we can apply the same reasoning that we used for rounds before $k$ and show that all agents assigned in Round $\tilde{k}$ such that $k < \tilde{k} < k''$ cannot be better off by misreporting. This is also true for Round $k''$.

Next we prove 2S-TTTC’s outcome cannot be Pareto dominated by an acceptable matching satisfying tolerable balance. Denote the outcome of the 2S-TTTC mechanism with $\mu$. Let $k > 1$ be the first round that we cannot locate a cycle. We consider the variant that we described above. In the first round, each student is pointing to her most preferred college among the ones considering her acceptable. If a student is assigned in this round, then she should get the same college under $\nu$. Now consider students assigned in Round $k' < k$. All the colleges that a student preferred to her assignment that consider her acceptable should have been removed in an earlier round, and we cannot make that student better off without making some students assigned in an earlier round worse off. If there does not exist a chain respecting tolerance interval, then 2S-TTTC reduces to the 2S-TTC mechanism. If college $c$ is removed because of either $b_\ell^c = u_c$ or $b_\ell^c = \ell_c$, then we cannot assign more students to $c$ in $\nu$.

Let $k'' > k$ be the first round after Round $k$ that we cannot locate a cycle. Then we can apply the same reasoning that we have used for rounds before $k$ and show that all agents assigned in Round $\tilde{k}$ such that $k < \tilde{k} < k''$ cannot be better off by misreporting.
This is also true for Round $k$".

Every student assigned to a college should be assigned the same college in $\nu$. If we assign fewer students, then at least one student would be better off under $\mu$ compared to $\nu$. If we assign more students in $\nu$ then either feasibility or the tolerance conditions are violated.

Proof of Theorem 8. We refer to the proof of Theorem 1.

Proof of Theorem 9. Recall that the 2S-TTC mechanism in the general domain can be run by using the set of colleges’ acceptable students, and that while proving the properties of 2S-TTC we consider only these sets of acceptable students. Hence, we can use the same proofs for the 2S-TTC in the general domain here. We refer to the proof of Theorem 1 and Theorem 3 for balance–efficiency, acceptability, individual rationality, strategy-proofness for students, and respecting internal priorities. We refer to the proof of Theorem 5 for uniqueness.

Immunity to Preference Manipulation by Colleges: Under Assumption 2, firms are indifferent between any balanced and acceptable matching that fills their quota. Since the 2S-TTC mechanism selects a balanced and acceptable matching that fills all firms’ quotas, firms cannot be better off by manipulating their preferences over the matchings and reporting quotas different from their true quotas.

Stability: Denote the outcome of the 2S-TTC by $\mu$. Recall that $q_c = |S_c|$ for all $c \in C$, all workers consider their current jobs acceptable, all firms consider their current workers acceptable, and workers who are not certified remain at their current jobs. Hence, $|\mu(c)| = q_c$ for all $c \in C$. Since $\mu$ fills all firms’ quotas, $\mu$ is nonwasteful. Since all employees in $\mu(c)$ are acceptable, replacing one of the employees in $\mu(c)$ with another one in $S \setminus \mu(c)$ cannot make $c$ better off. Hence, $\mu$ cannot be blocked by a worker-firm pair. Moreover, $\mu$ is individually rational.

References


Supplementary Appendices

Appendix C  Tuition-Exchange Programs

The Tuition Exchange Inc (TTEI): In addition to information provided in the Section 2, here we give more detail. In TTEI, every participating institution determines the number of outgoing students it can certify, as well as how many TTEI awards it will grant to incoming students each year. Each college determines its export and import quotas. Then each faculty member submits the TTEI application to the registration office of their college. If the number of applicants is greater than the number of outgoing students that the college is willing to certify, then the college decides whom to certify based on years of service or some other criterion (internal priority order).

Each student who is certified eligible submits a list of colleges to the liaison office of her home institution. Each liaison office sends a copy of the TTEI “Certificate of Eligibility” to the TTEI liaison officer at the participating colleges and universities listed by the eligible dependents. Certification only means that the student is eligible for a TTEI award; it is not a guarantee of an award. The eligible student must apply for admission to the college(s) in which she is interested, following each institution’s application procedures and deadlines. After admission decisions have been made, the admissions offices or TTEI liaisons at her listed institutions inform her whether she will be offered a TTEI award. TTEI scholarships are competitive, and some eligible applicants may not receive them. That is, the sponsoring institution cannot guarantee that an “export” candidate, regardless of qualifications, will receive a TTEI scholarship. Institutions choose their scholarship recipients (“imports”) based on the applicants’ academic profiles.

The Council of Independent Colleges Tuition Exchange Program (CIC-TEP): CIC-TEP is composed of almost 400 colleges. Most of the member colleges are from the Midwest or on the East Coast. All full-time employees of the member colleges and their dependents can benefit from this program. Each college certifies its own employees eligible based on its own rules. Each member college is required to accept at least three exchange students per year. There is no limitation on the number of exported students. Each certified student also applies for admission directly to the member colleges of her choice. Certified students must be admitted by the host college in order to be considered for the tuition exchange scholarship.

Catholic College Cooperative Tuition Exchange (CCCTE): CCCTE is composed of 70 member colleges. Each member college certifies its employees as eligible based on its own rules. Dependents must be admitted by the host college before applying for
the tuition exchange scholarship. Admission to the host college does not guarantee the scholarship. Each member college can have at most five more import students than its exports. There is no limitation on the number of students it exports in a given academic year.

**Great Lakes Colleges’ Association (GLCA):** GLCA is composed of thirteen liberal arts colleges in Pennsylvania, Michigan, Ohio, and Indiana. Each member college determines the eligibility of its employees based on its own rules. All other policies are determined by the host colleges. Each accepted student pays a fee equal to 15% of the GLCA mean tuition. The remaining tuition is paid by the home college.

**Associated Colleges of the Midwest (ACM):** ACM is composed of fourteen liberal arts colleges in Wisconsin, Minnesota, Iowa, Illinois, and Colorado. Eligibility is determined based on the home college rules. If a certified student is eligible for the host college’s tuition remission program, then she is considered eligible for the tuition exchange scholarship. Each host college compensates 50% tuition to all imported students. The remaining portion of the tuition is paid by the home college and the student. The share of the student cannot be more than 20% of tuition.

**Faculty and Staff Children Exchange Program (FACHEX):** FACHEX is composed of 28 Jesuit colleges. Each student first applies to be admitted by the host college. Admission to the host college does not guarantee receiving tuition exchange scholarship. For each college, the number of import cannot exceed the number of its export by three.

**Council for Christian Colleges and Universities Tuition-Waiver Exchange Program (CCCU-TWEP):** CCCU-TWEP is composed of 100 colleges. Each member college must accept at least one exchange student. In order to receive tuition exchange scholarship, each student needs to be admitted by the host college.

### Appendix D Temporary Worker-Exchange Programs

**D.1 Teacher Exchange**

The **Fulbright Teacher Exchange Program**, established by an act of the US Congress in 1946, provides opportunities to school teachers in US to participate in a direct exchange of positions with teachers from countries, including the Czech Republic, France, Hungary, India, Mexico, and the United Kingdom. Matching procedure is arranged by the Fulbright program staff, and each candidate and each school must be approved before the matchings are finalized.

The **Commonwealth Teacher Exchange Programme (CTEP)** was founded by the
League for the Exchange of Commonwealth Teachers more than 100 years ago. Participant teachers exchange their jobs and homes with each other usually for a year, and they stay employed by their own school. Countries participating to this program are Australia, Canada, and the UK, India, Jamaica, Malawi, New Zealand, and South Africa, and the United States. More than 40,000 teachers have benefited from the CTEP. Principals have the right to veto any proposed exchange they think will not be appropriate for their school.

The Educator Exchange Program is organized by the Canadian Education Exchange Foundation. The program includes reciprocal interprovincial and international exchanges. Destinations for international teacher exchanges are Australia, Denmark, France, Germany, Switzerland, the UK, and Colorado, US. Matches are determined based on the preferences, family needs, accommodations offered, and accommodations needed.

The Manitoba Teacher Exchange enables teachers in Manitoba to exchange their positions with teachers in Australia, the UK, the US, Germany, and other Canadian provinces. Once a potential match is found, the incoming teacher’s information is sent to the Manitoba applicant, the principal of the school, and the employing authority. Acceptance of all these agents is required for the completion of the exchange.

In the Saskatchewan Teacher Exchange, public school teachers with at least five years of experience can apply for exchange positions with teachers in the UK, the US, and Germany. Potential exchange candidates are determined based on similar teaching assignments and personal and professional interests and they are sent to applicant’s director of education. If the potential exchange candidates are considered acceptable by the the applicant’s director and principals, then the applicant will consider the candidate. The exchange is finalized once the applicant accepts it.

The Northern Territory Teacher Exchange Program is a reciprocal program in which teachers in Northern Territory exchange positions with teachers from the UK, Canada, the US, New Zealand, and the Australian States of New South Wales, Queensland, South Australia, and Western Australia. When a potential match is found for an applicant, the applicant and her school principal decide whether to accept or reject the proposal. The match is finalized when both sides accept it.

The Western Australian Teacher Exchange is a reciprocal program. The exchanges can be done with teachers from school systems in different states, territories, or countries. The match is finalized after the approval of the principals of both sides.

The Rural Teacher Exchange (TRTE) is a reciprocal program which gives opportunity to teachers in more than 800 rural and remote schools in New South Wales to exchange their positions. Exchanges are selected via centralized mechanism which considers
submitted preference list. However, if a teacher can find a possible exchange counterpart, then they can exchange their positions before entering the central mechanism.

D.2 Clinical Exchange

In the International Clinical Exchange Program, medical students exchange positions with other medical students from other countries. The program is run by the International Federation of Medical Students’ Association. Every year, approximately 10,000 students exchange their positions. The exchanges are done bilaterally: for every student who goes to a certain country, one student from that country can come to the student’s home country. In a county, the exact number of available positions available for another country is determined by the number of contracts signed between both countries. The MICEFA Medical Program enables medical students in France and US to exchange their positions for one to two months. Students are exchanged on a one-to-one basis and each exchange student pays tuition to her home institute. Faculty members can also benefit from this exchange program. Each faculty member receives salary from the host institute.

D.3 Student Exchange

The National Student Exchange (NSE), established in 1968, is composed of nearly 200 colleges from the United States, Canada, Guam, Puerto Rico, and the US Virgin Islands. More than 105,000 undergraduate students have exchanged their colleges through NSE. Exchange students pay either the in-state tuition of their host institution or the normal tuition of their home campus.

The University of California Reciprocal Exchange program (UCREP) enables the students of the University of California system to study in more than 120 universities from 33 countries. Around 4,000 students benefit from this program annually. Exchange students are selected by the home university. This is a reciprocal exchange program and it aims to balance the costs and benefits of import and export students for each university.

The University Mobility in Asia and the Pacific Exchange Program (UMAPEP), established in 1993, is a student exchange program between 500 universities in 34 Asia-Pacific countries. UMAPEP involves two programs: a bilateral exchange program and a multilateral exchange program. In the bilateral exchange program, home colleges select the exchange students and exchanges are done through bilateral agreements signed between the member colleges. In the multilateral exchange program, host universities select the incoming exchange students.
The International Student Exchange (ISE), founded in 1979, is a reciprocal program. Around 40,000 students from 45 countries have benefited from ISE. Each exchange student pays tuition to her home college. Reciprocity is based on the number of incoming and outgoing students.

The Erasmus Student Exchange Program is a leading exchange program between the universities in Europe. Close to 3 million students have participated since it started in 1987. The number of students benefiting from the program is increasing each year; in 2011, more than 231,408 students attended a college in another member country as an exchange student. The number of member colleges is more than 4,000. Each college needs to sign bilateral agreements with the other member institutions. In particular, the student exchanges are done between the member universities that have signed a bilateral contract with each other. The bilateral agreement includes information about the number of students who will be exchanged between the two universities in a given period. The selection process of the exchange students is mostly done as follows. The maximum number of students that can be exported to a partner university is determined based on the bilateral agreement with that partner and the number of students who have been exported since the agreement was signed. The students submit their list of preferences over the partner universities to their home university. Each university ranks its own students based on predetermined criteria, e.g., GPA and seniority. Based on the ranking, a serial dictatorship mechanism is applied to place students in the available slots. Finally, the list of students who received slots at the partner universities is sent to the partners. The partner universities typically accept all the students on the list. An exchange student pays her tuition to her own college, not the one importing her.

There are huge imbalances between the number of students exported and imported by each country. Moreover, countries with high positive balances are not often willing to match the quota requests of the net-exporter countries. This precautionary behavior may lead to inefficiencies as in tuition-exchange markets.

D.4 Scientific Exchange

The Mevlana Exchange Program aims to exchange academic staff between Turkish universities and universities in other countries. Turkish public universities are governed by the Turkish Higher Education Council and professors are public servants. Therefore, the part of the exchange that is among public universities can be seen as a staff-exchange program, while the exchange among public and private Turkish universities and foreign universities can be seen as a worker-exchange program. Any country can join this program. In 2013, around 1,000 faculty members benefited from this program.
Appendix E  Proofs of Appendix A

Proof of Proposition 3. We prove existence by showing that for any tuition-exchange market there exists an associated college admission market and the set of stable matchings are the same under both markets. Under Assumption 3, we fix a tuition exchange market \([q, e, \succeq]\). We first introduce an associated college admissions market, i.e., a Gale-Shapley (1962) two-sided many-to-one matching market, \([S, C, q, P_S, \overline{P}_C]\), where the set of students is \(S\); the set of colleges is \(C\); the quota vector of colleges for admissions is \(q\); the preference profile of students over colleges is \(P_S\), which are all the same entities imported from the tuition exchange market; and the preference profile of colleges over the set of students is \(\overline{P}_C\), which we construct as follows: for all \(T \subseteq E\) (i.e., the set of eligible students) such that \(|T| < q_c\) and \(i, j \in E \setminus T\), (i) \(i \in P_c \iff (T \cup i) \overline{P}_c(T \cup j)\), (ii) \(i \in P_c \iff (T \cup i) \overline{P}_c(T \cup j)\), and (iii) \(T \overline{P}_c(T \cup k)\) for all \(k \in S \setminus E\). We fix \(C\) and \(S\) and represent such a college admission market as \([q, P_S, \overline{P}_C]\). In this college admissions market, a matching \(\overline{P}\) is a correspondence \(\overline{P} : C \cup S \to C \cup S \cup c_0\) such that (1) \(\overline{P}(c) \subseteq S\) where \(|\overline{P}(c)| < q_c\) for all \(c \in C\), (2) \(\overline{P}(s) \subseteq C \cup c_0\) where \(|\overline{P}(s)| = 1\) for all \(s \in S\), and (3) \(s \in \overline{P}(c) \iff \overline{P}(s) = c\) for all \(c \in C\) and \(s \in S\). A matching \(\overline{P}\) is individually rational if \(\overline{P}(s)R_s c_0\) for all \(s \in S\), and, for all \(i \in \overline{P}(c)\), we have \(i \overline{P}_c s\), \(\overline{P}(c)\), and \(\overline{P}(c)\). A matching \(\overline{P}\) is blocked by a pair \((c, s) \in C \times S\) if \(c \in P_s\), \(\mu(s)\), and there exists \(s' \in \overline{P}(c)\) such that \(s \overline{P}_c s'\). A matching \(\overline{P}\) is stable in a college admission market if it is individually rational, nonwasteful, and not blocked by any pair.

By our construction \(\overline{P}_C\) is responsive; hence there exists at least one stable matching in \([q, P_S, \overline{P}_C]\) (cf. Gale and Shapley, 1962; Roth, 1985). Let \(\overline{P}\) be a stable matching in \([q, P_S, \overline{P}_C]\). We first show that \(\overline{P}\) is also a matching in \([q, e, \succeq]\). By the definition of a matching in a college admission market, the first three bullets of the definition of a matching in a tuition exchange market hold. Due to individual rationality, \(\mu(s) = c_0\) for all \(s \not\in E\). Hence, \(\overline{P}\) is a matching in \([q, e, \succeq]\).

Now, we show that \(\overline{P}\) is stable in \([q, e, \succeq]\). Due to individually rationality of \(\overline{P}\) in the college admission market, \(\mu(s)R_s c_0\) and \(s \overline{P}_c s\) for all \(s \in \overline{P}(c)\) and \(c \in C\). By Assumption 3 and the definition of individual rationality in the tuition-exchange market, \(\overline{P}\) is individually rational in \([q, e, \succeq]\). Whenever there exists \(s \in S\) such that \(c \overline{P}_s\overline{P}(s)\), then either \(s \in S \setminus E\) or \(\overline{P} \succ_e \mu'\) for all \(\mu' \in \mathcal{M}\), where \(s \in \mu'(c) \subset \overline{P}(c) \cup s\) and \(\overline{P}(s') = \mu'(s')\) for all \(s' \in S \setminus (\overline{P}(c) \cup s')\). This follows from the definition of stability and construction of the college preferences in the associated college admission market and Assumption 3. Hence, \(\overline{P}\) is stable in \([q, e, \succeq]\).

Finally, we show that if \(\overline{P}\) is not stable in \([q, P_S, \overline{P}_C]\), then it is either not a matching or
number of positions filled by each college is the same at every stable matching by Propo-
se. Then we have

college admissions market. If

outcome of the student-proposing DA algorithm in

Lemma 2

1986) it is shown that the number of students assigned to a college is the same in all stable

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In the rural hospital theorem (Roth,

it is shown that the number of students assigned to a college is the same in all stable

matchings, \(|\nu(c)| = |\mu(c)|\) for each \(c \in C\). Moreover, the set of students assigned to a real
college is the same in all stable matchings, i.e., \(S^\nu = S^\mu\). Since \(X_{c}^\mu = S^\mu \cap S_c\), \(X_{c}^\nu = S^\nu \cap S_c\), and
\(S^\nu = S^\mu\), we have \(X_{c}^\mu = X_{c}^\nu\). Then, \(b_{c}^\nu = |\mu(c)| - |S^\mu \cap S_c| = |\nu(c)| - |S^\nu \cap S_c| = b_{c}^\nu\) for all \(c \in C\).

Proof. Under Assumption 3, we fix a market \([q, e, \succsim]\). The case
in which we have a unique stable matching in \([q, e, \succsim]\) is trivial. Hence, we consider
the case in which there are at least two stable matchings. Let \(\nu\) and \(\mu\) be any two stable
matchings of \([q, e, \succsim]\). By the proof of Proposition 3, \(\nu\) and \(\mu\) are also stable under the
associated college admission market \([q, P_S, \overline{P}_C]\). Let \(S^\nu\) and \(S^\mu\) be the set of students
assigned to a college in \(\nu\) and \(\mu\), respectively. Due to Assumption 1 Part 3 and individual
rationality, \(M_{c}^\mu = \mu(c), M_{c}^\nu = \nu(c)\) for all \(c \in C\). In the rural hospital theorem (Roth,
1986) it is shown that the number of students assigned to a college is the same in all stable
matchings, \(|\nu(c)| = |\mu(c)|\) for each \(c \in C\). Moreover, the set of students assigned to a real
college is the same in all stable matchings, i.e., \(S^\nu = S^\mu\). Since \(X_{c}^\mu = S^\mu \cap S_c, X_{c}^\nu = S^\nu \cap S_c\), and
\(S^\nu = S^\mu\), we have \(X_{c}^\mu = X_{c}^\nu\). Then, \(b_{c}^\nu = |\mu(c)| - |S^\mu \cap S_c| = |\nu(c)| - |S^\nu \cap S_c| = b_{c}^\nu\) for all \(c \in C\).

We first state and prove the following Lemma, which is used in proving Proposition 6
and Theorem 11:

Lemma 2 Under Assumption 3, let \(\hat{\pi}\) be a stable matching of \([\hat{q}, \hat{e}, \succsim]\) and \(\pi\) be a stable
matching of \([(\hat{q}_c, \hat{q}_{-c}), (\hat{e}_c, \hat{e}_{-c}), \succsim]\) where \(\hat{e}_c = c + 1\), and \(\hat{q}_c = \hat{q}_c\) if \(|\pi(c)| = \hat{q}_c\) and \(\hat{q}_c \geq \hat{q}_c\) 
otherwise. Then we have \(b_{c}^{\hat{\pi}} \in \{b_{c}^{\pi} - 1, b_{c}^{\pi}\}\) and \(b_{c}^{\pi} \in \{b_{c}^{\hat{\pi}}, b_{c}^{\pi} + 1\}\) for all \(c' \in C \setminus c\).

Proof. Denote the newly certified student of \(c\) by \(i\) in \([(\hat{q}_c, \hat{q}_{-c}), (\hat{e}_c, \hat{e}_{-c}), \succsim]\). The number of positions filled by each college is the same at every stable matching by Proposition 5. Moreover, \(\hat{\pi}\) is stable in the associated college admissions market \([\hat{q}, P_S, \overline{P}_C]\)
by the proof of Proposition 3. Thus, without loss of generality, we assume \(\hat{\pi}\) to be the
outcome of the student-proposing DA algorithm in \([\hat{q}, P_S, \overline{P}_C]\).

First, consider the market \([(\hat{q}_c, \hat{q}_{-c}), (\hat{e}_c, \hat{e}_{-c}), \succsim]\). Let \([(\hat{q}_c, \hat{q}_{-c}), P_S, \overline{P}_C]\) be the associated
college admissions market. If \(|\hat{\pi}(c)| < \hat{q}_c\), then adding new seats to an underdemanded
college will not change the set of students assigned to \( c \), and the DA selects the same outcome in \([\hat{q}, P_S, \overline{P}_C]\) and \([\hat{q}_c, \hat{q}_{-c}, P_S, \overline{P}_C]\). If \(|\hat{\pi}(c)| = \hat{q}_c = \hat{q}_c\) by assumption. Hence, the DA selects the same outcome in \([\hat{q}_c, \hat{q}_{-c}, P_S, \overline{P}_C]\) and \([\hat{q}, P_S, \overline{P}_C]\).

Denote the associated college admissions market of \([\hat{q}_c, \hat{q}_{-c}, (\hat{c}_c, \hat{e}_{-c}), \hat{z}]\) by \([\hat{q}_c, \hat{q}_{-c}, P_S, \overline{P}_C]\). We will apply the sequential DA algorithm introduced by McVitie and Wilson (1971) in \([\hat{q}_c, \hat{q}_{-c}, P_S, \overline{P}_C]\), where the new agent \( i \) will be considered at the end. Let \( \hat{\pi} \) be the outcome of the DA for \([\hat{q}_c, \hat{q}_{-c}, P_S, \overline{P}_C]\).

Let \( C_\prec \) be the set of colleges that could not fill all their seats, and \( C_\succ \) be the set of colleges that did, in \( \hat{\pi} \). Formally, \( C_\prec = \{c \in C : |\hat{\pi}(c)| < \hat{q}_c\} \) and \( C_\succ = \{c \in C : |\hat{\pi}(c)| = \hat{q}_c\} \). Now, when it is the turn of \( i \) to apply in the sequential version of the student-proposing DA, the current tentative matching is \( \hat{\pi} \). After \( i \) starts making offers in the algorithm, let \( \hat{c} \) be the first college that does not reject \( i \). Since \( \emptyset P_c i, c \neq \hat{c} \).

In the rest of the proof, as we run the sequential DA, we run the following cases iteratively, starting with student \( i \):

1. If \( \hat{c} = c_0 \), then the algorithm terminates; \( b_{\hat{c}}^\hat{\pi} = b_{\hat{c}}^\hat{\pi} \).
2. If \( \hat{c} \in C_\prec \), then \( i \) will be assigned to \( \hat{c} \) and the algorithm terminates; \( b_{\hat{c}}^\hat{\pi} = b_{\hat{c}}^\hat{\pi} - 1, b_{\hat{c}}^\hat{\pi} = b_{\hat{c}}^\hat{\pi} + 1 \), and \( b_{\hat{c}}^\hat{\pi} = b_{\hat{c}}^\hat{\pi} \).
3. If \( \hat{c} \in C_\succ \), then student \( \hat{i} \) who has the lowest priority among the students in \( \hat{\pi}(\hat{c}) \) is rejected in favor of \( i \). We consider two cases:

   3.a. Case \( \hat{i} \in S_c \): The net balance of no college will change from the beginning, and we continue from the beginning above, again using student \( \hat{i} \) instead of \( i \).

   3.b. Case \( \hat{i} \notin S_c \): The instantaneous balance of \( c \) will deteriorate by 1 as \( i \) is tentatively accepted. Now, it is \( \hat{i} \)'s turn in the sequential DA to make offers. In this series of offers, suppose the first college that does not reject student \( \hat{i} \) is \( \hat{c} \). Denote the home college of \( \hat{i} \) by \( c' \) (note that \( c' \neq c \)).

   3.b.i. If \( \hat{c} \in c_0 \cup (C_\prec) \), then the algorithm will terminate, and \( b_{\hat{c}}^\hat{\pi} \in \{b_{\hat{c}}^\hat{\pi} - 1, b_{\hat{c}}^\hat{\pi}\}, b_{\hat{c}}^\hat{\pi} \in \{b_{\hat{c}}^\hat{\pi}, b_{\hat{c}}^\hat{\pi} + 1\} \) for all \( \hat{c} \in C \setminus c \).

   3.b.ii. If \( \hat{c} \in C_\succ \), then the lowest-priority student held by \( \hat{c} \) will be rejected in favor of \( \hat{i} \). Let this student be \( \tilde{i} \). There are two further cases:

   3.b.ii.A. Case \( \tilde{i} \in S_c \): Then, \( \tilde{c} \neq c \). The instantaneous balance of \( c \) will increase by 1, and we will start from the beginning again with \( \tilde{i} \) instead of \( i \). The total change in \( c \)'s balance since the beginning will be 0. Also, no other college’s balance has changed since the beginning.

   3.b.ii.B. Case \( \tilde{i} \notin S_c \): We start from Step 3.b above with student \( \tilde{i} \) instead of \( i \).

Thus, whenever we continue from the beginning, the instantaneous balance of \( c \) is \( b_{\hat{c}}^\hat{\pi} \), and whenever we continue from Step 3.b, the instantaneous balance of \( c \) is \( b_{\hat{c}}^\hat{\pi} - 1 \) or \( b_{\hat{c}}^\hat{\pi} \).
and the instantaneous balances of all other colleges either increase by one or stay the same. Due to finiteness, the algorithm will terminate at some point at Steps 1 or 2, or Steps 3.b.i or 3.b.ii; and the net balance of $c$ at the new DA outcome will be $b^\pi_c$ or $b^\hat{\pi}_c - 1$. Moreover, whenever the algorithm terminates, the net balance of any other college has gone up by one or stayed the same.

We are ready to prove the results stated in the main text:

**Proof of Proposition 6.** Let $[\hat{q}, P_S, \hat{P}_C]$ and $[(\hat{q}_c, \hat{q}_e-c), P_S, \hat{P}'_C]$ be the associated college admissions problems of $[\hat{q}, \hat{e}, \hat{\zeta}]$ and $[(\hat{q}_c, \hat{q}_e-c), (\hat{e}_c+1, \hat{e}_e-c), \hat{\zeta}]$, respectively. Let $\hat{\pi}$ and $\tilde{\pi}$ be the outcome of the DA in $[\hat{q}, P_S, \hat{P}_C]$ and $[(\hat{q}_c, \hat{q}_e-c), P_S, \hat{P}'_C]$, respectively. By Propositions 3 and 5, it is sufficient to prove the proposition for $\hat{\pi}$ and $\tilde{\pi}$. Note that $M^\pi_c = \hat{\pi}(c)$ by Assumption 1 Part 3, and $\tilde{\pi}$ is stable.

Two cases are possible:

**Case 1:** $b^\pi_c < 0$: We have $|\hat{\pi}(c)| = |M^\pi_c| < |X^\pi_e| \leq \hat{e}_c \leq \hat{q}_c$. Then, by Lemma 2, $b^\pi_c \in \{b^\pi_c - 1, b^\pi_c\}$.

**Case 2:** $b^\pi_c \geq 0$: We have two cases again:

2.a. $|\hat{\pi}(c)| < \hat{q}_c$ or $\hat{q}_c = \hat{q}_c$: By Lemma 2, $b^\pi_c \in \{b^\pi_c - 1, b^\pi_c\}$.

2.b. $|\hat{\pi}(c)| = \hat{q}_c$ and $\hat{q}_c = \hat{q}_c + k$ for $k > 0$: Denote the newly certified student of $c$ by $i$ in market $[(\hat{q}_c, \hat{q}_e-c), (\hat{e}_c+1, \hat{e}_e-c), \hat{\zeta}]$. We first consider the outcome of the DA in the associated college admissions market of $[(\hat{q}_c, \hat{q}_e-c), (\hat{e}_c, \hat{e}_e-c), \hat{\zeta}]$, which we denote by $\pi''$. We first show that the number of students imported by $c$ in $\pi''$ cannot be less than the one in $\hat{\pi}$. Let $C_0 = \{c \in C : |\hat{\pi}(c)| < \hat{q}_c\}$. By our construction, in any stable matching of an associated college admissions market all students in $S \setminus E$ are assigned to $c_0$. Due to the nonwastefulness of $\hat{\pi}$, $\hat{\pi}(s)P_s \hat{c}$ for all $s \in E \setminus \hat{\pi}(c)$ and $\hat{c} \in C_0$. We know that the DA is resource monotonic: when the number of seats increases, then every student will be weakly better off (cf. Kesten, 2006). That is, $\pi''(s)R_s \hat{\pi}(s)$ for all $s \in E$. By combining the resource monotonicity and individual rationality of the DA, we can say that if a student is assigned to a college in $\hat{\pi}$, then he will also be assigned to a college in $\pi''$. Hence, we can write:

$$\sum_{c' \in C} |\pi''(c')| \geq \sum_{c' \in C'} |\hat{\pi}(c')|.$$  (1)

Note that the difference between the left-hand side and the right-hand side of the equation can be at most $k$. This follows from the fact that in $\pi''$ no new student will be assigned to a college in $C_0$, the number of students assigned to other colleges can increase only for $c$, and the maximum increment is $k.$
By combining nonwastefulness and resource monotonicity we can write:

\[
\sum_{\tilde{c} \in C_<} |\pi''(\tilde{c})| \leq \sum_{\tilde{c} \in C_<} |\tilde{\pi}(\tilde{c})|.
\] (2)

Then, if we subtract the left–hand side of Equation 2 from the left–hand side of Equation 1 and the right–hand side of Equation 2 from the right–hand side of Equation 1, we get the following inequality:

\[
\sum_{c' \in C\setminus C_<} |\pi''(c')| \geq \sum_{c' \in C\setminus C_<} |\tilde{\pi}(c')|.
\] (3)

Given that each college in \(C \setminus C_<\) fills its seats in \(\tilde{\pi}\), when we subtract \(\sum_{c' \in C\setminus (C_< \cup c)} \hat{q}_{c'}\) from both sides of Equation 3, we get the following inequality:

\[
|\pi''(c)| + \sum_{c' \in C\setminus (C_< \cup c)} (|\pi''(c')| - \hat{q}_{c'}) \geq |\tilde{\pi}(c)|.
\] (4)

The term \(\sum_{c' \in C\setminus (C_< \cup c)} (|\pi''(c')| - \hat{q}_{c'})\) is nonnegative since \(|\pi''(c')| \leq \hat{q}_{c'}\) for all \(c' \in C \setminus (C_< \cup c)\). Therefore, \(|\pi''(c)| \geq |\tilde{\pi}(c)|\).

If \(|\pi''(c)| = |\tilde{\pi}(c)|\) then \(|\pi''(c')| = |\tilde{\pi}(c')|\) for all \(c' \in C\). This follows from Equation 4, Equation 2, and the fact that \(|\pi''(c')| \leq |\tilde{\pi}(c')|\) for all \(c' \in C \setminus \{c\}\). Therefore, \(c\) cannot export and import more students, and \(b''_c = b''_\hat{c}\). If \(|\pi''(c)| > |\tilde{\pi}(c)|\), then at most \(k\) more students can be assigned to a college in \(\pi''\) among the eligible students who were not assigned to a college in \(\hat{\pi}\). It is possible that some of the students belong to \(S_\hat{c}\). Thus, \(b''_c \in \{b''_\hat{c}, ..., b''_\hat{c} + k\}\).

Thus, by Lemma 2, as we increase the eligibility quota of college \(c\) by \(1\) and keep the admission quota at \(\hat{q}_c\), we have \(b''_c \in \{b''_\hat{c} - 1, b''_\hat{c}\}\), and hence, \(b''_c \in \{b''_\hat{c} - 1, b''_\hat{c}, ..., b''_\hat{c} + k\}\).

**Proof of Theorem 10.** Given Proposition 6, when \(c\) decreases its certification quota by one and keeps its admission quota the same, its balance in any stable outcome of the new market will either be the same or increase by one. Since \(c\) will have a nonnegative balance in any stable outcome of the market \([\bar{q}, (\tilde{e}_c - 1, \bar{e}_c), \bar{z}]\), there exists \(0 \leq \tilde{e}_c \leq \bar{e}_c\) such that \(c\) has a zero–balance in every stable matching of the market \([\bar{q}, (\tilde{e}_c, \bar{e}_c - 1), \bar{z}]\). **■**

**Proof of Theorem 11.** We consider two problems: \([\bar{q}, \hat{e}, \bar{z}]\) and \([\bar{q}', \hat{e}_c - 1, \bar{z}]\) with \(\hat{q}_c \geq \hat{e}_c\) and \(\hat{q}_c \geq \bar{q}' \geq \hat{e}_c - 1\) such that for \(c\), \(b''_c < 0\) for a stable matching \(\mu\) of the first market. Let \(\mu'\) be an arbitrary stable matching of the second market. We want to show that \(b''_{\hat{c}'_c} \geq b''_{\hat{c}'_c}\). From Proposition 6, we know that \(b''_{\hat{c}'_c} < 0\) or \(b''_{\hat{c}'_c} = 0\).
By Proposition 5, without loss of generality we assume that $\mu$ and $\mu'$ are the outcome of the sequential DA algorithm for the associated college admissions market of $[\hat{q}, \hat{e}, \hat{z}]$ and $[(\hat{q}_c, \hat{e}_c - 1, \hat{e}_c), \hat{z}]$, respectively. We have two cases:

**Case 1:** $b'_c < 0$. We have $|\mu'(c)| = |M_c^{\mu'}| < |X_c^{\mu'}| \leq e_c - 1 \leq \min\{\hat{q}_c, q'_c\}$. Hence, as $c$ did not fill its admission quota at $\mu'$ under both $\hat{q}_c$ and $q'_c$, in market $[\hat{q}, (\hat{e}_c - 1, \hat{e}_c), \hat{z}]$ $\mu'$ will still be the outcome of DA for the associated college admissions market. When we add a new student $i$ from $c$ to the set of eligible students, we obtain $[\hat{q}, \hat{e}, \hat{z}]$. By Lemma 2, we have $b'_c \in \{b'_c, b'_c + 1\}$ for all $c' \in C \setminus c$.

**Case 2:** $b'_c = 0$. There are two possibilities: (a) $|\mu'(c)| < q'_c$ and (b) $|\mu'(c)| = q'_c$.

2.a. If $|\mu'(c)| < q'_c$, then by Lemma 2, we have $b''_c \in \{b''_c, b''_c + 1\}$ for all $c' \in C \setminus c$.

2.b. If $|\mu'(c)| = q'_c$, then $|\mu'(c)| = \hat{e}_c - 1 = q'_c$. We first increase the admission quota of $c$ from $q'_c$ to $\hat{q}_c$ and keep its eligibility quota at $\hat{e}_c - 1$. Suppose the number of students assigned to $c$ increases at the outcome of the DA under the associated college admissions market of $[\hat{q}, (\hat{e}_c - 1, \hat{e}_c), \hat{z}]$, which we denote by $\mu''$, i.e., $|\mu''(c)| > |\mu'(c)| = \hat{e}_c - 1$. Thus, $b''_c > 0$. When we also increase the eligibility quota of $c$ from $\hat{e}_c - 1$ to $\hat{e}_c$, then by Lemma 2, $b''_c \in \{b''_c - 1, b''_c\}$, and hence, $b''_c \geq 0$. However, this contradicts the fact that $b''_c < 0$. Therefore, $|\mu''(c)| = |\mu'(c)| = q'_c \leq \hat{q}_c$. Hence, under both associated college admissions problems of $[(\hat{q}_c, \hat{e}_c - 1, \hat{e}_c), \hat{z}]$ and $[\hat{q}, (\hat{e}_c - 1, \hat{e}_c), \hat{z}]$, the DA chooses the same matching, i.e., $\mu'' = \mu'$. When we increase the eligibility quota of $c$ from $\hat{e}_c - 1$ to $\hat{e}_c$ and keep the admission quota at $\hat{q}_c$, the DA outcome changes from $\mu'' = \mu'$ to $\mu$ under the associated college admissions market. By Lemma 2, we have $b''_c \in \{b''_c, b''_c + 1\}$ for all $c' \in C \setminus c$.

In either case, $b''_c \leq b'_c$. ■

### Appendix F  Structure of Stable Matchings

In this Online Appendix, we inspect the structure of stable matchings, as our stability concept is novel. In the college admissions market, there always exist student–optimal and college–optimal Gale–Shapley–stable matchings (cf. Gale and Shapley, 1962; Roth, 1985). Under Assumption 3, we can guarantee the existence of college– and student–optimal stable tuition–exchange matchings. This result’s proof also uses the associated Gale–Shapley college admissions market for each tuition–exchange market and the properties of Gale–Shapley stable matchings in these markets.\(^{58}\)

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\(^{57}\)A matching is student–(or college–)optimal stable if it is preferred to all the other stable matchings by all students (or colleges).

\(^{58}\)The lattice property of Gale–Shapley–stable college–admissions matchings can also be used to prove an analogous lattice property for stable matchings in tuition–exchange markets under Assumption 3. We
Proposition 7 Under Assumption 3, there exist college- and student-optimal matchings in any tuition-exchange market.

Proof of Proposition 7. By the proof of Proposition 3, Gale and Shapley (1962), and Roth (1985), there exists a student-optimal stable matching for each tuition-exchange market. By Assumption 3 Part 1 and Proposition 5, colleges compare only the stable matchings through the admitted set of students. By Gale and Shapley (1962) and Roth (1985), there exists a college-optimal stable matching for each tuition-exchange market.

Appendix G An Example of the 2S-TTC Mechanism

We illustrate the dynamics of the 2S-TTC mechanism with an example below:

Example 1 (2S-TTC) Let $C = \{a, b, c, d, e\}$, $S_a = \{1, 2\}$, $S_b = \{3, 4\}$, $S_c = \{5, 6\}$, $S_d = \{7, 8\}$, and $S_e = \{9\}$. Let each college certify all its students as eligible and $q = (2, 2, 2, 1, 1)$. The internal priorities and the rankings of agents associated with their preferences over matchings are given as:

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Let $o_e$ and $o_a$ be the vectors representing the eligibility and admission counters of colleges, respectively. Then we set $o_e = (2, 2, 2, 1)$ and $o_a = (2, 2, 2, 1, 1)$.
**Round 1:** The only cycle formed is \((b, 3, a, 1)\). Therefore, \(1\) is assigned to \(b\) and \(3\) is assigned to \(a\). Observe that although college \(a\) is the most-preferred college of student \(6\), she is not acceptable to \(a\), and hence, she points to college \(b\) instead. The updated counters are \(o_e = (1, 1, 2, 2, 1)\) and \(o_a = (1, 1, 2, 1, 1)\).

**Round 2:** The only cycle formed in Round 2 is \((c, 6, b, 4)\). Therefore, \(6\) is assigned to \(b\) and \(4\) is assigned to \(c\). The updated counters are \(o_e = (1, 0, 1, 2, 1)\) and \(o_a = (1, 0, 1, 1, 1)\). College \(b\) is removed.

**Round 3:** The only cycle formed in Round 3 is \((a, 2, c, 5)\). Therefore, \(5\) is assigned to \(a\) and \(2\) is assigned to \(c\). The updated counters are \(o_e = (0, 0, 0, 2, 1)\) and \(o_a = (0, 0, 0, 1, 1)\). Colleges \(a\) and \(c\) are removed.

**Round 4:** The only cycle formed in Round 4 is \((c_9, 7)\). Therefore, \(7\) is assigned to \(c_9\). Given that we have a trivial cycle, we only update \(o_e\). The updated counters are \(o_e = (0, 0, 0, 1, 1)\) and \(o_a = (0, 0, 0, 1, 1)\).

**Round 5:** The only cycle formed at this round is \((c, 9, d, 8)\). Therefore, \(8\) is assigned to \(e\) and \(9\) is assigned to \(d\). The updated counters are \(o_e = (0, 0, 0, 0, 0)\) and \(o_a = (0, 0, 0, 0, 0)\).

All agents are assigned, so the algorithm terminates and its outcome is given by matching

\[
\mu = \begin{pmatrix}
a & b & c & d & e \\
3, 5 & 1, 6 & 2, 4 & 9 & 8
\end{pmatrix}.
\]
Appendix H  Independence of Axioms

- A student–strategy-proof, acceptable but not balanced–efficient mechanism that also respects internal priorities: A mechanism that always selects the null matching for any market.

- A student–strategy-proof, balanced–efficient, acceptable mechanism that does not respect internal priorities: Consider a variant of the 2S-TTC mechanism in which each college points to the certified student who has the lowest priority among the certified ones. This mechanism is strategy-proof for students, balanced–efficient, and individually rational, but it fails to respect internal priorities.

- A balanced–efficient, acceptable, but not student–strategy-proof mechanism that respects internal priorities: Consider the following market. There are three colleges $C = \{a, b, c\}$ and four students $S_a = \{1, 2\}$, $S_b = \{3\}$, and $S_c = \{4\}$. The ranking $P$ associated with preference preference profile $\succ_S$ is given as

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Let mechanism $\psi$ select the same matching as the 2S-TTC for each market except the market $[q = (2, 1, 1), e = (2, 1, 1), \succ]$, and for this market it assigns 1 to $c$, 2 to $b$, 3 to $a$, and 4 to $a$. This mechanism is balanced-efficient, acceptable, and respecting internal priorities. However, it is not student–strategy-proof, because when 1 excludes $c$, $\psi$ and 2S-TTC will assign 1 to $b$.

- A balanced–efficient, student–strategy-proof, but not individually rational mechanism that respects internal priorities: Consider a variant of the 2S-TTC in which students are not restricted to point to those that colleges consider them acceptable. This mechanism is balanced–efficient, strategy-proof, and respecting internal priorities, but it is not acceptable since an unacceptable student can be assigned to a college.

Appendix I  Omitted Simulation Results
Figure 3: Student welfare under simulations with 20 colleges each with 5 seats.

Figure 4: Student welfare under simulations with 10 colleges each with 10 seats.
Figure 5: Student welfare under simulations with 10 colleges each with 5 seats.

Figure 6: Excess balance under DA straightforward behavior simulations with 20 colleges each with 5 seats.
Figure 7: Excess balance under DA straightforward behavior simulations with 10 colleges each with 10 seats.

Figure 8: Excess balance under DA straightforward behavior simulations with 10 colleges each with 5 seats.