Constant Risk Aversion*

Zvi Safra

Faculty of Management, Tel Aviv University, Tel Aviv, 69978, Israel
safraz@post.tau.ac.il

and

Uzi Segal

Department of Economics, University of Western Ontario, London, Ontario, N6A 5C2, Canada
segal@sscl.uwo.ca

Received April 6, 1997; revised June 10, 1998

Constant risk aversion means that adding a constant to all outcomes of two distributions, or multiplying all their outcomes by the same positive number, will not change the preference relation between them. We prove several representation theorems, where constant risk aversion is combined with other axioms to imply specific functional forms. Among other things, we obtain a form of disappointment aversion theory without using the concept of reference point in the axioms, and a form of the rank dependent model without making references to the ranking of the outcomes. This axiomatization leads to a natural generalization of the Gini index.

Journal of Economic Literature Classification Number: D81.

1. INTRODUCTION

Constant risk aversion means that adding the same constant to all outcomes of two distributions, or multiplying all their outcomes by the same positive constant, will not change the preference relation between them. Within the expected utility framework, this assumption implies expected value maximization. But there are many (non-expected utility) functionals that satisfy this requirement, for example, the dual theory of Yaari [30], or functions offered by Roberts [23] and by Smorodinsky [27] (see Example 1 in Section 2 below).

* We thank Chew Soo Hong, Jim Davies, Larry Epstein, Joel Sobel, Shlomo Yitzhaki, and Peter Wakker for their useful comments and suggestions. Zvi Safra thanks the Johns Hopkins University and the Australian National University for their hospitality and the Israel Institute for Business Research for financial support. Uzi Segal thanks the Social Science and Humanities Research Council of Canada for financial support.
In this paper we prove several representation theorems, where constant risk aversion is combined with some other known axioms to imply specific functional forms. We first show that non-trivial (that is, non-expected value) functionals that satisfy constant risk aversion cannot be Fréchet differentiable. This differentiability is the key assumption in Machina’s analysis [19], and is widely used in the decision theoretic literature. Since they are not Fréchet differentiable, constant risk aversion functionals cannot be approximated (in the $L_1$ norm) by expected utility preferences. These results are presented in Section 3.

A possible relaxation of Fréchet differentiability is the requirement that representation functionals are only Gâteaux differentiable. This requires the derivative with respect to $\alpha$ of $V((1-\alpha)F + \alpha G)$ to exist, and to be continuous and linear in $G - F$. Many constant risk aversion functionals satisfy this assumption, but as we show in Section 4, adding betweenness to the list of axioms ($F \sim G$ implies $\alpha F + (1-\alpha) G \sim F$ for all $\alpha \in [0, 1]$) permits only one functional form, which is Gul’s disappointment aversion theory [16] with a linear utility function $u$. According to this theory, the decision maker evaluates outcomes that are better than the certainty equivalent of a lottery by using an expected utility functional with a utility function $u$. He similarly evaluates outcomes that are worse than the certainty equivalent (with the same function $u$). Finally, the value of a lottery is a weighted sum of these two evaluations. In this theory, the certainty equivalent serves as a natural reference point, to which Gul’s axioms make an explicit reference. Our axioms do not require any such explicit dependency, and the reference point is obtained as part of the results of the model, rather than as part of its assumptions.

One of the most popular alternatives to expected utility is the rank dependent model, given by $V(F) = \int u(x) dg(F(x))$. (This model has several different versions, see Weymark [29], Quiggin [21], and other citations in Section 5 below.) This functional form is consistent with constant risk aversion whenever $u$ is a linear function, which is Yaari’s dual theory [30]. Although many axiomatizations of the rank dependent model exist, they all depend on one crucial assumption, namely, that the value of an outcome depends on its relative position. This is of course a key feature of the rank dependent model, from which its name is derived. But it would be nice to be able to obtain this property as a result, rather than as an assumption of the model. In Section 5 we offer an axiomatization of a non-trivial special case of Yaari’s functional, where none of the axioms makes an explicit appeal to the relative position of any of the outcomes. The key added axiom is mixture symmetry, which states that if $F \sim G$, then for all $\alpha \in [0, 1]$, $\alpha F + (1-\alpha) G \sim (1-\alpha) F + \alpha G$ (see Chew, Epstein, and Segal

---

1 We have counted more than ten axiomatizations of this model.
The functional form we obtain is a weighted average of the expected value functional, and the Gini inequality index.

Two recent works on bargaining with non-expected utility preferences make use of homogeneity of preferences with respect to probabilities (see Rubinstein, Safra, and Thomson [24] and Grant and Kajii [14]). We offer a slightly stronger assumption, where \( F \sim G \) iff for every \( x \in [0, 1] \), \( xF + (1 - x)\delta_0 \sim xG + (1 - x)\delta_0 \) (\( \delta_0 \) is the distribution of the degenerate lottery that yields the outcome zero with probability one). Together with constant risk aversion, this axiom implies Yaari's representation with a probability transformation function of the form \( g(p) = 1 - (1 - p)^\lambda \). We prove this result in the last section of the paper.

2. DEFINITIONS

Let \( \Omega = (S, \Sigma, P) \) be a measure space and let \( \mathcal{X} \) be the set of real bounded random variables with non-negative outcomes on it. For \( X \in \mathcal{X} \), let \( F_X \) be the distribution function of \( X \). Denote by \( \mathcal{F} \) the set of distribution functions obtained from elements of \( \mathcal{X} \). With a slight abuse of notations, we denote by \( a \) the constant random variable with the value \( a \), and its distribution function by \( F_a \).

For \( X \in \mathcal{X} \), let \( X' \) be the lowest possible value of \( X \) (that is, \( X' \) is the supremum of the values of \( x \) for which \( F_X(x) = 0 \)). Observe that for \( X \in \mathcal{X} \) and \( a > -X', X + a \in \mathcal{X} \), and the distribution \( F_{X + a} \) is given by \( F_{X + a}(x) = F_X(x - a) \). Throughout the paper, when we use the notation \( X + a \) or \( F + a \) we assume that \( a > X' \). Also, for \( X \in \mathcal{X} \) and \( \lambda > 0 \), \( \lambda X \in \mathcal{X} \), and we define the distribution \( \lambda \times F_X := F_{\lambda X} \) by \( (\lambda \times F_X)(x) = F_X(x/\lambda) \).

On \( \mathcal{X} \) we assume the existence of a complete and transitive preference relation \( \succcurlyeq \). We assume throughout the paper that if \( F_X = F_Y \), then \( X \sim Y \). Therefore, \( \succcurlyeq \) induces an order on \( \mathcal{F} \), which we also denote \( \succcurlyeq \). Assume further that \( \succcurlyeq \) is continuous (with respect to the weak topology), and monotonic (with respect to first order stochastic dominance). It then follows that every \( F \in \mathcal{F} \) has a unique certainty equivalent \( x \in [0, \infty) \), satisfying \( F \sim \delta_x \) (recall that for every \( F \in \mathcal{F} \) there exists \( x \) such that \( F(x) = 1 \)). We restrict attention to preference relations satisfying the following assumption.

**Constant Risk Aversion.** \( X \succcurlyeq Y \) iff for every \( a > \max\{-X, -Y\} \) and for every \( \lambda > 0 \), \( \lambda(X + a) \succcurlyeq \lambda(Y + a) \). Or equivalently, \( F \succcurlyeq G \) iff for every such \( a \) and \( \lambda \), \( \lambda \times (F + a) \succcurlyeq \lambda \times (G + a) \).

Note that \( \succcurlyeq \) satisfies constant risk aversion if it satisfies both constant absolute risk aversion and constant relative risk aversion. Also, its representation functional \( V: \mathcal{F} \rightarrow \mathbb{R} \) which is defined implicitly by \( F \sim \delta_{V(F)} \).
(that is, $V(F)$ is the certainty equivalent of $F$), satisfies $V(\lambda \times (F + a)) = \lambda [V(F) + a]$. In such a case we say that $V$ satisfies constant risk aversion.

The following are examples for such functionals.

**Example 1.**
1. $V(F) = \int x \, d\mu(F(x))$ (Yaari’s dual theory [30]).
2. $V(F) = \mu_F + \sigma_F W(\lambda F - \mu_F)$ for some functional $W$, where $\mu_F$ is the expected value of $F$ and $\sigma_F$ is its standard deviation (Roberts [23]).
3. $V(F) = \arg \min \{ x - t \}^{c+1} \, dF(x)$ for some $c > 0$ (Smorodinsky [27]).

The next lemma shows that the set of functionals satisfying constant risk aversion is much larger than the above list. Moreover, from two such functionals more functionals can be created.

**Lemma 1.** Let $\mathcal{V}$ be the set of all functionals that satisfy constant risk aversion. Then for every $\mathcal{V} \subseteq \mathcal{V}$, the functionals $V^\downarrow$ and $V^\uparrow$ are in $\mathcal{V}$, where $V^\downarrow(F) = \inf \{ V(F) : V \in \mathcal{V} \}$ and $V^\uparrow(F) = \sup \{ V(F) : V \in \mathcal{V} \}$.

**Proof.** For a given $F$, there is a sequence $V_i$ in $\mathcal{V}$ such that $V^\downarrow(F) = \lim V_i(F)$. Hence $V^\downarrow(\lambda \times (F + a)) \leq \lim V_i(\lambda \times (F + a)) = \lim \lambda V_i(F + a) = \lambda V^\downarrow(F + a)$. On the other hand, there is a sequence $V'_i$ in $\mathcal{V}$ such that for $G = \lambda \times (F + a)$, $V^\uparrow(G) = \lim V'_i(G)$, hence $V^\uparrow(F) \leq \lim V'_i(F) = \lim V'_i(\lambda^{-1} \times (G - \lambda a)) = \lim \lambda^{-1} (V'_i(G) - \lambda a) = \lambda^{-1} (V^\uparrow(G) - \lambda a)$. These two inequalities imply $V^\uparrow(F) \leq \lambda^{-1} (V^\downarrow(\lambda \times (F + a)) - \lambda a) \leq V^\downarrow(F)$, hence $V^\downarrow(\lambda \times (F + a)) = \lambda V^\downarrow(F + a)$.

The proof of the sup case is similar. $\square$

3. **SMOOTH PREFERENCES**

Machina [19] introduced the concept of smooth representations, that is, representations that are Fréchet differentiable.

**Definition 1.** The function $V : \mathcal{F} \to \mathbb{R}$ is Fréchet differentiable if for every $F \in \mathcal{F}$ there exists a “local utility” function $u(\cdot; F) : \mathbb{R} \to \mathbb{R}$ such that for every $G \in \mathcal{F}$,

$$V(G) - V(F) = \int u(x; F) \, d\mu[G(x) - F(x)] + o(\|G - F\|), \quad (1)$$

where $\| \cdot \|$ is the $L_1$-norm.
In other words, $V$ is Fréchet differentiable if for every $F$, the functional $V$ behaves around $F$ like an expected utility representation with the von Neumann and Morgenstern utility function $u(\cdot; F)$. Machina [19] demonstrated how under this assumption, many results in decision theory can be extended to non-expected utility models. Obviously, if $V(F)$ always equals $E[F]$, the expected value of $F$, then $V$ is smooth and satisfies constant risk aversion. But we have seen above that many other functionals satisfy constant risk aversion. It is therefore natural to ask whether any of them is Fréchet differentiable.

**Theorem 1.** The following two conditions are equivalent.

1. $V$ is an expected value functional, that is, $V(F) = E[F] = \int x \, dF(x)$.
2. $V$ satisfies constant risk aversion and is Fréchet differentiable.

**Proof.** Obviously, (1) $\implies$ (2). To see why (2) $\implies$ (1), consider first the family of local utilities $u(\cdot; \delta_x)$. Since local utility functions are unique up to scalar addition, we assume that for every $x$, $u(x; \delta_x) = x$.

Consider now the lottery $(y, p; 1, 1 - p)$ for arbitrary $y$ and $p$. Using the local utility $u(\cdot; \delta_1)$ we obtain (recall that $u(1; \delta_1) = 1$)

$$V(y, p; 1, 1 - p) - V(\delta_1) = [pu(y; \delta_1) + 1 - p] - 1 + o(p(y - 1)).$$

(2)

Similarly, for $(y + x, p; 1 + x, 1 - p)$ we obtain

$$V(y + x, p; 1 + x, 1 - p) - V(\delta_{1+x}) = [pu(y + x; \delta_{1+x}) + (1 - p)(1 + x)] - (1 + x) + o(p(y - 1)).$$

(3)

By constant risk aversion, the left hand sides of Eqs. (2) and (3) are the same. Subtract Eq. (2) from Eq. (3) to obtain

$$0 = p \left[ u(y + x; \delta_{1+x}) - u(y; \delta_1) - x + \frac{o(p(y - 1))}{p} \right].$$

Divide by $p$, and then take the limit as $p \to 0$ to obtain

$$u(y + x; \delta_{1+x}) = u(y; \delta_1) + x.$$

Set $x = y + x$ and $k = 1 + x$, and obtain for $x \geq k - 1$,

$$u(x; \delta_k) = u(x - k + 1; \delta_1) + k - 1.$$ (4)

For this and other statements concerning local utilities, see Machina [19].
Similarly to Eq. (2) we obtain for $\lambda > 0$
\[
V(\lambda y, \lambda, 1 - p) - V(\lambda) = \left[ pu(\lambda y; \lambda_1) + \lambda(1 - p) \right] - \lambda + o(p\lambda(y - 1)). \tag{5}
\]

By constant risk aversion, the left hand side of Eq. (5) equals $\lambda$ times the left hand side of Eq. (2). Hence
\[
pu(\lambda y; \lambda_2) - \lambda p + o(p\lambda(y - 1)) = \lambda [ pu(y; \lambda_1) - p + o(p(y - 1))].
\]
Divide both sides of the last equation by $p$, and then take the limit as $p \to 0$ to obtain
\[
u(\lambda y; \lambda_2) = \lambda u(y; \lambda_1).
\]
Set $x = \lambda y$ and $k = \lambda$ to obtain for $x \geq k - 1$,
\[
u(x; \lambda_2) = ku \left( \frac{y}{k}; \lambda_1 \right) \tag{6}
\]
Let $h(\cdot) = u(\cdot; \lambda_1)$ and obtain from Eqs. (4) and (6)
\[
h(x - k + 1) + k - 1 = kh \left( \frac{x}{k} \right). \tag{7}
\]
Since $h$ is increasing, it is almost everywhere differentiable. Pick a point $x^* > 1$ at which $h'$ exists. Differentiate both sides of Eq. (7) and obtain that for $x = kx^*$,
\[
h'(k(x^* - 1) + 1) = h'(x^*).
\]
Since $x^* > 1$, it follows that $h'(z)$ is constant for $z > 1$.

By similar arguments, there is $x^* < 1$ at which $h$ is differentiable. We now obtain that $h$ is differentiable on $(0, 1)$, and that its derivative there is constant. In other words, there are two numbers, $s$ and $t$ such that
\[
\frac{\partial u(x; \lambda_2)}{\partial x} = \begin{cases} 
  s, & x < 1 \\
  t, & x > 1.
\end{cases}
\]
From Eq. (6) it now follows that
\[
\frac{\partial u(x; \lambda_2)}{\partial x} = \begin{cases} 
  s, & x < k \\
  t, & x > k. \tag{8}
\end{cases}
\]

**Claim 1.** Let $V$ be a Fréchet differentiable functional, and let $u(\cdot; \delta_{x^*})$ be its local utility at $\delta_{x^*}$. Then for almost all $x^*$, $u(x; \delta_{x^*})$ is differentiable with respect to its first argument at $x = x^*$. 
Proof of Claim 1. By monotonicity, the functional $V$ satisfies $V(x) > V(y)$ if $x > y$, hence the set of points where $\partial V(\delta_x) / \partial x$ does not exist is of measure zero. The claim now follows from the equivalence of the following two conditions.

1. The derivative $\partial V(\delta_x) / \partial x$ exists at $x = x^*$.
2. The local utility $u(x; \delta_{x^*})$ is differentiable with respect to its first argument at $x = x^*$.

To see why (1) and (2) are equivalent, note that

$$V(x^* + \varepsilon) = u(x^* + \varepsilon; \delta_{x^*}) - u(x^*; \delta_{x^*}) + o(\varepsilon).$$

Divide both sides by $\varepsilon$ and let $\varepsilon \to 0$ to obtain that $V(\delta_x)$ is differentiable with respect to $x$ at $x = x^*$ if $u(x; \delta_{x^*})$ is differentiable with respect to its first argument at $x = x^*$.

Since $V$ is Fréchet differentiable, it follows from Claim 1 that for almost all $k$, $u(\cdot; \delta_k)$ is differentiable with respect to the first argument at $k$. Hence in Eq. (8), $s = t$, and since $V(\delta_k) = k$, it follows that $u(x; \delta_k) = sx + (1 - s) k$.

Fix the probabilities $p_1, \ldots, p_n$, and consider the space of lotteries $(x_1, p_1; \ldots; x_n, p_n)$. These lotteries can be represented as vectors of the form $(x_1, \ldots, x_n) \in \mathbb{R}^n$. For $y \geq 0$, let $d_y = (y, \ldots, y)$ be the point on the main diagonal corresponding to the lottery $\delta_y$. Pick a point $x^*$ not on the main diagonal of $\mathbb{R}^n$ and $y$ such that $x^* \sim \delta_y$, and let $H^*$ be the two dimensional plane containing $x^*$ and the main diagonal. It follows from Roberts [23, p. 430] that indifference curves on $H^*$ below the main diagonal are linear and parallel to each other, and so are indifference curves above the main diagonal. In other words, $x^* \sim d_y$ iff for all $x \in [0, 1]$, $sx^* + (1 - s) y \sim d_y$.

Note that this mixture is with respect to outcomes, not with respect to distributions.

The local utility function at $\delta_y$ is given by $u(x; \delta_y) = sx + (1 - s) y$, hence

$$0 = V(x^*) - V(d_y) = V(sx^* + (1 - s) y) - V(d_y)$$

$$= \sum_{i=1}^n p_i (sx_i^* + (1 - s) y) - (sy + (1 - s) y) + o(x)$$

$$= sx \left[ \sum_{i=1}^n p_i x_i^* - y \right] + o(x).$$

3 Up to this point, we only used Gâteaux, rather than Fréchet, differentiability (see Section 4).
Divide by $\alpha$, and then let $\alpha \to 0$ to obtain $\sum_{i=1}^{n} p_i x_i = y$, which is the expected value functional. By continuity, $V(F)$ is the expected value functional for all $F$.

Following this result, Chambers and Quiggin [3] proved that, assuming differentiability, decreasing absolute risk aversion implies increasing relative risk aversion.

4. BETWEENNESS AND GÂTEAUX DIFFERENTIABILITY

The last section suggests that in the presence of constant risk aversion, the assumption of Fréchet differentiability is too strong. Weaker notions of differentiability exist, and at least one of them got special attention in the literature.

**Definition 2.** A functional $V$ is Gâteaux differentiable at $F$ (Zeidler [32, p. 191]) if for every $G$,

$$\delta V(F, G - F) := \frac{\partial}{\partial t} V((1 - t) F + t G) \bigg|_{t=0}$$

exists and if $\delta V(F, G - F)$ is a continuous linear function of $G - F$. $V$ is Gâteaux differentiable if it is Gâteaux differentiable at $F$ for every $F$.

If $V$ is Fréchet differentiable, then it is also Gâteaux differentiable, but the opposite is not true. For example, the rank dependent model is Gâteaux, but not Fréchet differentiable (see Chew, Karni, and Safra [8]).

In this section we assume that preferences can be represented by Gâteaux differentiable functionals, and that they satisfy the following betweenness assumption.

**Betweenness.** $F \geq G$ implies that for every $\alpha \in [0, 1]$, $F \geq \alpha F + (1 - \alpha) G \geq G$ (see Chew [4, 5] and Dekel [9]).

We say that the functional $V$ satisfies betweenness if $V(F) \geq V(G)$ implies that for every $\alpha \in [0, 1]$, $V(F) \geq V(\alpha F + (1 - \alpha) G) \geq V(G)$. In this section we characterize functionals that satisfy constant risk aversion, Gâteaux differentiability, and betweenness. It turns out that the only functional to satisfy these three axioms is a special case of Gul [16] disappointment aversion theory (which by itself is a special case of Chew’s semi-weighted utility theory [5]).

---

4 The minimum of two Gâteaux differentiable functionals is not necessarily Gâteaux differentiable. Suppose that for $\alpha \in [0, x^*], V^1(\alpha F + (1 - \alpha) G) > V^2(\alpha F + (1 - \alpha) G)$, but for $\alpha \in [x^*, 1], V^1(\alpha F + (1 - \alpha) G) < V^2(\alpha F + (1 - \alpha) G)$. Let $H = x^* F + (1 - x^*) G$ and let $V^* = \min\{V^1, V^2\}$ to obtain that $\delta V^*(H, F - H) \neq -\delta V^*(H, G - H)$.
**Definition 3.** \( V \) is a Disappointment Aversion functional (see Gul [16]) if it is given by

\[
V(F) = \frac{\gamma(x)}{x} \int_{x > C(F)} u(x) \, dF(x) + \frac{1 - \gamma(x)}{1 - x} \int_{x < C(F)} u(x) \, dF(x),
\]

where \( \alpha \) is the probability that \( F \) yields an outcome above its certainty equivalent \( C(F) \), and \( \gamma(x) = \alpha[1 + (1 - \alpha) \beta] \) for some number \( \beta \).

According to disappointment aversion theory, the decision maker evaluates outcomes that are better than the certainty equivalent of a lottery by using an expected utility functional with a utility function \( u \). He similarly evaluates outcomes that are worse than the certainty equivalent. Finally, the value of a lottery is a weighted sum of these two evaluations.

**Theorem 2.** The following two conditions are equivalent.

1. \( V \) is Gâteaux differentiable, and satisfies constant risk aversion and betweenness.

2. \( V \) is a disappointment aversion functional with the linear utility \( u(x) = x \).

**Proof.** If \( V \) satisfies betweenness, then each of its indifference curves can be obtained from an expected utility functional. Assuming as before that for every \( k \), \( V(\delta_k) = k \), it follows that there are utility functions \( u_k : [0, \infty) \rightarrow \mathbb{R} \) such that \( F \sim \delta_k \) iff

\[
\int u_k(x) \, dF(x) = u_k(k) = k.
\]

Choose \( k > m > 0 \), and let \( (x, p; 0, 1 - p) \sim (k, 1) \). Then \( (mx/k, p; 0, 1 - p) \sim (m, 1) \). By Eq. (10), \( pu_k(x) = k \) and \( pu_m(mx/k) = m \), hence

\[
u_k(x) = \frac{k}{m} u_m \left( \frac{mx}{k} \right).
\]

Let \( \mathcal{F}_{k,m} = \{ F \sim \delta_k : F(k - m) = 0 \} \). That is, \( \mathcal{F}_{k,m} \) consists of those distributions whose certainty equivalent is \( k \), and whose lowest outcome is not less than \( k - m \). On \( \mathcal{F}_{k,m} \) the preference relation \( \succeq \) satisfies \( \int u_k(x) \, dF = k \) and \( \int u_m(x - k + m) \, dF = m \), or equivalently, \( \int u_m(x - k + m) \, dF + k - m = k \). The reason is that \( F \sim \delta_k \) iff \( F - k + m \sim \delta_m \). Consider the two expected utility preferences on \( \{ F : F(k - m) = 0 \} \) that are represented by \( \int u_k(x) \, dF(x) \) and \( \int v(x) \, dF(x) \), where \( v(x) = u_m(x - k + m) + k - m \). Since they share an indifference curve (the one that goes through \( \delta_k \)), they are the same, hence
v is a linear transformation of \( u_k \). Also, since \( u_k(k) = v(k) = k \), there exists \( \theta > 0 \) such that

\[
v(x) = u_m(x - k + m) + k - m = \theta u_k(x) + (1 - \theta) k.
\]

Together with Eq. (11), this implies

\[
u_m(x - k + m) + k - m = \frac{\theta k}{m} u_m \left( \frac{m x}{k} \right) + (1 - \theta) k.
\]

Let \( y = x - k + m \) and obtain

\[
u_m(y) = \frac{\theta k}{m} u_m \left( \frac{m y + m k - m^2}{k} \right) + k - \theta k. \tag{12}
\]

We want to show that \( \theta = 1 \).

Since \( V \) is Gâteaux differentiable, it follows from the proof of Theorem 1 (see Footnote 3) that

\[
\frac{\partial u_m(x)}{\partial x} = \begin{cases} 
s, & x < m \\
t, & x > m.
\end{cases}
\]

For \( y \neq m \) we obtain

\[
u_m'(y) = \theta u_m' \left( \frac{m y + m k - m^2}{k} \right).
\]

But \( m + (m/k)(y - m) \in (y, m) \) (or \( y \in (m, y) \)), hence \( \theta = 1 \). This is the case of disappointment aversion theory, where \( u(x) = x \) and \( \beta = (s/t) - 1 \).

In the Appendix we show that disappointment aversion function with linear utility function \( u \) is Gâteaux differentiable. 

Theorem 2 strongly depends on the assumption that the functional is Gâteaux differentiable. For a functional that satisfies betweenness and is not disappointment aversion (and therefore, by Theorem 2, is not Gâteaux differentiable), see Example 1(3), which is taken from Smorodinsky [27]. In disappointment aversion theory, the certainty equivalent of a lottery serves as a natural reference point, which Gul's axioms explicitly use. Our axioms do not refer to any special point, and the reference point is obtained as part of the results of the model, rather than as part of its assumptions, even if only for a special case of this theory.
5. MIXTURE SYMMETRY

As mentioned in the Introduction, Yaari’s dual theory \[30\], given by
\[ V(F) = \int x \, dg(F(x)) \]
satisfies constant risk aversion. This functional is a special case of the rank dependent model, \( V(F) = \int u(x) \, dg(F(x)) \) where the utility function \( u: \mathbb{R} \to \mathbb{R} \) and the probability transformation function \( g: [0, 1] \to [0, 1] \) are strictly monotonic, \( g(0) = 0 \) and \( g(1) = 1 \) (see Weymark \[29\] and Quiggin \[21\]). This family of functionals received many different axiomatizations (e.g., Chew and Epstein \[6\], Segal \[26\], Quiggin and Wakker \[22\], Wakker \[28\] and Fishburn and Luce \[12\]), but all these axiomatizations make use of the order of the outcomes. For example, Quiggin’s axiom 4 of \[21\] implies expected utility if non-ordered outcomes are allowed.\(^5\)

Since rank dependent functionals evaluate outcomes not only by their value, but also by their relative rank as compared to other possible outcomes, axioms that presuppose attitudes that are based on outcomes’ relative rank are arguably less convincing than axioms that do not make an explicit appeal to such ranks. The aim of this section is to offer what we believe to be the first axiomatization of a non-trivial set of rank dependent functionals where none of the axioms refers to the order of the outcomes, or treats an outcome differently based on its rank. We will use the following terms.

**Mixture Symmetry.** \( F \sim G \) implies for all \( x \in [0, 1], \ xF + (1 - x) G \sim (1 - x) F + xG \) (see Chew, Epstein, and Segal \[7\]).

**Non-betweenness.** There exist \( F \sim G \) and \( x \in [0, 1] \) such that \( F \not> xF + (1 - x) G \).

**Quasi Concavity/Quasi convexity.** \( F \not< G \) implies that for every \( x \in [0, 1], \ xF + (1 - a) G \not< GF/F \not> xF + (1 - x) G \).

Similarly to betweenness, these definitions can be extended to the functional \( V \). The next lemma shows that quasi concavity/betweenness/quasi convexity along one indifference curve implies global quasi concavity/betweenness/quasi convexity. Formally,

**Lemma 2.** Suppose that \( \not< \) satisfies constant risk aversion. If there is \( a^* > 0 \) such that \( F \sim G \not< a^* \) implies for all \( x \in (0, 1), \ (1) \ xF + (1 - x) G \not< F; \ (2) \ F \sim xF + (1 - x) G; \ (3) \ F \not> xF + (1 - x) G \), then the preference relation \( (1) \) is quasi concave; \( (2) \) satisfies the betweenness assumption; \( (3) \) is quasi convex, respectively.

\(^5\) Miyamoto and Wakker \[20\] combine the rank dependent model with absolute and, separately, relative risk aversion. They too assume ordering of outcomes.
Proof. We prove the case of betweenness; the other two cases are similar. Let $F \sim G \sim \delta_a$ for $a \neq 0$. Then

$$\frac{a^*}{a} \times F \sim \frac{a^*}{a} \times G \sim \frac{a^*}{a} \times \delta_a = \delta_{a^*}.$$ 

Hence, by the betweenness assumption,

$$\forall a, \quad \frac{a^*}{a} \times F \sim \frac{a^*}{a} \times F$$

$$\Rightarrow \forall a, \quad \frac{a}{a^*} \times \left( \frac{a^*}{a} \times F \right) \sim \frac{a}{a^*} \times \left( \frac{a^*}{a} \times F \right) + (1 - \alpha) \left( \frac{a^*}{a} \times G \right)$$

$$\Rightarrow \forall a, \quad F \sim aF + (1 - \alpha) G.$$ 

Since all outcomes are non-negative, $F \sim \delta_0$ implies $F = \delta_0$, hence the lemma.

**Theorem 3.** The following two conditions on $\succsim$ are equivalent.

1. It satisfies constant risk aversion, non-betweenness, and mixture symmetry.

2. It can be represented by a rank dependent functional with linear utility (that is, Yaari’s dual theory functional [30]) and quadratic probability transformation function of the form $g(p) = p + cp - cp^2$ for some $c \in (-1, 0]$. 

Proof. Obviously, (2) implies (1), so we show that (1) implies (2). As is proved in [7], mixture symmetry implies that the domain of $\succsim$ can be divided into three regions $A, B$, and $C$, $A > B > C$, such that on $B$, $\succsim$ satisfies betweenness, on $A$ and on $C$, $\succsim$ can be represented by (not necessarily the same) quadratic functional of the form

$$V(x_1, p_1; \ldots; x_n, p_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \varphi_i(x_i, x_j), \quad \ell = A, C,$$ 

where $\varphi_i$ is symmetric, $\ell = A, C$. Moreover, $V$ is quasi concave on $A$ and quasi convex on $C$.

By Lemma 2, only one of the three regions is not empty, and by the non-betweenness assumption, $V$ is quadratic throughout. In other words, $\succsim$ can be represented by a strictly quadratic function $\sum_{i,j} p_i p_j \varphi_i(x_i, x_j)$. Define a function $c: \mathbb{R} \times [0, 1] \to \mathbb{R}$ implicitly by

$$(x, p; 0, 1 - p) \sim (v(x, p), 1).$$  

(14)
For \( \lambda > 0 \), \( r(\lambda x, p) = \lambda r(x, p) \). So for a fixed \( p \) the function \( r \) is homogeneous of degree 1, hence

\[
 r(x, p) = \rho(p) \cdot x. \tag{15}
\]

Assume without loss of generality that \( \varphi(0, 0) = 0 \). Let \( q(x) := \varphi(x, x) \) and \( r(x) := \varphi(x, 0) \), hence \( q(0) = 0 \). Then from Eqs. (13) and (14) it follows that

\[
 p^2 q(x) + 2p(1 - p) r(x) = q(xp(p)). \tag{16}
\]

In Lemma 3 below we prove that the two functions \( q \) and \( \rho \) are trice differentiable. Differentiate both sides three times with respect to \( p \) to obtain

\[
 0 \equiv \rho'''(p) x q''(xp(p)) + 3 \rho''(p) \rho'(p) x^2 q''(xp(p))
  + [\rho'(p)]^3 x^3 q'''(xp(p)). \tag{17}
\]

Since this equation holds for every \( x \), it follows that all the coefficients of \( x \) on the right-hand side of Eq. (17) are zero. In particular, \( \rho'''(p) q''(xp(p)) \equiv 0 \). By monotonicity, \( q' > 0 \), hence \( \rho \) is quadratic. Since by Eq. (14), \( \rho(0) = 0 \) and \( \rho(1) = 1 \), it follows that

\[
 \rho(p) = cp^2 + (1 - c) \cdot p. \tag{18}
\]

Also, \([\rho']^3 q'' \equiv 0 \). Since \( \rho \) is not constant, it follows that \( q''' \equiv 0 \), hence \( q \) is quadratic. In other words, together with the assumption that \( q(0) = 0 \), we obtain

\[
 q(x, x) = ax^2 + bx. \tag{19}
\]

From Eqs. (14) and (15) it follows that \((x, p; 0, 1 - p) \sim (\rho(p)x, 1)\). By constant absolute risk aversion we obtain for every \( x > y \)

\[
 (x - y, p; 0, 1 - p) \sim (\rho(p)[x - y], 1)
 \Rightarrow (x, p; y, 1 - p) \sim (\rho(p)x + (1 - \rho(p)) y, 1)
 \Rightarrow p^2 \varphi(x, x) + (1 - p)^2 \varphi(y, y) + 2p(1 - p) \varphi(x, y)
 = \varphi(\rho(p)x + (1 - \rho(p)) y, \rho(1 - p)) y)
 \Rightarrow 2p(1 - p) \varphi(x, y)
 = a[\rho(p)x + (1 - \rho(p)) y]^2 + b[\rho(p)x + (1 - \rho(p)) y]
 - p^2[ax^2 + bx] - (1 - p^2 [ay^2 + by]].
\]
Substitute Eq. (18) into this last equality to obtain

\[
(2p - 2p^2) \varphi(x, y) = a \left( cp^2 + (1 - c) p \right) x + \left[ 1 - cp^2 - (1 - c) p \right] y^2 \\
+ b \left[ cp^2 + (1 - c) p \right] x + \left[ 1 - cp^2 - (1 - c) p \right] y \\
- p^2 \left[ ax^2 + bx \right] - (1 - p^2) \left[ ay^2 + by \right].
\] (20)

Comparing the coefficients of powers of \( p \) on both sides of this last equation we get for \( p^4 \)

\[
0 = ac^2(x - y)^2.
\]

Hence either \( a = 0 \) or \( c = 0 \). Suppose first that \( c = 0 \), then \( \rho(p) = p \).

Comparing the coefficients of \( p^2 \) in Eq. (20) we obtain

\[
\varphi(x, y) = axy + b \left( x + y \right).
\]

It follows from Eq. (13) that for \( X(x_1, p_1; \ldots; x_n, p_n) \),

\[
V(F_X) = a \sum_i \sum_j p_i p_j x_i x_j + b \sum_i \sum_j p_i p_j (x_i + x_j) \\
= a \mathbb{E}[F_X])^2 + b \mathbb{E}[F_X]
\]

hence \( V \) is an expected value functional, but this contradicts the non-betweenness assumption.

On the other hand, if \( c \neq 0 \) and \( a = 0 \), then by Eq. (19), \( b \neq 0 \). By comparing the coefficients of \( p^2 \) in Eq. (20) we get for \( x > y \)

\[
\varphi(x, y) = \frac{b}{2} \left( (1 - c) x + (1 + c) y \right).
\]

Similarly, for \( y > x \),

\[
\varphi(x, y) = \frac{b}{2} \left( (1 - c) y + (1 + c) x \right).
\]

By the monotonicity of \( \varphi \) we may assume, without loss of generality, that \( b = 1 \). We thus obtain that

\[
\varphi(x, y) = \frac{b}{2}(1 - c)(x + y) + c \min\{x, y\}.
\]

Following Chew, Epstein, and Segal [7, Sect. 3, especially footnote 5], we obtain

\[
V(F_X) = (1 - c) \mathbb{E}[F_X] + c \int x \, d\text{g}^*(F_X(x)),
\] (21)
where $g^*(p) = 1 - (1 - p)^2$. Alternatively, $V(F_X) = \int x \, d(g(F_X(x)))$, where $g(p) = p + cp - cp^2$.

**Lemma 3.** The functions $q$ and $p$ of the proof of Theorem 3 are trice differentiable.

**Proof.** By monotonicity, both $q$ and $p$ are increasing functions, hence almost everywhere differentiable. The left hand side of Eq. (16) is always differentiable with respect to $p$, hence so is the right hand side of this equation. Suppose $p$ is not differentiable at $p^*$. Since $q$ is almost everywhere differentiable, there is $x$ such that $q$ is differentiable at $xp(p^*)$, a contradiction. Since $p$ is differentiable, it follows by the differentiability of the left hand side of Eq. (16) that so is $q$. Differentiating both sides of Eq. (16) with respect to $p$ we thus obtain

$$2pq(x) + (2 - 4p) r(x) = p'(p) xq'(xp(p)). \quad (22)$$

The right-hand side of Eq. (16) is differentiable with respect to $x$, and since $q$ is differentiable, so is $r$. It follows that the left-hand side of Eq. (22) is differentiable with respect to $x$, hence so is the right-hand side of this equation, in other words, $q''$ exists. Since $p$ is differentiable, $\partial q'(p)/\partial p$ exists, and since the left-hand side of Eq. (22) is differentiable with respect to $p$, so must be $p'(p)$. In other words, $p$ is twice differentiable. Differentiating both sides of Eq. (22) with respect to $p$ we obtain

$$2q(x) - 4r(x) = [p'(p)]^2 x^2 q''(xp(p)) + p''(p) xq'(xp(p)). \quad (23)$$

Similarly to the above analysis, since both sides of Eq. (23) are differentiable with respect to $x$, $q'''$ exists, and since both sides are differentiable with respect to $p$, $p'''$ exists.

Since betweenness implies mixture symmetry, and since preferences satisfy either betweenness or non-betweenness, Theorems 2 and 3 imply the following corollary.

**Corollary 1.** Let $V$ represent the preferences $\succsim$. Then the following two conditions are equivalent.

1. The preferences $\succsim$ satisfy constant risk aversion and mixture symmetry, and $V$ is a Gâteaux differentiable functional.

2. The preferences $\succsim$ can be represented either by Gul’s disappointment aversion functional with the linear utility $u(x) = x$ or by a dual theory functional with a quadratic probability transformation function of the form $g(p) = p + cp - cp^2$ for some $c \in [-1, 0) \cup (0, 1]$. 

---

**CONSTANT RISK AVERSION**
Consider again Eq. (21). When \( c = 1 \), \( V(F_X) = \int x \, dg(F_X(x)) \), where \( g(p) = p^2 \), and when \( c = 1 \), \( V(F_X) = \int x \, dg^*(F_X(x)) \), which is the Gini measure of income inequality. Since \( \rho(p) \) is monotonic (see Eq. (15)), it follows by Eq. (18) that \( c \in [-1, 1] \). Since \( g^* \) is concave it represents risk aversion (see Yaari [30] and Chew, Karni, and Safra [8]), hence \( \int x \, dg^*(F(x)) < E[F_X] \). In other words, the Gini measure is the lower bound of all the monotonic functionals that satisfy the assumptions of Theorem 3.

6. ZERO INDEPENDENCE

Suppose \( F \sim G \). What will be the relation between \( qF + (1 - q) \delta_0 \) and \( qG + (1 - q) \delta_0 \)? Some existing empirical evidence suggests that if \( x > y > 0 \) and \( (x, p; 0, 1 - p) \sim (y, 1) \), then \( (x, qp; 0, 1 - qp) \sim (y, q; 0, 1 - q) \) (this is called the common ratio effect—see Allais [2], MacCrimmon and Larsson [18], or Kahneman and Tversky [17]). Assuming the rank dependent model, Segal [25, 26] connected this effect to the elasticity of the probability transformation function \( g \), and showed that if \( qF \sim G \) iff \( F + (1 - q) \delta_0 \sim qG + (1 - q) \delta_0 \) for all \( F \) and \( G \) with two outcomes at most, then \( g(p) = 1 - (1 - p)^t \) for some \( t > 0 \). Stronger results were achieved by Grant and Kajii [15], who proved the same for a wider set of initially possible functionals. They assume that preferences can be represented by a measure of the epigraphs of cumulative distribution functions, \( \sigma \) and that if \( F(z) = G(z) = 1 \), then \( F \succeq G \) implies that for all \( q \in (0, 1) \), \( qF + (1 - q) \delta_x \succeq qG + (1 - q) \delta_x \). In this section we prove that to a certain extent, it is enough to assume constant risk aversion (although in that case the utility function will have to be linear). The formal assumption we use is the following.

Zero Independence. \( F \sim G \) iff for all \( q \in [0, 1] \), \( qF + (1 - q) \delta_0 \sim qG + (1 - q) \delta_0 \).

Slightly weaker assumptions were introduced by Rubinstein, Safra, and Thomson [24], and later by Grant and Kajii [14], for the derivation of a preference-based Nash bargaining solution that applies to generalized expected utility preferences. The outcome of zero is considered there to be the disagreement outcome (the outcome that the bargainers receive if they fail to reach an agreement). In [24] the similar assumption is called homogeneity and it requires that zero independence should hold for \( G = \delta_x \). In [14] it is called weak homogeneity and it requires that zero independence should hold for \( G = \delta_x \) and for \( F = p \delta_x + (1 - p) \delta_y \). These assumptions play a crucial role in establishing the existence and the uniqueness of the ordinal Nash solution.

The rank dependent model is a special case of this family.
To prove Theorem 4 we will modify a result from Gilboa and Schmeidler [13] and assume that preferences satisfy diversification (see below). As in Section 2, let $\Omega = (\mathcal{S}, \Sigma, P)$ be a measure space and let $\mathcal{X}$ be the set of all measurable bounded random variables on it.

**Diversification.** Let $X, Y \in \mathcal{X}$. If $X \sim Y$, then $\alpha X + (1 - \alpha) Y \succcurlyeq X$ for all $0 < \alpha < 1$. Equivalently, for $G, H \in \mathcal{F}$, if $G \sim H$, then for all $X$ and $Y$ such that $G = F_X$ and $H = F_Y$, and for all $0 < \alpha < 1$, $F_{\alpha X + (1 - \alpha) Y} \succcurlyeq G$.

**Lemma 4.** The following two conditions are equivalent.

1. $V$ satisfies constant risk aversion and diversification.
2. There exists a unique set $T$ of increasing, concave, and onto functions over $[0, 1]$ such that $\succcurlyeq$ can be represented by

   $$V(F_X) = \min_{g \in T} \left\{ \int x \, d(F_X^g(x)) \right\}.$$ 

**Proof.** Clearly (2) implies (1) (see Lemma 1 in Section 2 above), so we prove here that (1) implies (2). First note that for each $X \in \mathcal{X}$ with certainty equivalent $x$, the set $\{ Y : Y \succcurlyeq X \}$ is a convex cone with a vertex at $x$. The reason is that for $X \sim x$, constant risk aversion implies that for all $\alpha \geq 0$, $\alpha X + (1 - \alpha) x \sim x$, and diversification implies convexity of upper sets. Constant risk aversion also implies that all these cones are parallel shifts of each other.

The conditions in (1) imply the axioms of Gilboa and Schmeidler [13]. These authors discuss preference relations over a set of “horse-lotteries” that is, acts with subjective probabilities whose consequences are (possibly degenerate) roulette lotteries. They show, in their Theorem 1 and Proposition 4.1, that a preference relation satisfies their axioms if, and only if, it can be represented as a minimum over a family of subjective expected utility functionals. Translated into our framework, their results imply the existence of a compact set $\mathcal{C}$ of finitely additive measures on $\Omega$ such that $X \succ Y$ iff $\min_{Q \in \mathcal{C}} \{ X \, dQ \} \geq \min_{Q \in \mathcal{C}} \{ Y \, dQ \}$. For a given $X$, let $F^P_X$ denote the distribution function of $X$ with respect to the measure $Q$ ($F^P_X$ is denoted by $F_X$ as before) and let $V : \mathcal{F} \to \mathbb{R}$ be defined by $V(F_X) = \min_{Q \in \mathcal{C}} \{ z \, dF^Q_X(z) \}$. Then $F_X \succcurlyeq F_Y$ iff $V(F_X) \geq V(F_Y)$. The following claim explains the relationship between the measures in $\mathcal{C}$ and the given probability measure $P$.

**Claim 2.** Every $Q \in \mathcal{C}$ is absolutely continuous with respect to $P$. That is, for all $Q$ and $E \in \mathcal{S}$, $P(E) = 0 \Rightarrow Q(E) = 0$. 

Proof of Claim 2. Let $E$ be such that $P(E) = 0$ and let $[z : w]$ stand for the random variable $[z$ on $E; w$ on $E^c]$ ($E^c$ is the complement of $E$). By monotonicity with respect to first-order stochastic dominance, $w_1 > w_2$ implies $[z_1 : w_1] > [z_2 : w_2]$. Let $q = \max_{Q \in \mathcal{G}} \{Q(E)\}$. If there exists $Q \in \mathcal{G}$ such that $Q(E) > 0$, then $q > 0$ and $[4; 5] \ni [5 - q; 5 - (q/2)]$.

On the other hand, for $z \leq w$,

$$\min_{Q \in \mathcal{G}} \{ zQ(E) + w(1 - Q(E)) \} = \min_{Q \in \mathcal{G}} \{ w + (z - w)Q(E) \} = w + (z - w)q.$$  

Therefore

$$\min_{Q \in \mathcal{G}} \left\{ (5 - q)Q(E) + \left( 5 - \frac{q}{2} \right)(1 - Q(E)) \right\} = 5 - \frac{q}{2} - \frac{q^2}{2}.$$  

On the other hand,

$$\min_{Q \in \mathcal{G}} \{ 4Q(E) + 5(1 - Q(E)) \} = 5 - q.$$  

Hence $[5 - q; 5 - (q/2)] \ni [4; 5]$, a contradiction.  

Claim 2 implies that, for all $X, Q, z_1,$ and $z_2$,

$$F_X(z_1) = F_X(z_2) \Rightarrow F^Q_X(z_1) = F^Q_X(z_2).$$

Consider now a given $X \in \mathcal{X}$ and a measure $Q \in \mathcal{G}$, and define a function $g_{Q,X} : [0, 1] \rightarrow [0, 1]$ by

$$g_{Q,X}(p) = \begin{cases} F_X^{-1}(F_Q^{-1}(p)), & p \in \text{Image}(F_X) \\ 0, & p = 0 \\ l(p), & \text{otherwise}, \end{cases}$$

where $l$ is the piece-wise linear function, defined on the complement of the image of $F_X$, that makes $g_{Q,X}$ continuous. By the claim, $g_{Q,X}$ is well defined. Clearly, it is onto and non-decreasing, and it satisfies $\int z \, dF^Q_X(z) = \int z \, dg_{Q,X}(F_X(z)).$

For each $X \in \mathcal{X}$ there exists $Q(X) \in \mathcal{G}$ such that

$$V(F_X) = \min_{Q \in \mathcal{G}} \left\{ \int z \, dF^Q_X(z) \right\} = \int z \, dF^Q_X(z) = \int z \, dg_{Q,X}(F_X(z)),$$

where $g_X = g_{Q(X),X}$.

Let $\text{Com}(X) = \{ Y : \forall s_1, s_2 \in S, (X(s_1) - X(s_2))(Y(s_1) - Y(s_2)) \geq 0 \}$ (that is, the set of all random variables that are comonotone with $X$). Restricted
to Com(\(X\)), indifference sets of \(\{z \ d g_Z(F_Y(z))\}\) are hyperplanes in \(\mathcal{X}\). Therefore, for \(Y \in \text{Com}(X)\),

\[
V(F_Y) = \min_{Z \in \text{Com}(X), Z \neq Y} \left\{ \left\| z \ d P^Y_Z(z) \right\| \right\},
\]

\[
= \min_{Z \in \text{Com}(X), Z \neq Y} \left\{ \left\| z \ d g_Z(F_Y(z)) \right\| \right\}.
\]

Next, we discuss the case of non-comonotone random variables. Define \(W^*: \mathcal{F} \to \mathbb{R}\) by

\[
W^*(F_X) = \min_{Z \in \mathcal{F}, Z \neq X} \left\{ \left\| z \ d g_Z(F_X(z)) \right\| \right\}.
\]

By definition, \(W^*(F_X) \leq V(F_X)\). Suppose there exists \(X\) such that \(W^*(F_X) = \bar{x} < x = V(F_X)\). Then there exists \(\bar{X} \notin \text{Com}(X)\) such that \(W^*(F_{\bar{X}}) = \bar{x}\),

Let \(\bar{\mathcal{F}}^n\) be the set of all random variables \(Y \in \mathcal{F}\) that have at most \(n\) different outcomes \(y_1, \ldots, y_n\) and satisfy \(\Pr(Y = y_i) = 1/n\). Assume first that there exist \(n\) and \(X\) such that \(X \in \bar{\mathcal{F}}^n\) and \(F_X = F_{\bar{X}}\). Clearly, \(Q(X) = Q(\bar{X})\). Therefore, \(g_{\bar{X}} = g_X\), which implies \(W^*(F_{\bar{X}}) = V(F_X)\).

If there is no such \(n\) then, by continuity, there exists \(n\) large enough for which there exist \(X, \bar{X} \in \bar{\mathcal{F}}^n\) that satisfy \(W^*(F_{\bar{X}}) = \bar{x}\) and \(V(F_{\bar{X}}) > \bar{x} + 1/2(x - \bar{x})\), contradiction. Hence, for \(\mathcal{F} = \{g_Z: Z \in \mathcal{F}\}\),

\[
V(F_X) = \min_{g \in \mathcal{F}} \left\{ \left\| z \ d g(F_X(z)) \right\| \right\}.
\]

It remains to show that the functions \(g_Z\) are concave. Assume, without loss of generality, that there exist \(n\) and \(X \in \text{int}(\bar{\mathcal{F}}^n)\) such that, for some \(i_0 \in \{2, \ldots, n - 1\}\),

\[
2g_X \left( \frac{i_0}{n} \right) < g_X \left( \frac{i_0 - 1}{n} \right) + g_X \left( \frac{i_0 + 1}{n} \right).
\]

Let \(Q = Q(X)\) and denote \(q_i = Q(X = x_i)\). By definition, \(g_X(i/n) = \sum_{j=1}^{i} q_j\). Therefore, \(q_{i_0} < q_{i_0 + 1}\). Take \(\varepsilon > 0\) small enough and consider \(X(\varepsilon) \in \text{int}(\bar{\mathcal{F}}^n)\) with the values \(x_1 < \cdots < x_{i_0} - \varepsilon < x_{i_0 + 1} + \varepsilon < \cdots < x_n\). By the construction of \(Q\), \(X(\varepsilon) > X\). This, however, contradicts risk aversion (note that risk aversion is implied by diversification, see Dekel [10]).
Theorem 4. The following two conditions on \( p \) are equivalent.

1. It satisfies constant risk aversion, diversification, and zero independence.
2. It can be represented by \( V(F) = \int x \, dg(F(x)), \) where \( g(p) = 1 - (1 - p)^t \) for some \( t \geq 1 \).

Proof. Obviously (2) \( \Rightarrow \) (1). We prove that (1) \( \Rightarrow \) (2) for finite lotteries (that is, for lotteries with a finite number of different outcomes) by induction on the number of the nonzero outcomes. Continuity is then used to get the desired representation for all \( F \in \mathcal{F} \).

By Lemma 4 there is a family of probability transformation functions \( \{g_x : x \in \mathcal{X}\} \) such that for every \( F \), \( V(F) = \min_x \int x \, dg_x(F(x)) \). For lotteries of the form \( (x, p; 0, 1 - p) \) we obtain

\[
V(x, p; 0, 1 - p) = \min_x [1 - g_x(1 - p)].
\]

Define \( f_x(p) = 1 - g_x(1 - p) \) and \( h(p) = \min_x f_x(p) \) and obtain that \( V(x, p; 0, 1 - p) = xh(p) \).

By zero independence, \( (x, p; 0, 1 - p) \sim \delta_x \) implies for all \( q \in [0, 1] \), \( (x, pq; 0, 1 - pq) \sim (y, q; 0, 1 - q) \), hence \( xh(p) = y \) and \( xh(pq) = yh(q) \). Combining the two we obtain

\[
h(pq) = h(p)h(q).
\]

The solution of this functional equation is \( h(p) = p^t \) (see Aczél [1, p. 41]).

Suppose we have already proved that for lotteries with at most \( n \) prizes \( V(F) = \int x \, dg(F(x)) = \int x \, d[1 - (1 - F(x))^t] = \int x \, d[(1 - F(x))^t] \). That is, for \( x_1 \leq \cdots \leq x_n, \)

\[
V(x_1, p_1; \ldots; x_n, p_n) = x_n p_n^t + \sum_{i=1}^{n-1} x_i \left[ \left( \sum_{j=i+1}^{n} p_j \right)^t - \left( \sum_{j=i}^{n} p_j \right)^t \right].
\]

We will now prove it for \( n + 1 \). Let \( x_1 \leq \cdots \leq x_{n+1} \), and consider the lottery \( X = (x_1, p_1; \ldots; x_{n+1}, p_{n+1}) \). By constant risk aversion,

\[
V(F_X) = x_1 + V(F_{X - x_1}) = x_1 + V(0, p_1; \ldots; x_{n+1} - x_1, p_{n+1}).
\]

By continuity there is \( y \) such that \( F_X - x_1 \sim (y, 1 - p_1; 0, p_1) \). By zero independence \( F_{X - x_1} \sim \delta_y \), where

\[
X^* = \left( x_2 - x_1, \frac{p_2}{1 - p_1}; \ldots; x_{n+1} - x_1, \frac{p_{n+1}}{1 - p_1} \right).
\]
By the induction hypothesis, $V(F_{x+}) = V(\delta_x)$ implies

$$
(x_{n+1} - x_1) \left( \frac{p_{n+1}}{1-p_1} \right) + \sum_{i=2}^{n} (x_i - x_1) \left[ \left( \frac{n}{\sum_{j=i}^{n+1} p_j} \right) \right] = y. \quad (24)
$$

Also, since $F_{x-x_1} \sim (y, 1-p_1; 0, p_1)$, it follows by the constant risk aversion properties of $V$ that $V(F_{x}) = x_1 + y(1-p_1)'.$ Substitute into Eq. (24) to obtain

$$
V(F_{x}) = x_1 + (1-p_1)' \left\{ (x_{n+1} - x_1) \left( \frac{p_{n+1}}{1-p_1} \right) \right\} \left( \sum_{i=1}^{n} x_i \right) \left[ \left( \frac{n}{\sum_{j=i}^{n+1} p_j} \right) \right] = x_{n+1} p'_{n+1} + \sum_{i=1}^{n} x_i \left[ \left( \frac{n}{\sum_{j=i}^{n+1} p_j} \right) \right]$$

which proves the induction hypothesis.

The functional form of Theorem 4 has been previously appeared in the literature on income distribution under the name of “the S-Gini family” (see Donaldson and Weymark [11] and Yitzhaki [31]).

APPENDIX: DISAPPOINTMENT AVERSION THEORY AND GÂTEAU X DIFFERENTIABILITY

In this appendix we prove that the disappointment aversion with linear utility is Gâteaux differentiable. In our case, we assume that $V(\delta_k) = k$. Therefore, for fixed $s$ and $t$, the value of $V$ at $F$ is the number $k$ that solves the implicit equation

$$
\int_0^k \left[ s x + (1-s) k \right] dF(x) + \int_k^\infty \left[ t x + (1-t) k \right] dF(x) - k = 0.
$$

For given $F$ and $G$, let $H_x = xG + (1-x) F$, and obtain that the value of $V(H_x)$ is the number $k(x)$ that solves

$$
\int_0^{k(x)} \varphi(x, k(x)) dH_x(x) + \int_{k(x)}^\infty \psi(x, k(x)) dH_x(x) - k(x) = 0,
$$

CONSTANT RISK AVERSION

39
where \( \varphi(x, k(x)) = sx + (1-s) k(x) \) and \( \psi(x, k(x)) = tx + (1-t) k(x) \).

Define

\[
J(x, k) = \int_0^k \varphi(x, k) \, dH_x(x) + \int_k^\infty \psi(x, k) \, dH_x(x) - k
\]

\[
= \int_0^\infty \psi(x, k) \, dH_x(x) + (s-t) \int_0^k (x-k) \, dH_x(x) - k.
\]

By the implicit function theorem, \( \partial k/\partial x = -J_x/J_k \). Trivially,

\[
\frac{\partial J}{\partial x} = \int_0^k \varphi(x, k) \, d[G(x) - F(x)] + \int_k^\infty \psi(x, k) \, d[G(x) - F(x)].
\]

(25)

We compute next the derivative of \( J \) with respect to \( k \). We need only the derivative at \( x = 0 \), in which case \( H_x = F \). Therefore, the derivative of \( J \) with respect to \( k \) at \( x = 0 \) is given by

\[
(1-t) \int_0^\infty dF(x) + (s-t) \lim_{\varepsilon \to 0} \int_0^{k+\varepsilon} (x-k-\varepsilon) \, dF(x) - \int_0^k (x-k) \, dF(x) - 1
\]

\[
= t + (s-t) \lim_{\varepsilon \to 0} \int_0^{k+\varepsilon} (x-k-\varepsilon) \, dF(x) - \int_0^k (x-k) \, dF(x)
\]

\[
+ (s-t) \lim_{\varepsilon \to 0} \int_0^{k+\varepsilon} (x-k) \, dF(x)
\]

\[
= -t - (s-t) \lim_{\varepsilon \to 0} F(k+\varepsilon) + (s-t) \lim_{\varepsilon \to 0} \int_0^{k+\varepsilon} (x-k) \, dF(x).
\]

The expression

\[
\int_0^{k+\varepsilon} (x-k) \, dF(x)
\]

(26)

is bounded from above by the maximal change in \( x - k \) multiplied by the maximal change in the cumulative probability, that is, by \( e[F(k+\varepsilon) - F(k)] \).

The integral at Eq. (26) is bounded from below by \( \varepsilon \) multiplied by the minimal possible change in the value of \( F \), namely by \( \lim_{\varepsilon \to 0} F(k+\varepsilon) - F(k) \).

Hence, \( e^{-1} \) times the integral at Eq. (26) is between \( \lim_{\varepsilon \to 0} F(k+\varepsilon) - F(k) \) and \( F(k+\varepsilon) - F(k) \). It thus follows that

\[
(s-t) \lim_{\varepsilon \to 0} \int_0^{k+\varepsilon} (x-k) \, dF(x) = (s-t) \lim_{\varepsilon \to 0} F(k+\varepsilon) - F(k)
\]
and
\[ \frac{\partial J}{\partial k} \bigg|_{x=0} = -t - (s - t) F(k) = t(1 - F(k)) - sF(k) < 0. \]

From Eq. (25) it now follows that
\[ \frac{\partial k}{\partial x} \bigg|_{x=0} = \frac{\int_0^x \rho(x, k) d[G(x) - F(x)] + \int_k^\infty \rho(x, k) d[G(x) - F(x)]}{t + (s - t) F(k)}. \]

Obviously, this expression is continuous in $G - F$ and also linear in $G - F$.

REFERENCES