A sufficient condition for additively separable functions

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This paper presents a set of sufficient conditions under which a completely separable function on an open \( S \subset \mathbb{R}^N \) is additively separable. The new condition is that different connected elements of intersections of parallel-to-the-axes hyperplanes with the domain \( S \) are intersected by common indifference surfaces.

Key words: Completely separable functions; Additively separable functions

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1. Introduction

Additively separable functions on \( \mathbb{R}^N \) are of the form \( \sum_{i=1}^{N} u_i(x_i) \). These functions have the obvious property that they are completely separable. That is, the induced orders on the \((N-1)\)-dimensional sets \( \{(x_1, \ldots, x_N): x_i_0 = c\} \) are independent of \( c \). A natural question is whether the opposite is also true, that is, whether complete separability of a function \( V \) implies additive separability. Debreu (1954) proved this to be the case whenever the domain of \( V \) is a product of intervals \( \pi_1 \times \cdots \times \pi_N \subset \mathbb{R}^N \). Wakker (1989) extended this result to ordered cones (i.e., sets in \( \mathbb{R}^N \) where \( x_1 \geq \cdots \geq x_N \geq 0 \)). He also pointed out the importance of the requirement that indifference surfaces are connected. In Segal (1991) I showed that one can relax the assumption that the domain \( S \) is an ordered cone and that it is sufficient to require that intersections of the domain of the function with parallel-to-the-axes hyperplanes are connected sets. Chateauneuf and Wakker (1993) extended these results to general product spaces. For further references, see this last paper. See also Blackorby et al. (1978) for a survey of applications of additively separable functions in economics.

In this paper I offer a further relaxation of the above mentioned requirements. Intersections of parallel-to-the-axes hyperplanes with the
domain do not have to be connected, as long as connected elements of such intersections can be tied up through the indifference surfaces of $V$ (see next section for details and section 3 for a proof). In section 4 I discuss the necessity of this condition and present some relevant examples. Topological definitions and claims are taken from Dugundji (1966).

2. Definitions

Let $S$ be connected open subset of $\mathbb{R}^N$, $N \geq 3$, and let $V$ be a strictly monotonic function on it. That is, $\forall i \in \{1, \ldots, N\}$ \( y_i \geq x_i \) and $\exists i \in \{1, \ldots, N\}$ such that $y_i > x_i \Rightarrow V(y) = V(y_1, \ldots, y_N) > V(x) = V(x_1, \ldots, x_N)$. For $x \in S$, let $I(x) = \{ y \in S : V(y) = V(x) \}$ be the indifference surface of $V$ through $x$. We assume throughout that all indifference surfaces of $V$ are connected subset of $\mathbb{R}^N$.

Let $\pi_i(S)$ be the projection of $S$ on the $i$th axis,

$$\pi_i(S) = \{ x_i : \exists (x_1, \ldots, x_i, \ldots, x_N) \in S \}.$$

Since $S$ is an open connected subset of $\mathbb{R}^N$, $\pi_i(S)$ is an open interval, $i = 1, \ldots, N$. In the sequel, the term open box means a set of the form $\prod_{i=1}^N J_i$, where $J_i$ is a nonempty bounded open interval in $\pi_i(S)$, $i = 1, \ldots, N$.

**Definition 1.** The function $V: S \to \mathbb{R}$ is called completely separable if for every $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N)$, $(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N)$, $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N)$, and $(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N)$ in $S$,

$$V(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) \geq V(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_N)$$

$$\Leftrightarrow V(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N) \geq V(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N).$$

**Definition 2.** The function $V: S \to \mathbb{R}$ is called additively separable if there exist continuous and strictly increasing functions $u_i: \pi_i(S) \to \mathbb{R}$, $i = 1, \ldots, N$, and a strictly increasing function $\zeta: \text{Rng}(\sum_{i=1}^N u_i(\cdot)) \to \mathbb{R}$ such that $V(x_1, \ldots, x_N) = \zeta(\sum_{i=1}^N u_i(x_i))$, where $\text{Rng}(\sum_{i=1}^N u_i(\cdot)) = \{ t \in \mathbb{R} : \exists (x_1, \ldots, x_N) \in \mathbb{R}^N$ such that $t = \sum_{i=1}^N u_i(x_i) \}$.

**Definition 3.** The function $V: S \to \mathbb{R}$ is called locally separable if, for every $x \in S$, there is neighborhood $N(x)$ of $x$ on which the function $V$ is additively separable.
sufficient to prove that local separability implies additive separability there. One should note, however, that local separability is strictly weaker than complete separability. The function in Example 1 is locally separable, but not completely separable. This distinction becomes relevant in the proof of Lemma 2 below.

**Example 1.** Let

\[ A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 10 < x_1 < 15, 0 < x_2 < 2, x_3 \geq 0, x_1 + x_3 < 16\}, \]

\[ B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < 15, 0 < x_2 < 8, -1 < x_3 < 0\}, \]

\[ C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < 5, 6 < x_2 < 8, x_3 \geq 0, x_1 + 2x_3 < 10\}; \]

Let \( S = A \cup B \cup C \) and define \( V : S \to \mathbb{R} \) by

\[
V(x_1, x_2, x_3) = \begin{cases} 
x_1 + x_2 + x_3, & (x_1, x_2, x_3) \in A \cup B \\
x_1 + x_2 + 2x_3, & (x_1, x_2, x_3) \in C
\end{cases}
\]

This function is locally separable, but not completely separable. For example, \( V(1, 7, 4) = V(11, 1, 4) = 16 \), but \( V(1, 7, 3) = 14 \) while \( V(11, 1, 3) = 15 \).

For every \( i \in \{1, \ldots, N\} \) and \( c \in \mathbb{R} \), let \( S(i, c) = \{(x_1, \ldots, x_N) \in S : x_i = c\} \). Since \( S \) is open, \( S(i, c) \) is relatively open in \( \mathbb{R}^{N-1} \). We say that the connected set \( T \subseteq S \) is a maximal connected element of \( S(i, c) \) if there is no connected set \( T' \) such that \( T \subseteq T' \subseteq S(i, c) \). Let \( \mathcal{A} \) be the set of maximal connected elements of \( S(i, c) \). That is,

1. \( S(i, c) = \bigcup_{T \in \mathcal{A}} T \); and
2. for every \( T \in \mathcal{A} \), \( T \) is a maximal connected element of \( S(i, c) \).

The set \( \mathcal{A} \) is countable. To see this, observe that if the intersection of two connected sets is not empty, then their union is connected. Since \( S(i, c) \) is (relatively) open, each point in it has a (connected) neighborhood around it in \( S(i, c) \). Therefore, a maximal connected subset of \( S(i, c) \) is open. It is well known that \( \mathbb{R}^{N-1} \) contain at most countable disjoint nonempty open sets. Hence \( \mathcal{A} = \{T^1, T^2, \ldots\} \). For \( T \in \mathcal{A} \), define

\[
\mathcal{A}^1(T) = \{T' \in \mathcal{A} : \text{there exists } x \in T \text{ and } y \in T' \text{ such that } V(x) = V(y)\},
\]

\[
\mathcal{A}^d(T) = \bigcup_{T' \in \mathcal{A}^{d-1}(T)} \mathcal{A}^1(T'),
\]

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\]
Lemma 1. \( T' \in \mathcal{A}(T) \Leftrightarrow T \in \mathcal{A}(T') \).

Proof. \( T' \in \mathcal{A}(T) \Leftrightarrow \exists l \) such that \( T' \in \mathcal{A}^l(T) \). By definition there are \( T^{i_1}, \ldots, T^{i_l} \) such that \( T^{i_1} = T \) and \( T^{i_l} = T' \), and there are \( x^{2,1}, x^{1,2}, x^{2,2}, \ldots, x^{2,1}, x^{1,1} \) such that \( x^{2,1} \in T^{i_1}, x^{1,1} \in T^{i_l} \), for every \( 2 \leq m \leq l-1 \), \( x^{1,m}, x^{2,m} \in T^{i_m} \), and \( V(x^{2,m}) = V(x^{1,m+1}) \), \( m = 1, \ldots, l-1 \). Therefore, \( T \in \mathcal{A}(T') \), hence \( T \in \mathcal{A}(T') \). \( \square \)

The set \( \mathcal{A} \) can thus be partitioned into equivalence classes \( \mathcal{A}_1, \ldots \) where \( T, T' \) belong to the same equivalence class \( \mathcal{A}_m \) if and only if \( T \in \mathcal{A}(T') \) [and \( T' \in \mathcal{A}(T) \)]. For a given \( i \) and \( c \), denote this set of equivalence classes by \( \mathcal{C}(i,c) \) and denote its cardinality by \( |\mathcal{C}(i,c)| \). In the sequel we will be interested in functions and sets where \( |\mathcal{C}(i,c)| = 1 \), that is, in cases where for every \( T \) and \( T' \), \( T' \in \mathcal{A}(T) \).

3. Sufficient conditions

This section presents sufficient conditions under which a completely separable function must also be additively separable. These conditions are weaker than those discussed in Segal (1991). Necessity is discussed in the next section.

Let \( V \) be a continuous and strictly monotonic completely separable (hence locally separable) function on \( S \), an open subset of \( \mathbb{R}^N \). Let \( x^* = (x_1^*, \ldots, x_N^*) \) and \( y^* = (y_1^*, \ldots, y_N^*) \) in \( S \) such that \( V(x^*) = V(y^*) \) and let \( R^* \) be two open boxes in \( S \) such that \( x^* \in R^* \) and \( y^* \in R^* \). By local separability it follows that for \( z = x^*, y^* \), \( V \) on \( R^2 \) is given by

\[
V(x_1, \ldots, x_N) = \zeta^z \left( \sum_{i=1}^{N} u_i^z(x_i) \right)
\]

for some strictly monotonic functions \( \zeta^x \) and \( \zeta^y \).

Lemma 2. Under the above conditions, there are \( a^* > 0, b^*, \) and \( \epsilon^* > 0 \), such that for every \( x_{i_0} = (x_{i_0}^* - \epsilon^*, x_{i_0}^* + \epsilon^*) \), \( u_{i_0}^z(x_{i_0}) = a^* u_{i_0}^z(x_{i_0}) + b^* \).

Proof. Suppose, without loss of generality, that \( i_0 \neq 1 \). For \( i \in \{1, \ldots, N\} \), let \( e_i = (e_1^i, \ldots, e_N^i) \), where \( e_j^i \) equals 1 if \( i = j \) and 0 otherwise. Let \( \epsilon^*, \epsilon^{k,2} > 0 \), \( z = x^*, y^* \), \( k = 1,2 \), such that

1. \( z + (-1)^k \epsilon^{k,2} e^1 + (-1)^k \epsilon^* e^{i_0} \in R(z), z = x^*, y^*, k = 1,2 \); and
2. \( V(x^* +(-1)^k \epsilon^{k,2} e^1) = V(y^* +(-1)^k \epsilon^{k,2} e^1) \), \( k = 1,2 \).
For $z = x^*, y^*$, let $E^z = \{ (w_1, \ldots, w_N) \in \mathbb{R}^N: w_i \in [z_i - \varepsilon_i, z_i + \varepsilon_i], w_{i_0} \in [z_{i_0} - \varepsilon^*, z_{i_0} + \varepsilon^*] \}$, and $w_i = z_i, i \in \{1, \ldots, N \} \setminus \{1, i_0\}$. By condition 1 above, $E^z \subset E(z), z = x^*, y^*$. By condition 2, continuity, and monotonicity it follows that there is a strictly increasing, continuous, and onto function $\rho: [y^*_1 - \varepsilon^1, y^*_1 + \varepsilon^2, y^*] \rightarrow [x^*_1 - \varepsilon^1, x^*_1 + \varepsilon^2]$ such that for $w_i \in [y^*_1 - \varepsilon^1, y^*_1 + \varepsilon^2, y^*], V(w_1, y^*_2, \ldots, y^*_N) = V(\rho(w_1), x^*_2, \ldots, x^*_N)$. Since the function $V$ is completely separable, it follows that for every $w_{i_0} \in [x_{i_0} - \varepsilon^*, x_{i_0} + \varepsilon^*]$, $V(w_1, y^*_2, \ldots, y^*_N) = V(\rho(w_1), x^*_2, \ldots, w_{i_0}, \ldots, x^*_N)$. (2)

(This last equation requires complete, and not only local, separability.)

The functions $\zeta^x$ and $\zeta^y$ are strictly monotonic and therefore have an inverse. It follows therefore by eq. (1) and (2) that on $E^x$:

$$u_i^x(w_1) + u_{i_0}^x(w_{i_0}) + \sum_{i \neq 1, i_0} u_i^y(y^*) = \zeta^x\left(\sum_{i \neq 1, i_0} u_i^y(y^*), x_{i_0}, + b^*ight).$$

This last equation implies that $\zeta^x \circ \zeta^y$ is affine [see Aczél (1966)]. That is, $\zeta^x \circ \zeta^y(x) = ax + b$. By the monotonicity of $\zeta^x$ and $\zeta^y$, $a > 0$. Denote $a^* = a$, and $b^* = a \sum_{i \neq i_0} u_i^x(x^*) - \sum_{i \neq i_0} u_i^y(y^*) + b$ [recall that $\rho(x^*) = y^*$] to obtain that for every $x_{i_0} \in [x_{i_0} - \varepsilon^*, x_{i_0} + \varepsilon^*]$, $u_{i_0}^x(x_{i_0}) = a^* u_{i_0}^y(x_{i_0}) + b^*$.

**Theorem 1.** Let $N \geq 3$ and let $(S, V)$ satisfy the following conditions:

1. The set $S$ is an open and connected subset of $\mathbb{R}^N$.
2. For every $i$ and $c$, $|\mathcal{G}(i, c)| = 1$.
3. The function $V$ on $S$ is continuous and strictly monotonic.
4. All indifference surfaces of $V$ are connected subsets of $\mathbb{R}^N$.

Then the function $V$ is completely separable if and only if it is additively separable.

**Remark.** This theorem is a weaker version of Theorem 1 in Segal (1991), where the second condition requires that for every $i$ and $c$, $S(i, c)$ is a connected set.

**Proof.** If $V$ is additively separable, then it is of course completely separable. Suppose therefore that it is completely separable. Let $x^0 = (x^0_1, \ldots, x^0_{i_0}, \ldots, x^0_N), y^0 = (y^0_1, \ldots, y^0_{i_0}, \ldots, y^0_N) \in S$ and let $R^x$ and $R^y$ be two open boxes in $S$ such that $x^0 \in R^x$ and $y^0 \in R^y$. Since $V$ is completely separable, it follows by...
Debreu (1960) that on $\mathbb{R}^n$ it is equal to $\zeta^z(\sum_{i=1}^N u_i^0(x_i))$, $z = x^0, y^0$. Next I show that there are $a > 0$, $b$, and $\varepsilon > 0$, such that for every $x_{i0} \in (x_{i0}^0 - \varepsilon, x_{i0}^0 + \varepsilon)$, $u_i^{x_0}(x_{i0}) = ax_i^0 u_i^{x_0}(x_{i0}) + b$.

If $x^0$ and $y^0$ belong to the same connected element of $S(i_0, x_{i0}^0)$, then the proof of the above claim is the same as the proof of Lemma 1 in Segal (1991). Suppose therefore that $T^{x^0}$ and $T^{y^0}$ are different maximal connected elements of $S(i_0, x_{i0}^0)$ such that $x^0 \in T^{x^0}$ and $y^0 \in T^{y^0}$. Assume first that $T^{y^0} \in \mathcal{A}(T^{x^0})$. Let $x^* \in T^{x^0}$ and $y^* \in T^{y^0}$ such that $V(x^*) = V(y^*)$. Let $R x^*$ and $R y^*$ be two open boxes in $S$ such that $x^* \in R x^*$ and $y^* \in R y^*$. Let $V$ on $\mathbb{R}^n$ be given by $\zeta^z(\sum_{i=1}^N u_i^0(x_i))$, $z = x^*, y^*$. By Lemma 1 in Segal (1991) and Lemma 2 above, there are $a^2 > 0$, $b^2$, and $\varepsilon^2 > 0$, $z = x^0, x^*, y^*$, such that for every $x_{i0} \in (x_{i0}^0 - \varepsilon^2, x_{i0}^0 + \varepsilon^2)$, $u_i^{x_0}(x_{i0}) = ax_i^0 u_i^{x_0}(x_{i0}) + b^2$, and for every $x_{i0} \in (x_{i0}^0 - \varepsilon^2, x_{i0}^0 + \varepsilon^2)$, $u_i^{y_0}(x_{i0}) = ay_i^0 u_i^{y_0}(x_{i0}) + b^2$. Let $\varepsilon = \min\{\varepsilon^2, \varepsilon^2, \varepsilon_0\}$, $a = ax^0 a^2 a^3$, and $b = bx^0 a^2 a^3 b^2$. It follows that for every $x_{i0} \in (x_{i0}^0 - \varepsilon, x_{i0}^0 + \varepsilon)$, $u_i^{x_0}(x_{i0}) = ax_i^0 u_i^{x_0}(x_{i0}) + b$.

By finite induction, such $\varepsilon$, $a$, and $b$ exist whenever $T^{y^0} \in \mathcal{A}(T^{x^0})$. Since for every $i$ and $c$, $|\mathcal{E}(i, c)| = 1$, it follows that for every $x^0, y^0 \in S(i_0, x_{i0}^0)$ there exists $l$ such that $T^{y^0} \in \mathcal{A}(T^{x^0})$. Therefore, for every $x^0 = (x_1^0, \ldots, x_l^0, \ldots, x_N^0)$, $y^0 = (y_1, \ldots, x_l^0, \ldots, y_N^0) \in S$ there are $a > 0$, $b$, and $\varepsilon > 0$, such that for every $x_{i0} \in (x_{i0}^0 - \varepsilon, x_{i0}^0 + \varepsilon)$, $u_i^{x_0}(x_{i0}) = ax_i^0 u_i^{x_0}(x_{i0}) + b$. This property is proved in Segal (1991) under the assumption that for every $i$ and $c$, the set $S(i, c)$ is a connected subset of $\mathbb{R}^N$. Moreover, this connectedness assumption is used in no other place in the proof of Theorem 1 there. Therefore, we can now use the rest of the proof in Segal (1991) (i.e., Lemmas 2–4) to complete the proof of Theorem 1.

$\Box$

4. Necessity of the conditions

A natural question is whether the second condition in Theorem 1 [i.e., that for every $i$ and $c$, $|\mathcal{E}(i, c)| = 1$] can be replaced by a weaker condition. First, notice that this condition may be implied, for some sets $S$, by the rest of the conditions of the Theorem.

Example 2. Let

$$A = \{(x_1, x_2, x_3) \in \mathbb{R}^3: 0 < x_1 < 3, 0 < x_2 < 3, 0 < x_3 < 1\},$$

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3: x_1 < 3, x_2 > 0, x_1 - x_2 > 2, 1 \leq x_3 < 2\},$$

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3: 0 < x_1 < 3, 0 < x_2 < 3, x_1 - x_2 < 1, 1 \leq x_3 < 2\}.$$

Let $S = A \cup B \cup C$. For every $c \in (0, 3)$, $S(1, c)$ and $S(2, c)$ are connected sets and for every $c \in (0, 1)$, $S(3, c)$ is connected. For $c \in [1, 2)$, $S(3, c)$ is discon-
nected. However, if $V$ is a strictly monotonic function, then for every $c^* \in [1, 2]$ and for every $x \in S(3, c^*) \cap B$ there exists a point $y \in S(3, c^*) \cap C$ such that $V(x) = V(y)$. Therefore, for $c \in [1, 2]$, $|\Phi(3, c)| = 1$.

The next example shows that Theorem 1 may hold even if condition 2 is not satisfied.

**Example 3.** Let

$$A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < 4, 0 < x_2 < 4, 0 < x_3 < 4\},$$

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 6, x_3 \geq 3\}.$$

Let $S = A \setminus B$. For all the pairs $(i, c)$, except for the pairs $(3, c)$ where $c \geq 3$, the set $S(i, c)$ is connected. Let the completely additive function $V$ satisfy all the conditions of Theorem 1, except for condition 2. This is possible. For example, let $V(x_1, x_2, x_3) = x_1 + x_2 + x_3$. I prove next that such a function $V$ must also be additively separable. The first step is to extend $V$ to a continuous function $V^*$ on $A$.

Let $(x_1^*, x_2^*, x_3^*) \in B$ and let \(\{(x_1^n, x_2^n, x_3^n)\}_{n=1}^{\infty} \subset A\), such that \(x_1^n, x_2^n, x_3^n \rightarrow (x_1^*, x_2^*, x_3^*)\), $i = 1, 2$. Assume, without loss of generality, that for every $n$, $i = 1, 2$, and $j = 1, 3$,

$$|x_i^n - x_i^*| \leq \frac{x_j^*}{8}.$$

Let $a_n^* = 2 + \frac{1}{2} \max \{x_1^*, x_2^*, x_3^*\}$ and $a_n^* = \frac{1}{2} \min \{x_1^*, x_2^*, x_3^*\}$. Since $V$ is monotonic, we can assume, without loss of generality, that for every $n$, $i = 1, 2$,

$$V(a_n^*, a_n^*, a_n^*) \leq V(x_1^n, x_2^n, x_3^n) \leq V(a^*, a^*, a^*),$$

Assume first that $\lim_{n \rightarrow \infty} V(x_1^n, x_2^n, x_3^n)$ exists and equals $\theta^*$. I prove next that $\theta^1 = \theta^2$. Therefore all converging subsequences of $V(x_1^n, x_2^n, x_3^n)$ have the same limit and the sequences converge.

Suppose that $\theta^1 < \theta^2$. Since $V$ is continuous and strictly monotonic, there is a point $(w, x_2^*, w) \in S$ such that $\theta = V(w, x_2^*, w) \in (\theta^1, \theta^2)$. Therefore, there exists $\varepsilon > 0$ such that for $n \geq n^*$ (without loss of generality, $n^* = 1$), $V(x_1^n, x_2^n, x_3^n) \leq V(w - \varepsilon, x_2^n, w - \varepsilon)$ and $V(x_1^n, x_2^n, x_3^n) \geq V(w + \varepsilon, x_2^n, w + \varepsilon)$. Since $V$ is completely separable and by eq. (3), it follows that these two last inequalities are also satisfied when $x_2^*$ is replaced by $x_2^n/2$. Since $(x_1^n, x_2^n/2, x_3^n) \in A$, these new inequalities violate the continuity of $V$. Define the value of the function $V^*$ at the point $(x_1^*, x_2^n, x_3^n)$ as this common limit and let $V^* \equiv V^*$ on $S$.

Let, as before, $(x_1^*, x_2^n, x_3^n) \in B$ and let $(x_1^n, x_2^n, x_3^n) \rightarrow (x_1^*, x_2^n, x_3^n)$ such that for every $n$, $(x_1^n, x_2^n, x_3^n) \in S$. Suppose, for example, that $V(x_1^n, x_2^n, x_3^n) \rightarrow V^*(x_1^*, x_2^n, x_3^n) - \zeta$ for some $\zeta > 0$. By the continuity of $V$ on $S$ and the
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construction of $V^*$ on $B$, there are $q > 0$ and $\alpha \in (0, \frac{1}{2})$ such that
$V(x^*_1 - \eta, x^*_2, x^*_3 - \eta) = V^*(x^*_1, x^*_2, x^*_3) - \alpha \xi$. For a sufficiently large $n$, $x^*_1 > x^*_1 - \eta$,
$x^*_2 > x^*_2 - \eta$, $V(x^*_1, x^*_2, x^*_3) < V^*(x^*_1, x^*_2, x^*_3) - 3 \xi/4$, and by the continuity of $V$
on $S$, $V(x^*_1 - \eta, x^*_2, x^*_3 - \eta) > V(x^*_1 - \eta, x^*_2, x^*_3 - \eta) - \xi/4 > V^*(x^*_1, x^*_2, x^*_3) - 3 \xi/4$, a
ccontradiction with the assumption that $V$ on $S$ is strictly monotonic.

Finally, let $(x^*_1, x^*_2, x^*_3) \rightarrow (x^*_1, x^*_2, x^*_3) \in B$ such that for every $n$, $(x^*_1, x^*_2, x^*_3) \in B$.
Let $(x^*_1', x^*_2', x^*_3') \in S$ satisfy
1. $|V(x^*_1', x^*_2', x^*_3') - V^*(x^*_1, x^*_2, x^*_3)| \leq 2^{-n}$ (this is possible by the preceding
paraaphraph),
2. $d((x^*_1', x^*_2', x^*_3'), (x^*_1, x^*_2, x^*_3')) \leq 2^{-n}$ where $d(\cdot, \cdot)$ is the Euclidean metric on
$\mathbb{R}^3$.

Obviously, $(x^*_1', x^*_2', x^*_3') \rightarrow (x^*_1, x^*_2, x^*_3)$ and limit $V(x^*_1', x^*_2', x^*_3') = lim V^*(x^*_1, x^*_2, x^*_3)$. By
the last paragraph, this common limit exists and equals $V^*(x^*_1, x^*_2, x^*_3)$. This
establishes the continuity of $V^*$.

The function $V^*$ is clearly monotonic. Also, it follows by continuity that it
is completely separable. All the conditions of Debreu’s (1960) are satisfied,
hence $V^*$ is additively separable. Since on $S$, $V^* = V$, so is $V$. $\square$

A variant of this last example shows that even a combination of the
second and fourth conditions in Theorem 1 is not a necessary condition for
the theorem. Let $A$ be as in Example 3 and let

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 3, x_3 = 3\}.$$ 

Let $S = A \setminus C$. If $|\mathcal{C}(3, 3)| = 2$, then there must be a disconnected indifference
surface. Let $\alpha^* = \inf \{V(x_1, x_2, 3) : x_1 + x_2 > 3\}$ and let $\alpha_* = \sup \{V(x_1, x_2, 3) : x_1 + x_2 < 3\}$. By the monotonicity of $V$, $\alpha_* \leq \alpha^*$. Let $\beta \in [\alpha_*, \alpha^*]$. By
the continuity and monotonicity of $V$ there are $x_1, x_2, x_1', x_2'$ and $\epsilon > 0$ such that
$V(x_1, x_2, 3 + \epsilon) = V(x_1', x_2', 3 - \epsilon) = \beta$. The indifference surface
$\{(x_1, x_2, x_3) \in S : V(x_1, x_2, x_3) = \beta\}$ is therefore disconnected. Nevertheless, similarly
to the proof of Example 3, it can be shown here as well that a
completely separable function $V$ satisfying all the rest of the conditions of
Theorem 1, must also be additively separable.

Example 3 seems to suggest that the second condition of Theorem 1 will
become necessary if we require in addition that $S = \text{Int}(\text{Cl}(S))$. This conjecture is false, as is demonstrated by the following example.

Example 4. Let $A$ and $B$ be as in Example 3. For $n = 1, \ldots, \infty$ let

$$C^n = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 6 - 1/2^n, x_1 \geq 1/2^n, x_2 \geq 1/2^n,$n
$3 + 1/2^n \leq x_3 \leq 4 - 1/2^n\},$$
\[ D^n = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : d((x_1, x_2, x_3), (x'_1, x'_2, x'_3)) \leq 1/4^n \text{ for some} \\
(x'_1, x'_2, x'_3) \in C^n \}, \]

\[ D = \bigcup_{n=1}^{\infty} D^n. \]

Let \( S = A \setminus (B \cup D) \). Clearly \( S = \text{Int} (\text{Cl} (S)) \) and \( S(i, c) \) is connected except for \( S(3, c) \) for \( c \geq 3 \) where \( |\mathcal{C}(3, c)| \) may equal 2. It nevertheless follows, as in Example 3, that a completely separable function on \( S \) is also additively separable there. \( \square \)

**References**


Dugundji, J., 1966, Topology (Allyn and Bacon, Boston, MA).
