Dominance Axioms and Multivariate Nonexpected Utility Preferences

Zvi Safra; Uzi Segal


Stable URL:
http://links.jstor.org/sici?sici=0020-6598%28199305%2934%3A2%3C321%3ADAAMNU%3E2.0.CO%3B2-L

*International Economic Review* is currently published by Economics Department of the University of Pennsylvania.

Your use of the JSTOR archive indicates your acceptance of JSTOR’s Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR’s Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/ier_pub.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.
DOMINANCE AXIOMS AND MULTIVARIATE NONEXPECTED
UTILITY PREFERENCES*

BY ZVI SAFRA AND UZI SEGAL.1

This paper deals with nonexpected utility preferences over multivariate distributions. We present two equivalent dominance axioms, implying an additively separable structure of the local utility functions. They also imply that nonexpected utility functionals directly depend on the marginals of the multivariate distributions. We define an invariance axiom, show that it is equivalent to the property that all local utility functions are ordinally equivalent, and that it implies an additively separable expected utility functional when the dominance axiom is assumed. An interesting property of multivariate preferences is that risk neutrality does not imply affinity of the utility function over nonstochastic outcomes.

1. INTRODUCTION

It is well known that preference relations over multivariate distributions require an analysis which is different from that of preferences over distributions on the real line. This is the case even within the expected utility framework. For example, basic concepts such as first-order stochastic dominance and risk aversion require, in the multivariate case, a treatment that is much more sophisticated than that of the univariate case. (See Kihlstrom and Mirman 1974, and Levhari, Paroush, and Peleg 1975.) The major reason for this is that in the multivariate case preferences over distributions depend both on the decision maker’s attitude towards risk and on his preference relation over bundles of commodities, while in the univariate case all preferences over sure outcomes have the same order. As was shown by Kihlstrom and Mirman, some interpersonal comparisons of risk aversion with many commodities are possible only if the two decision makers’ preferences coincide over \( \mathbb{R}^m \).

In the context of nonexpected utility, extensions of the concept of risk aversion to the case of multivariate distributions were given by Yaari (1986) and Karni (1989). Yaari, using an axiomatic approach, has presented a specific multivariate nonexpected utility functional that is the multivariate analog of his univariate dual theory (see Yaari 1987). In his approach the decision maker first evaluates the \( m \) marginals of the multivariate distribution and then uses an \( m \)-dimensional real-valued function to evaluate the \( m \)-dimensional vector that is achieved. Yaari then defined a notion of risk aversion and applied it to some economic problems.

Karni took a different approach and discussed general Fréchet differentiable nonexpected utility functionals. He shows that, in this case, interpersonal comparisons of risk aversion require more than the equivalence of the two decision

* Manuscript received January 1991.
1 We are grateful to an anonymous referee for comments and to the Social Sciences and Humanities Research Council of Canada for financial support.
makers’ preferences over $\mathbb{R}^m$; rather, it is necessary that all their local utility functions are ordinally equivalent as well. Karni also discusses the notion of decreasing risk aversion assuming that all the local utility functions are ordinally equivalent and homothetic.

In this paper we take an approach that might be considered as being in between those of Yaari and Karni. We impose an axiom which, on the one hand, is weak enough to allow for a wide class of nonexpected utility functionals and, on the other hand, is tight enough to give an additively separable structure to all of the local utility functions. This axiom has two different representations. The first, called dominance, is reduced in the univariate case to the usual stochastic dominance axiom. The second, called weak dominance, is completely meaningless in the univariate case. Yet, these two representations happen to be equivalent in our framework. These axioms also allows for Yaari’s functional; we show what additional properties are required for that. This is done in Section 2.

In Section 3 we check for the relationships between several properties of the utility function and the structure of the local utilities and show, for example, that Karni’s assumption on the ordinal equivalence of the local utility functions is equivalent to a sort of invariance assumption that we then present. We also show that the invariance and the dominance axioms together imply an additively separable expected utility functional. Also, we show the equivalence of two monotonicity conditions at the presence of a weaker notion of invariance. In Section 4 we discuss the issue of risk neutrality and show that, unlike the univariate case, this property does not imply an affine utility function. Instead, it implies that all the local utility functions are affine, but not necessarily ordinally equivalent. Hence the utility on sure outcomes might be nonaffine.

2. DOMINANCE AXIOMS

Let $D = [0, M]^m \subset \mathbb{R}^m$, and let $L^m$ be the set of random variables with outcomes in $D$. Let $x, y \in \mathbb{R}^m$. We say that $x \leqslant y$ if $x_i \leqslant y_i$, $i = 1, \ldots, m$. For $X \in L^m$, let $F_X(x) = \Pr(X \leqslant x)$. Finite random variables are also written as $(x_1, p_1; \ldots; x_n, p_n)$. Such a lottery yields the vector $x_i$ with probability $p_i$, $i = 1, \ldots, n$. We adopt throughout the common notation $\delta_x = (x, 1)$.

On $L^m$ there exists a complete and and transitive preference relation, $\succeq$. We say that $X \sim Y$ if and only if $X \succeq Y$ and $Y \succeq X$, and $X > Y$ if and only if $X \succeq Y$ but not $Y \succeq X$. We assume that $\succeq$ is continuous in the topology of weak convergence. It is called monotonic if $[x \succeq y, x \neq y] \Rightarrow \delta_x \succeq \delta_y$.

The functional $V: L^m \rightarrow \mathbb{R}$ represents the relation $\succeq$ if $V(X) \succeq V(Y)$ if and only if $X \succeq Y$. We assume that $V$ is Fréchet differentiable. Following Machina (1982) and Karni (1989), this implies the existence of local utility functions $\{U(\cdot; F_X): X \in L^m\}$ such that

$$V(Y) - V(X) = \int_D U(x; F_X) \, dF_Y(x)$$

$$- \int_D U(x; F_X) \, dF_X(x) + o(\|F_Y - F_X\|).$$
We assume throughout the paper that $U$ is twice differentiable with respect to $x$, and that these partial derivatives are continuous in $F^2$.

The local utility function $U(\cdot; F)$ is additively separable if there are $U^1(\cdot; F), \ldots, U^m(\cdot; F): [0, M] \to \mathbb{R}$ such that $U(x; F) = U((x^1, \ldots, x^m); F) = \sum_{j=1}^m U^j(x^j; F)$. In this section we present conditions implying that the local utility function is additively separable.

Let $B \subseteq D$ be a Borel set. For $X \in L^m$, let $P_X(B) = \int_B dF_X(x)$ be the probability that the outcome of the random variable $X$ is in $B$. For $x \in D$, let $x^* = \{y \in D : y \geq x\}$ and $x_* = \{y \in D : x \geq y\}$. The function $u:D \to \mathbb{R}$ is increasing if $u(x) \geq u(y)$ whenever $x \geq y$. The random variable $X$ dominates the random variable $Y$ by first-order stochastic dominance if, for every increasing function $u:D \to \mathbb{R}$, $\int_D u(x) dF_X(x) \leq \int_D u(x) dF_Y(x)$. The preference relation $\geq$ satisfies the first-order stochastic dominance axiom if $X \geq Y$ whenever $X$ stochastically dominates $Y$. It satisfies the strict first-order stochastic dominance axiom if $X > Y$ whenever $X$ stochastically dominates $Y$ and $F_X \neq F_Y$. Obviously, if a preference relation satisfies the first-order stochastic dominance axiom, then it is monotonic. The opposite, however, is not true even in $L^1$. For example, let $V(X) = E[X] - \text{var}[X]$. Trivially $\delta_x \geq \delta_y$ if and only if $x \geq y$, but $\delta_0 > (0, 1/2; 4, 1/2)$. Following Machina (1982) it can be shown that $\geq$ satisfies the first-order stochastic dominance axiom if and only if all the local utility functions $U(\cdot; F)$ are increasing on $D$.

One possible interpretation of the first-order stochastic dominance axiom for real random variables is that if for every possible outcome $x$, the probability of receiving less than $x$ in the random variable $X$ is not greater than it is in the random variable $Y$, then $X$ is preferred to $Y$. This interpretation does not naturally extend to lotteries over $\mathbb{R}^m$, as there is no natural complete order over bundles of commodities. One possible solution to this problem is due to Levhari, Paroush, and Peleg (1975). They showed that for $X, Y \in L^m$, $X$ stochastically dominates $Y$ if and only if for every comprehensive set $B$, $P_X(B) \leq P_Y(B)$. This condition replaces the notion of lower-set of a point, which was used for $L^1$, by a lower set of a surface. (Recall that in $\mathbb{R}$, a surface is a point.) If we wish to extend this property with regard to single points, we have to compare the values of $P_X(x^*)$ and $P_Y(x^*)$. The first-order stochastic dominance axiom for lotteries in $L^1$ also assumes that if, for every $x$, the probability of receiving more than $x$ under $X$ is not less than the corresponding probability under $Y$, then $X$ is preferred to $Y$. For the extension of this property with regard to single points we need to compare the values of $P_X(x^*)$ and $P_Y(x^*)$. We now present three axioms for $L^m$ that are based on the former observations. Note that when $m = 1$, they all coincide with first-order stochastic dominance on the real line.

**Upper Dominance.** If for every $x \in D$, $P_X(x^*) \geq P_Y(x^*)$, then $X \geq Y$.

**Lower Dominance.** If for every $x \in D$, $P_X(x^*) \leq P_Y(x^*)$, then $X \geq Y$.

2 See Allen (1987) for conditions on the preference relation implying the existence of local utility functions.

3 $B$ is comprehensive if $[x \in B, y \in D, and y \leq x] \Rightarrow y \in B$. 


**Dominance.** If for every $x \in D$, $P_X(x^*) \geq P_Y(x^*)$, or, if for every $x \in D$, $P_X(x^*) \leq P_Y(x^*)$, then $X \succeq Y$.4

Clearly, the dominance axiom implies both the upper and the lower dominance axioms. Also, each of these three axioms implies the first-order stochastic dominance axiom. If $X$ stochastically dominates $Y$, then by Levhari, Peleg, and Paroush (1975), $P_X(B) \leq P_Y(B)$ whenever $B$ is a comprehensive set. In particular, $P_X(x^*_e) \leq P_Y(x^*_e)$ for all $x$, hence by lower dominance, $X \succeq Y$. The proof that upper dominance implies first-order stochastic dominance is similar—note that $D \setminus x^*$ is a comprehensive set.

However, these three axioms are not implied by the first-order stochastic dominance axiom. For example, let $X = ((1, 2), 1/2; (2, 1), 1/2)$ and $Y = ((1, 1), 1/2; (2, 2), 1/2)$. These two lotteries cannot be compared by first-order stochastic dominance. By upper dominance, $Y \succeq X$, by lower dominance, $X \succeq Y$, and by dominance, $X \sim Y$.

Next, we present an axiom that will be proved to be equivalent to the dominance axiom (for $m > 1$). For this we use the following notations for $x, y \in D$: Let $(x, y) = (\max \{x^1, y^1\}, \ldots, \max \{x^m, y^m\}) \in D$, and let $(x, y) = (\min \{x^1, y^1\}, \ldots, \min \{x^m, y^m\})$.

**Weak Dominance.** Let $X \in L^m$ and let $x, y \in D$ such that $P_X(\{x\}), P_X(\{y\}) \geq p > 0$. If $Y$ is the same as $X$, except that

- $P_Y(\{z\}) = P_X(\{z\}) - p$, $z = x, y$; and
- $P_Y(\{z\}) = P_X(\{z\}) + p$, $z = (x, y)$

then $X \sim Y$. (See Figure 1.)

The weak dominance axiom is meaningless when $m = 1$, since then $(x, y) = \min \{x, y\}$ and $(x, y) = \max \{x, y\}$. Hence every preference relation on $L^1$ vacuously satisfies this axiom.

Several authors have already discussed this axiom within the expected utility framework. Epstein and Tanny (1980) called this axiom correlation neutrality, and Richard (1975) called it multivariate risk neutrality. Scarsini (1988) and Mosler (1987) generalized Richard’s notion of risk to what they called $k$-variate risk aversion ($k > 2$). For a more natural definition of multivariate risk neutrality, see Section 4 below.

**Theorem 1.** Let $m \geq 2$. The following three conditions are equivalent.

- a. The preference relation $\succeq$ satisfies the dominance axiom.
- b. The preference relation $\succeq$ satisfies the weak dominance and the first-order stochastic dominance axioms.
- c. The local utility functions $U(\cdot; F)$ are monotonic and additively separable. That is, $U(\cdot; F) = \sum_{j=1}^m U_j(x^j; F)$ where the functions $U_j$ are all increasing.

4 For a discussion of these axioms for two-stage lotteries, see Segal (1990).
Remark. In the case of expected utility theory, Levy and Paroush (1974) proved the implication (c) ⇒ (b) and Epstein and Tanny (1980, Theorem 4-c), and Richard (1975), with stronger assumptions, proved the equivalence (b) ⇔ (c).

Proof. (a) ⇔ (b). We have already shown that the dominance axiom implies the first-order stochastic dominance axiom. Let $X$ and $Y$ be as in the definition of the weak dominance axiom. We show first that $\forall z, P_Y(z^*) \geq P_X(z^*)$. This follows from the fact that if $x \in z^*$ or $y \in z^*$, then $(\overline{x}, \overline{y}) \in z^*$, and if $x \in z^*$ and $y \in z^*$, then $(x, y) \in z^*$, hence $Y \succeq X$. Similarly, we show that $\forall z, P_Y(z_\ast) \leq P_Y(z_\ast)$. Indeed, if $x \in z_\ast$ or $y \in z_\ast$, then $(x, y) \in z_\ast$. If $x \in z_\ast$ and $y \in z_\ast$, then $(\overline{x}, \overline{y}) \in z_\ast$, hence $X \succeq Y$. It thus follows that $X \sim Y$.

(b) ⇒ (c). Let $\Delta$ be a countable dense set in $[0, M]$. Then $\Delta^m$ is a countable dense set in $D$. Let $X \in L^m$ such that $\forall z \in \Delta^m, P_X\{z\} > 0$. Note that the set of these lotteries, $\hat{L}^m$, is dense in $L^m$. Let $X \in \hat{L}^m$. Let $x = (x^1, \ldots, x^i, \ldots, x^m), y = (y^1, \ldots, y^i, \ldots, y^m) \in \Delta^m$, where $y^i > x^i$, and let $p = \min\{P_X\{x\}, P_X\{y\}\} > 0$. Let $X_i(\varepsilon) \in L^m$, $i = 1, 2$ be the same as $X$, except for a $p$-probability mass being shifted from $y$ to $y(\varepsilon) = (x^1, \ldots, y^i, \ldots, x^i + \varepsilon, \ldots, x^m)$ to obtain $X_1(\varepsilon)$ and a $p$-probability mass being shifted from $x$ to $x(\varepsilon) = (x^1, \ldots, x^i, \ldots, x^i + \varepsilon, \ldots, x^m)$ to create $X_2(\varepsilon)$ (see Figure 2). Obviously, $y(\varepsilon) = (\overline{x(\varepsilon)}, \overline{y})$ and $x = (x(\varepsilon), y)$, hence, by weak dominance, $X_1(\varepsilon) \sim X_2(\varepsilon)$.

$$0 = V(X_1(\varepsilon)) - V(X_2(\varepsilon))$$

$$= [V(X_1(\varepsilon)) - V(X)] - [V(X_2(\varepsilon)) - V(X)]$$
= \int_D U(z; F_X) \, dF_{X_1(\varepsilon)}(z) - \int_D U(z; F_X) \, dF_{X_2(\varepsilon)}(z) + o(\varepsilon) \\
= p[[U(y(\varepsilon); F_X) - U(y; F_X)] - [U(x(\varepsilon); F_X) - U(x; F_X)]] + o(\varepsilon).

Divide both sides by $\varepsilon$ and take to the limit to obtain

$$
\lim_{\varepsilon \to 0} \frac{U(y(\varepsilon); F_X) - U(y; F_X)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{U(x(\varepsilon); F_X) - U(x; F_X)}{\varepsilon} \Rightarrow
$$

$$
\frac{\partial U}{\partial x_j}(y; F_X) = \frac{\partial U}{\partial x_j}(x; F_X).
$$

This is true for all $y = (x^1, \ldots, y^j, \ldots, x^m) \in \Delta^m$, hence as $\Delta$ is dense in $[0, M]$ it follows that $\partial^2 U/\partial x^i \partial x^j(x; F_X) = 0$. Since this holds for all $i$ and $j$ and for every $x$, it follows that $U(x; F_X) = \sum U^j(x^j; F_X)$ where $U^j(\cdot; F_X): [0, M] \to \mathbb{R}$. This holds true for all $X \in L^m$ which is dense in $L^m$, and since the second order derivatives of $U(\cdot; F_X)$ are continuous in $X$, the additively separable form clearly follows for all $X$.

We assumed that the order satisfies the first-order stochastic dominance axiom. It thus follows, similarly to Machina (1982), that the local utility functions are
increasing. Since they are additively separable, each of the functions $U^j$ is increasing.

(c) $\Rightarrow$ (a). Let $X$ and $Y$ be such that for every $x$, $P_X(x^*) \geq P_Y(x^*)$. Let $Z(\alpha)$ satisfy $F_{Z(\alpha)} = \alpha F_X + (1 - \alpha) F_Y$. For $\alpha > \beta$, $Z(\alpha)$ dominates $Z(\beta)$ by upper dominance. Indeed, $P_{Z(\alpha)}(x^*) = \alpha P_X(x^*) + (1 - \alpha) P_Y(x^*) \geq \beta P_X(x^*) + (1 - \beta) P_Y(x^*) = P_{Z(\beta)}(x^*)$. To prove that $X \succeq Y$ it is therefore sufficient to show that $\frac{\partial V(Z(\alpha))}{\partial \alpha} \geq 0$.

$$
\frac{\partial V}{\partial \alpha} V(Z(\alpha)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ V(Z(\alpha + \varepsilon)) - V(Z(\alpha)) \right]$$

$$
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \int_D \sum_{j=1}^{m} U^j(x^j; F_{Z(\alpha)}) \ dF_{Z(\alpha + \varepsilon)}(x) \right. - \left. \int_D \sum_{j=1}^{m} U^j(x^j; F_{Z(\alpha)}) \ dF_{Z(\alpha)}(x) + o(\varepsilon) \right]$$

$$
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \int_D \sum_{j=1}^{m} U^j(x^j; F_{Z(\alpha)}) \varepsilon \ d(F_X - F_Y)(x) + o(\varepsilon) \right]$$

(2) $$
\int_D \sum_{j=1}^{m} U^j(x^j; F_{Z(\alpha)}) \ dF_X(x) - \int_D \sum_{j=1}^{m} U^j(x^j; F_{Z(\alpha)}) \ dF_Y(x).$$

For $X \in L^m$, let $F_X^1, \ldots, F_X^m$ be the induced distributions on the $m$ axes. Formally, for $x \in [0, M]$,

$$F_X^j(x^j) = P_X((M, \ldots, x^j, \ldots, M)_*).$$

If $X$ dominates $Y$ by upper or by lower dominance, then for every $j$, $F_X^j$ dominates $F_Y^j$ by first order stochastic dominance. (To see this, note that $F_X^j(x^j) = 1 - \lim_{y^j \to x^j} P_X(0, \ldots, y^j, \ldots, 0)$.) The expression at equation (2) now equals

$$\sum_{j=1}^{m} \int_0^{m} U^j(x^j; F_{Z(\alpha)}) \ d(F_X^j - F_Y^j)(x^j) \geq 0.$$ 

It is nonnegative since, by familiar results for the univariate case, every element is nonnegative. Similar results hold if for every $x$, $P_X(x_*) \leq P_Y(x_*)$, hence $\succeq$ satisfies the dominance axiom.

The next conclusion follows immediately from the proof of Theorem 1.

**Conclusion.** Let $m \geq 2$. The following two conditions for $\succeq$ on $L^m$ are equivalent.

a. The preference relation $\succeq$ satisfies the weak dominance axiom.

b. The local utility functions $U(\cdot; F)$ are additively separable.
A lottery $X$ on $D$ induces distribution functions on the axes by $F_{X}^{i}(x_{i}) = P_{X}(M, \ldots, x_{i}, \ldots, M)$. Let $X$ and $Y$ induce the same distributions on the axes, that is, $F_{X}^{j} = F_{Y}^{j}, j = 1, \ldots, m$. If for every two such lotteries $X \sim Y$, when $\succeq$ satisfies the weak dominance axiom, and the local utility functions are additively separable.

Suppose now that $\succeq$ satisfies the dominance axiom, and let $X$ and $Y$ induce the same distributions on the axes. Similarly to the proof of the third part of Theorem 1, it follows that $X \sim Y$. (Note that for every $\alpha$, $\alpha X + (1 - \alpha)Y$ induces the same distributions on the axes as $X$ and $Y$.) It thus follows that if $\succeq$ satisfies the dominance axiom, then it can be represented by a function of the distributions it induces on the axes. Formally it is stated in the following theorem.

**Theorem 2.** The preference relation $\succeq$ on $L^{m}$ satisfies the dominance axiom if, and only if, it is monotonic and can be represented by a function $V(X) = f(F_{X}^{1}, \ldots, F_{X}^{m})$.

An interesting subset of the functionals of Theorem 2 consists of functionals of the form $V(X) = g(V_{1}(F_{X}^{1}), \ldots, V_{m}(F_{X}^{m}))$. It is well known (see Debreu 1960) that the additional necessary and sufficient condition for such a representation is that for every $i$ and $j$, $(F_{X}^{1}, \ldots, F_{X}^{i}, \ldots, F_{X}^{j}, \ldots, F_{X}^{m}) \succeq (F_{X}^{1}, \ldots, F_{X}^{i}, \ldots, F_{X}^{j}, \ldots, F_{X}^{m})$ if and only if $(F_{X}^{i}, \ldots, F_{X}^{j}, \ldots, F_{X}^{j}, \ldots, F_{X}^{m}) \succeq (F_{X}^{1}, \ldots, F_{X}^{i}, \ldots, F_{X}^{i}, \ldots, F_{X}^{m})$. Such a functional is especially plausible in multiperiod decision problems. Suppose that a financial asset generates uncertain yield during $m$ periods. In other words, it produces $m$ lotteries, one at each period. The last functional suggests that the decision maker first evaluates each one of these lotteries, not necessarily with respect to the same representation functional for single-variable distributions, and then aggregates the values of the lotteries by the function $g$. Such a model is flexible enough to allow for changes in attitude towards risk between different parts of the decision maker’s life cycle. On the other hand, it makes the analysis of multivariable attitude towards risk relatively simple as it can use known results concerning preference relations on lotteries over $\mathbb{R}$.

3. **THE INVARIANCE AXIOM**

Let $\succeq$ be an order on $D$, given by $x \succeq y$ if and only if $\delta_{x} \succeq \delta_{y}$. In expected utility theory, this order is carried over to preferences between general lotteries in the following way. Let $Y = (x_{1}; p_{1}; \ldots; y, p_{i}; \ldots; x_{n}, p_{n})$ and let $Y' = (x_{1}; p_{1}; \ldots; y', p_{i}; \ldots; x_{n}, p_{n})$. Then

$$Y \succeq Y' \iff \sum_{j \neq i} p_{j}u(x_{j}) + p_{i}u(y) \succeq \sum_{j \neq i} p_{j}u(x_{j}) + p_{i}u(y')$$

$$\iff u(y) \succeq u(y') \iff \delta_{x} \succeq \delta_{x'} \iff y \succeq y'.$$

Let $Y$ and $Y'$ be two lotteries as above. These lotteries can be rewritten as $Y = \alpha X + (1 - \alpha)\delta_{y}$ and $Y' = \alpha X + (1 - \alpha)\delta_{y'}$, where $\alpha = \sum_{j \neq i} p_{j}$ and

---

5 This is Yaari’s (1986) Axiom A.
\[ X = \left( x_1, \frac{p_1}{\alpha}; \ldots; x_{i-1}, \frac{p_{i-1}}{\alpha}; x_{i+1}, \frac{p_{i+1}}{\alpha}; \ldots; x_n, \frac{p_n}{\alpha} \right). \]

Define an order \( \succeq_{(a, X)} \) on \( D \) by \( y \succeq_{(a, X)} y' \) if and only if \( \alpha X + (1 - \alpha)\delta_y \succeq \alpha X + (1 - \alpha)\delta_y' \). As was just claimed, in expected utility theory, the orders \( \succeq_{(a, X)} \) are all the same. We call this property invariance (see Segal 1989 axiom e). It is stated formally below.

**INvariance.** The induced relations \( \succeq_{(a, X)} \) on \( D \) do not depend on \( a \) or \( X \).\(^6\)

Nonexpected utility models do not necessarily satisfy this axiom which, to a certain extent, trivializes the preference relation on \( L^m \) to a preference relation on \( L^1 \). If invariance is satisfied then under continuity and first-order stochastic dominance, each outcome in \( D \) can be replaced by an outcome on the main diagonal of \( D \), and the lottery \( X \) is reduced to a lottery on a line. Formally, denote the common relation \( \succeq_{(a, X)} \) by \( \succeq^{*} \). For \( x \in D \), let \( \xi(x) \in [0, M] \) satisfy \( (\xi(x), \ldots, \xi(x)) \sim^{*} x \) and denote the vector \((\xi(x), \ldots, \xi(x)) \in D\) by \( \xi^v(x) \). Then

\[
(x_1, p_1; \ldots; x_n, p_n)
\]

\[ = [\sum_{k=1}^{n-1} p_k] \left( x_1, \frac{p_1}{\sum_{k=1}^{n-1} p_k}; \ldots; x_{n-1}, \frac{p_{n-1}}{\sum_{k=1}^{n-1} p_k} \right) + \left[ 1 - \sum_{k=1}^{n-1} p_k \right] \delta_{x_n}
\]

\[ \sim [\sum_{k=1}^{n-1} p_k] \left( x_1, \frac{p_1}{\sum_{k=1}^{n-1} p_k}; \ldots; x_{n-1}, \frac{p_{n-1}}{\sum_{k=1}^{n-1} p_k} \right) [1 - \sum_{k=1}^{n-1} p_k] \delta_{\xi^v(x_n)}
\]

\[ \sim \ldots \sim \sum_{k=1}^{n} p_k \delta_{\xi^v(x_k)} = (\xi^v(x_1), p_1; \ldots; \xi^v(x_n), p_n).
\]

All the outcomes of this last lottery are on the main diagonal of \( D \).

Nevertheless, the invariance axiom by itself does not imply expected utility and should therefore be investigated. This is done in the next theorem.

**Theorem 3.** Suppose that for every \( F \), the local utility function \( U(\cdot; F) \) is not constant. Then the following two conditions are equivalent:

a. The preference relation \( \succeq \) on \( L^m \) satisfies the invariance and the strict first-order stochastic dominance axioms.

b. All the local utility functions \( U(\cdot; F) \) are ordinally equivalent and increasing.

**Proof.** (a) \( \Rightarrow \) (b). Let \( F \) and \( G \) be such that \( U(\cdot; F) \) and \( U(\cdot; G) \) are not ordinally equivalent. There are therefore \( x, y \in D \) such that \( U(x; F) > U(y; F) \), but \( U(y; G) \succeq U(x; G) \). Since the preference relation \( \succeq \) satisfies the strict first-order stochastic dominance axiom, it follows, similarly to Machina (1982), that

\(^6\) For some applications of this axiom, see Schlee (1991).
the local utility functions are strictly monotonic. Therefore, by continuity of the local utility functions, we can assume, without loss of generality, that \( U(y; G) > U(x; G) \). By equation (1) it follows that for \( \alpha \) sufficiently close to 1, \( \alpha F + (1 - \alpha) \delta_x > \alpha F + (1 - \alpha) \delta_y \), but \( \alpha G + (1 - \alpha) \delta_y > \alpha G + (1 - \alpha) \delta_x \), in contradiction to (a).

(b) \( \Rightarrow \) (a). Suppose that all the local utility functions are ordinarily equivalent but there are \( x, y, \alpha_i, \) and \( X_i, i = 1, 2 \), such that \( \alpha_1 X_1 + (1 - \alpha_1) \delta_x > \alpha_1 X_1 + (1 - \alpha_1) \delta_y \), but \( \alpha_2 X_2 + (1 - \alpha_2) \delta_y > \alpha_2 X_2 + (1 - \alpha_2) \delta_x \). Let \( X \) and \( Y \) be any two lotteries. By standard techniques we obtain

(3) \[
\frac{\partial}{\partial \alpha} V(\alpha X + (1 - \alpha) Y) = \int_D U(z; F_{\alpha X + (1 - \alpha) Y}) \, d [F_X(z) - F_Y(z)].
\]

Now let \( X = \alpha_i X_i + (1 - \alpha_i) \delta_x \) and \( Y = \alpha_i X_i + (1 - \alpha_i) \delta_y \). Substitute into equation (3) to obtain for \( i = 1, 2 \),

\[
\frac{\partial}{\partial \alpha} V(\alpha_i X_i + (1 - \alpha_i)[\alpha \delta_x + (1 - \alpha) \delta_y])
\]

\[= (1 - \alpha_i) \int_D U(z; F_{X_i + (1 - \alpha_i)[\alpha \delta_x + (1 - \alpha) \delta_y]}) \, d [F_{\delta_x} - F_{\delta_y}] \]

\[= (1 - \alpha_i)[U(x; F_{X_i + (1 - \alpha_i)[\alpha \delta_x + (1 - \alpha) \delta_y]}) - U(y; F_{X_i + (1 - \alpha_i)[\alpha \delta_x + (1 - \alpha) \delta_y]})].\]

Since all the local utility functions are ordinarily equivalent, the sign of this last expression is always the same. However, for \( i = 1 \) it must be somewhere positive, and for \( i = 2 \), it must be somewhere negative—a contradiction. It thus follows that the preference relation \( \succ \) satisfies the invariance axiom. \( \Box \)

If all the local utility functions are identical up to affine transformations, the preference relation \( \succ \) on \( L^m \) can be represented by an expected utility functional. It is well known that if a transformation \( \psi \) transforms an additively separable function into an additively separable function, then \( \psi \) is affine (see Aczél 1966). Therefore, if \( \succ \) satisfies both dominance and invariance, all its local utility functions are the same. It is stated formally below.

**Theorem 4.** The following two conditions concerning a preference relation \( \succ \) on \( L^m \) are equivalent:

a. It satisfies both the dominance and the invariance axioms.

b. It can be represented by an expected utility functional with an increasing additively separable utility function.

In Section 2 we proved that a monotonic preference relation does not have to satisfy the first-order stochastic dominance axiom. However, if it satisfies a weaker version of the invariance axiom, then these two axioms are equivalent. Formally, it is stated as follows.
WEAK INVARIANCE. Let $x \succeq y$. Then for every $X$ and $\alpha \in [0, 1]$, $x \succeq_{(\alpha, X)} y$.

THEOREM 5. Let the preference relation $\succeq$ on $L^m$ satisfy the weak invariance axiom. Then the following two conditions are equivalent.

a. The relation $\succeq$ is monotonic.

b. The relation $\succeq$ satisfies the first-order stochastic dominance axiom.

PROOF. We have already seen in Section 2 that (b) $\implies$ (a). To prove that (a) $\implies$ (b), we invoke the following theorem from Kamae, Krengel, and O'Brien (1977):

Let $X$ and $Y$ be two lotteries with finite support. Then $X$ stochastically dominates $Y$ if and only if $X = (x_1, p_1; \ldots; x_n, p_n)$ and $Y = (y_1, p_1; \ldots; y_n, p_n)$ such that $x_i \succeq y_i$, $i = 1, \ldots, n$.

If $X$ and $Y$ satisfy the above conditions, then by using the weak invariance axiom $n$ times we obtain that $X \succeq Y$. If $X$ and $Y$ do not have a finite support, but $X$ stochastically dominates $Y$, then we can approach both of them by lotteries with finite supports such that for every $k$, $X_k$ stochastically dominates $X$ and $Y$ stochastically dominates $Y_k$. The theorem now follows by continuity. $\Box$

4. RISK NEUTRALITY

Let $\succeq$ be a preference relation on $L^1$, the space of lotteries on the real segment $[0, M]$. If for every $X \in L^1$, $X \sim (E[X], 1)$, then the decision maker is an expected value maximizer, and his representation function is $\int_0^M x \, dF_X(x)$. In this section we discuss an extension of this axiom to the case of multi-commodities outcomes. It turns out that risk neutrality is much less restrictive in the multivariate case and it does not imply affinity of the sure-outcomes utility. It is stated formally.

RISK NEUTRALITY.$^8$ For $X \in L^m$, $X \sim (E[X], 1)$, where

$$E[X] = \int_D x \, dF_X(x) = \left( \int_0^M x_1 \, dF_X^1(x_1), \ldots, \int_0^M x_m \, dF_X^m(x_m) \right).$$

For every $x, y \in [0, M]^m$, $x + y = (\overline{x}, \overline{y}) + (\underline{x}, \underline{y})$. It follows immediately that if a decision maker is risk neutral, then he satisfies the weak dominance axiom. By the conclusion from Theorem 1 it also follows that for every distribution $F$, the local utility function $U(\cdot; F)$ is additively separable. The risk neutrality condition is nevertheless stronger, as it implies that the local utility functions are affine.$^9$

THEOREM 6. The following three conditions are equivalent.

$^7$ The original theorem holds for general spaces and preferences. We present here only a limited version of it.

$^8$ For different definitions of risk attitude see the references in Section 2 above. See also Safra and Zilcha (1988).

$^9$ Note that local utility functions are determined up to an affine transformation (see Machina 1987).
a. The preference relation $\succeq$ satisfies the risk neutrality axiom.

b. The local utility functions $U(\cdot; F)$ are additively separable and affine, that is, $U(x; F) = \sum_{j=1}^m a^j(F)x^j$.

(c). The preference relation $\succeq$ can be represented by the functional $V(X) = U(\int dF X(x))$ for a certain function $U: \mathbb{R}^m \rightarrow \mathbb{R}$.

**Proof.** (a) $\Rightarrow$ (b). As in the proof of Theorem 1, let $\Delta$ be a countable dense set in $[0, M]$, and let $X \in L^m$ such that $\forall z \in \Delta^m, P_X(\{z\}) > 0$. Let $x = (x^1, \ldots, x^i, \ldots, x^m), y = (x^1, \ldots, y^i, \ldots, x^m) \in \Delta^m$, and let $p = \min \{P_X(\{x\}), P_X(\{y\})\} > 0$. Let $X_1(\epsilon), X_2(\epsilon) \in L^m$ be the same as $X$, except for a $p$-probability mass being shifted from $x$ to $(x^1, \ldots, x^i + \epsilon, \ldots, x^m)$ to create $X_1(\epsilon)$ and a $p$-probability mass being shifted from $y$ to $(x^1, \ldots, y^i + \epsilon, \ldots, x^m)$ to create $X_2(\epsilon)$. Since $E[X_1(\epsilon)] = E[X_2(\epsilon)]$, it follows by risk neutrality that $X_1(\epsilon) \sim X_2(\epsilon)$. Hence, by the additivity of the local utility function $U(\cdot; F_X)$,

$$0 = V(X_1(\epsilon)) - V(X_2(\epsilon)) = [V(X_1(\epsilon)) - V(X)] - [V(X_2(\epsilon)) - V(X)]$$

$$= p[[U^i(x^i + \epsilon; F_X) - U^i(x^i; F_X)] - [U^i(y^i + \epsilon; F_X) - U^i(y^i; F_X)]] + o(\epsilon).$$

Divide both sides by $\epsilon$ and take to the limit to obtain

$$U^i_1(x^i; F_X) = U^i(y^i; F_X).$$

Since this is true for a dense set in $[0, M]$ and since we assumed that the function $\partial U(\cdot; F)/\partial x^i$ is continuous in its first argument, it follows that $U^i$ is affine.

(b) $\Rightarrow$ (a). Let $X \in L^m$ and define $X(\alpha) = (1 - \alpha)X + \alpha E[X]$.

$$\frac{\partial}{\partial \alpha} V(X(\alpha)) \bigg|_{\alpha = 0} = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [V(X(\alpha)) - V(X)]$$

$$= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left[ \sum_{j=1}^m a_j^i(F_X) \int_0^M (-x^j + E^j[X]) dF_X(x) + o(\alpha) \right] = 0$$

where $E^j[X]$ is the $j^{th}$ component of the vector $E[X]$.

Since this derivative is zero at all starting points on the path $(1 - \alpha)X + \alpha E[X]$, it follows that $V(X) = V(E[X])$.

(a) $\Leftrightarrow$ (c) The proof is obvious. \qedsymbol

Risk neutrality does not imply invariance, nor does it imply that the preference relation on bundles of commodities with no uncertainty can be represented by an additively separable affine utility function. To see this, consider the functional $V: L^1 \rightarrow \mathbb{R}, V(F) = \int v(x) dF(x) + \int w(x) dF(x)^2$ where $v(x_1, x_2) = x_1 + x_2$ and $w(x) = 2x_1 + x_2$. This functional satisfies risk neutrality, and the utility function $U$ of condition (c) of Theorem 6 is given by $U(s, t) = 4s^2 + t^2 + 4st + s + t$. Its local utility functions are given by $U(x; F) = v(x) + 2[\int w(z) dF(z)]$. 


Since the local utility functions change with \( F \), it follows from Theorem 5 that \( V \) does not satisfy the invariance axiom. However, assuming invariance we get that the function \( U \) in condition (c) above is a (transformation of an) affine function.

**Theorem 7.** The following two conditions concerning a preference relation \( \succeq \) are equivalent:

a. It satisfies the risk neutrality and the invariance axioms.

b. It can be represented by \( V(X) = \sum_{j=1}^{m} a^j E[X^j] \).

Even in this restricted case, there are many different preference relations satisfying these axioms. All are expected utility-type relations, but they differ from one another by the slopes of the affine indifference curves of the utility function \( U: \mathbb{R}^m \to \mathbb{R} \).

Tel Aviv University, Israel  
University of Toronto, Canada

**REFERENCES**


