Additively Separable Representations on Non-convex Sets

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This paper proves sufficient conditions under which a completely separable order on non-convex sets can be represented by an additively separable function. The two major requirements are that indifference curves are connected and that intersections of the domain of the order with parallel-to-the-axes hyperplanes are connected. Journal of Economic Literature Classification Numbers: 020, 022.

1. INTRODUCTION

It follows from a famous theorem by Debreu [4] that a reflexive, complete, transitive, and continuous order $\succeq$ on a product of intervals $\pi_1 \times \cdots \times \pi_N \subseteq \mathbb{R}^N$, $N \geq 3$, is completely separable (in the sense that the induced order on the product $\prod_{i \neq j} \pi_i$ for any $j \in \{1, \ldots, N\}$ does not depend on the outcome in $\pi_j$) if and only if it can be represented by an additively separable function of the form $V(x_1, \ldots, x_N) = \sum_{i=1}^{N} u_i(x_i)$.

There are situations where the domain of the function $V$ is not necessarily a Cartesian product. Some recent generalizations of expected utility theory make use of additively separable functions that are defined on such domains; see, for example, [2]. The assumption that the domain of preferences is a Cartesian product is common. However, less restrictive assumptions (e.g., convexity) appear in general equilibrium theory. Moreover, considerations of survival suggest that the consumption possibilities set is often not expressible as a Cartesian product, for example, when leisure is one of the goods and greater labor supply requires

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1 As pointed out by Wakker [7], there are mistakes in the proofs of some of the results in this work. I believe that the representation theorems proved below may help in correcting these mistakes.
increased nourishment. (See Blackorby, Primont, and Russell [1] for a survey of applications of additively separable functions in economics.)

Debreu's theorem was recently extended by Wakker [6] to ordered cones (i.e., sets in \( \mathbb{R}^N \), where \( x_1 \geq \cdots \geq x_N \geq 0 \)). In Theorem 1 below I add a monotonicity assumption and prove this theorem for more general non-convex sets. The crucial conditions concerning the domain of the order \( \succeq \) are that all its indifference surfaces are connected sets (see [7] for the importance of this condition) and that the intersection of this domain with any parallel-to-the-axes hyperplane is a connected set. Section 2 deals with the case of an open domain, and Section 3 proves a sufficient condition for the existence of an additively separable representation on a closed set given that such a representation exists on its interior. Section 4 concludes with three examples illustrating the importance of some of the assumptions used throughout. Topological definitions and claims are taken from Dugundji [5].

2. Open Domain

Let the connected [5, p. 107] and open subset \( S \subset \mathbb{R}^N \), \( N \geq 3 \), satisfy the condition that for every \( i \) and \( c \), the set \( S(i, c) = S \cap \{(x_1, \ldots, x_N) : x_i = c\} \) is a connected subset of \( \mathbb{R}^N \). Of course, the connectedness of \( S \) does not imply, nor is it implied, by the connectedness of the sets \( S(i, c) \), as is demonstrated by the sets \( S = (0, 3)^3 \setminus [(1, 3) \times (0, 3) \times [1, 2)] \) and \( S = (0, 1)^3 \cup (2, 3)^3 \).

Let \( \succeq \) be a reflexive, complete, and transitive order on \( S \). For \( x, y \in S \), define \( x \sim y \) if and only if \( x \succeq y \) and \( y \succeq x \), and \( x \succ y \) if and only if \( x \succeq y \) but not \( y \succeq x \). The order \( \succeq \) is called strictly monotonic if \( \forall i \in \{1, \ldots, N\} \) \( y_i \geq x_i \) and \( \exists i \in \{1, \ldots, N\} \) such that \( y_i > x_i \Rightarrow y = (y_1, \ldots, y_N) \succ x = (x_1, \ldots, x_N) \). It is called continuous if for every \( x \in S \) the two sets \( \{y \in S : y \succ x\} \) and \( \{y \in S : x \succ y\} \) are open in \( S \). Assume throughout that \( \succeq \) is strictly monotonic and continuous. For every \( x \in S \), let \( I(x) = \{y \in S : y \sim x\} \) be the indifference surface of \( \succeq \) through \( x \). Assume further that all indifference surfaces of \( \succeq \) are connected subsets of \( \mathbb{R}^N \). By Debreu [3] there exists a continuous function \( V : S \to \mathbb{R} \) representing the order \( \succeq \), that is, \( x \succeq y \) if and only if \( V(x) \geq V(y) \).

**Definition 1.** The function \( V : S \to \mathbb{R} \) is called completely separable if, for every \((x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N)\), \((y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_N)\), \((x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N)\), and \((y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N)\) in \( S \),

\[
V(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) \geq V(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_N)
\]

\(\iff\)

\[
V(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N) \geq V(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N).
\]

\(^2\) When \( N = 2 \), strict monotonicity implies complete separability (see Definition 1).
Let $\pi_i(S)$ be the projection of $S$ on the $i$th axis,
\[ \pi_i(S) = \{ x_i; \exists (x_1, \ldots, x_i, \ldots, x_N) \in S \}. \]
Since $S$ is an open connected subset of $\mathbb{R}^N$, $\pi_i(S)$ is an open interval, $i = 1, \ldots, N$.

**Definition 2.** The function $V: S \to \mathbb{R}$ is called additively separable if there exist continuous and strictly increasing functions $u_i: \pi_i(S) \to \mathbb{R}$, $i = 1, \ldots, N$ and a strictly increasing function $\zeta: \text{Rng}(\sum_{i=1}^{N} u_i(\cdot)) \to \mathbb{R}$ such that
\[ V(x_1, \ldots, x_N) = \zeta \left( \sum_{i=1}^{N} u_i(x_i) \right), \]
where \( \text{Rng}(\sum_{i=1}^{N} u_i(\cdot)) = \{ x \in \mathbb{R} : \exists (x_1, \ldots, x_N) \in \mathbb{R}^N \text{ such that } x = \sum_{i=1}^{N} u_i(x_i) \} \).

**Theorem 1.** Let $N \geq 3$ and let $(S, \succeq)$ satisfy the following conditions:
- The set $S$ is an open and connected subset of $\mathbb{R}^N$
- For every $i$ and $c$, the set $S(i, c)$ is a connected subset of $\mathbb{R}^N$
- The order $\succeq$ on $S$ is continuous and strictly monotonic
- All indifference surfaces of $\succeq$ are connected subsets of $\mathbb{R}^N$.

Then a representation function $V$ of $\succeq$ is completely separable if and only if it is additively separable.

Obviously, if $V$ is additively separable, then it is completely separable. The proof that the opposite is also true follows from Lemmas 1–4. In the sequel, the term open box means a set of the form $\prod_{i=1}^{N} J_i$, where $J_i$ is a nonempty bounded open interval in $\pi_i(S)$, $i = 1, \ldots, N$. For each $x^0 \in S$, let $R(x^0)$ be an open box in $S$ containing $x^0$.² By Debreu’s [4] theorem, there are strictly increasing and continuous functions $u_i(\cdot; x^0)$ and $\zeta(\cdot; x^0)$ such that on $R(x^0)$,
\[ V(x_1, \ldots, x_N) = \zeta \left( \sum_{i=1}^{N} u_i(x_i; x^0); x^0 \right). \]  

**(Lemma 1.)** Let $x^0 = (x^0_1, \ldots, x^0_i, \ldots, x^0_N)$, $y^0 = (y^0_1, \ldots, x^0_i, \ldots, y^0_N) \in S$ and let $R^x$ and $R^y$ be two open boxes in $S$ such that $x^0 \in R^x$ and $y^0 \in R^y$. Let the order $\succeq$ on $\mathbb{R}^2$ be represented by $\sum_{i=1}^{N} u_i^2(x_i) + z = x, y$.² Then there are $a > 0$, $b$, and $\varepsilon > 0$, such that for every $x_{i_0} \in (x^0_{i_0} - \varepsilon, x^0_{i_0} + \varepsilon)$, $u_{i_0}^y(x_{i_0}) = au_{i_0}^x(x_{i_0}) + b$.

**Proof.** The set $S(i_0, x^0_{i_0})$ is (isomorphic to) an open set in $\mathbb{R}^{N-1}$ and

² There are, of course, a lot of open boxes containing $x^0$ in $S$. For every $x^0$ we choose one of these possible boxes to be $R(x^0)$.

² Such representations exist by Debreu [4].
connected, hence path connected. (A set $T \in \mathbb{R}^N$ is called path connected if for every $x, y \in T$ there exists a continuous mapping $f: [0, 1] \to T$ such that $f(0) = x$ and $f(1) = y$; see [5, pp. 114–116].) There is, therefore, a continuous mapping $f: [0, 1] \to S(i_0, x_0^0)$ such that $f(0) = x^0$ and $f(1) = y^0$.

Denote its image by $L$. The curve $L \subset S$ is compact [5, p. 224]; hence there is a finite set of open boxes $R^1, \ldots, R^m \subset S$ covering $L$. Assume, without loss of generality, that $R^1 = R^x, R^m = R^y$, and $R^j \cap R^{j+1} \neq \emptyset$, $j = 1, \ldots, m-1$.

By the above-mentioned theorem of Debreu, on $R^j$ the order can be represented by $U^j(x_1, \ldots, x_N) = \sum_{i=1}^N u^j_i(x_i)$, $j = 1, \ldots, m$, where $u^j_i \equiv u^x_i$ and $u^m_i \equiv u^y_i$, $i = 1, \ldots, N$.

Let $1 \leq j < m$. On the set $R^j \cap R^{j+1}$ the order can be represented by both $U^j$ and $U^{j+1}$; hence there exists a positive linear transformation such that $U^{j+1} = a^jU^j + b^j$ (see [4]). In particular, on this intersection, $u^{j+1}_i = a^j u^j_i + b^j$. Let $(x_0^0 - \varepsilon, x_0^0 + \varepsilon, \varepsilon > 0)$ be contained in the projection on the $i_0$th axis of the open boxes $R^1, \ldots, R^m$. This $\varepsilon$ satisfies the conditions of the lemma.

Q.E.D.

Remark. The proof of Lemma 1 is the only place in the proof of Theorem 1 where the assumption concerning the connectedness of the sets $S(i, c)$ is used.

**Lemma 2.** There are $N$ strictly increasing and continuous functions $u_i: \pi_1(S) \to \mathbb{R}$, $i = 1, \ldots, N$, such that for every $x^0$, on $R(x^0)$, $V(x_1, \ldots, x_N) = \sum_{i=1}^N u_i(x_i); x^0$.

**Proof.** I show first how to construct $u_i$. Let $x_1 = \inf \pi_1(S)$ and $\bar{x}_1 = \sup \pi_1(S)$. If $S$ is unbounded, $x_1$ may be $-\infty$ and $\bar{x}_1$ may be $\infty$. Let $x_1^n \to x_1$ and $\bar{x}_1^n \to \bar{x}_1$ such that for every $n$, $x_1^{n+1} < x_1^n < \bar{x}_1^n < \bar{x}_1^{n+1}$. Let $Y^n = [x_1^n, \bar{x}_1^n]$. For every $t \in Y^n$ there is a point $x(t) - (t, x_2, \ldots, x_N) \in S$ with the open box $R(x(t)) \subset S$. Let the open interval $X(t)$ be the projection of $R(x(t))$ on the $\pi_1(S)$. Since $Y^n$ is compact, there is a finite set of these open intervals covering $Y^n$. Denote them by $X^1, \ldots, X^m$, and assume, without loss of generality, that $x_1^j \in X^1, \bar{x}_1^j \in X^m, X^j \cap X^{j+1} \neq \emptyset$, $j = 1, \ldots, m-1$, and $X^j \cap X^{j+2} = \emptyset$, $j = 1, \ldots, m-2$. Otherwise, if for some $j$, $X^j \cap X^{j+2} \neq \emptyset$, then $\{X^1, \ldots, X^j, X^{j+2}, \ldots, X^m\}$ is also a cover of $Y^n$. Note that this implies that for $k < m-1$, $X_k \cap (\bigcup_{j=k+2}^m X^j) = \emptyset$. For every $1 \leq j \leq m$ there is $t^j \in X^j$ such that $X^j$ is the projection on $\pi_1(S)$ of the open box $R(x(t^j))$. On $R(x(t^j))$ the order $\geq$ can be represented by $\sum_{i=1}^N u_i^n(x_i)$, where the functions $u_i^n$ are strictly increasing and continuous, $i = 1, \ldots, N, j = 1, \ldots, m$.

Define $u_i^n: Y^n \to \mathbb{R}$ as follows. For $x_1 \in Y^n \cap X^j$, let $u^n_i(x_1) = u^n_i(x_1)$. Let $k \in \{1, \ldots, m-1\}$ and suppose that we have defined $u^n_i$ on $Y^n \cap (\bigcup_{j=k+1}^m X^j)$ and redefine the functions $u_i^n$, $i = 1, \ldots, N$ and $j = k+1, \ldots, m$, such that for $x_1 \in X^j$, $u^n_i(x_1) = u^n_i(x_1)$, $j = k+1, \ldots, m$. Extend $u^n_i$ now to $Y^n \cap (\bigcup_{j=k}^m X^j)$ as follows: Let $t \in X^k \cap X^{k+1}$. By Lemma 1 it
follows that there is an open interval $T$ around $\tilde{t}$ in $\pi_1(S)$ on which $u_{k+1}^{k,n}(x_1) = a^k u_{k,n}^n(x_1) + b^k$ with $a^k > 0$. Assume therefore, without loss of generality, that on $R(x(t^*))$ the order is represented by $a^k \sum_{i=1}^{N} u_{i}^{k,n}(x_i) + b^k$ rather than by $\sum_{i=1}^{N} u_{i}^{k,n}(x_i)$, and redefine $u_{k,n}^{k,n}, \ldots, u_{N,n}^{k,n}$ accordingly. Hence on $T$, $u_{k+1}^{k,n} = u_{k+1,n}^{k,n} = u_{k,n}^n$.

Suppose now that there exists $x_1^* \in X_k \cap X^{k+1}$ such that $u_{k,n}(x_1^*) \neq u_{k+1,n}^{k,n}(x_1^*)$. Assume first that $x_1^* > \tilde{t}$. Let $t^* = \inf \{ x_1 > \tilde{t} : u_{k,n}^{k,n}(x_1) \neq u_{k+1,n}^{k,n}(x_1) \}$. Since on $T$, $u_{k+1}^{k,n} = u_{k,n}^{k,n}$, it follows that $t^* > \tilde{t}$. By Lemma 1 there is an open interval $T^*$ around $t^*$ on which $u_{k,n}$ is a positive linear transformation of $u_{k}^{k,n}$. Since these two functions coincide on $[\tilde{t}, t^*]$, they must also coincide on $T^*$, a contradiction. Similar proof deals with the case $x_1^* < \tilde{t}$. It thus follows that on $X^k \cap X^{k+1}$, $u_{k,n}^{k,n} = u_{k+1,n}^{k,n} = u_{k,n}^n$. Extend now the function $u_{k,n}^n$ such that on $X^k$, $u_{k,n}^n = u_{k+1,n}^{k,n}$. For every $k < m - 1$, $X^k \cap (\bigcup_{j=k+1}^{m} X^j) = \emptyset$, hence this extension of $u_{k,n}^n$ does not violate its already defined values. In a finite number of steps we obtain $u_{k,n}^n$ defined for every $x_1 \in Y^n$. Since for every $k$, $u_{k,n}^{k,n}$ is strictly increasing and continuous, so is $u_{k,n}^n$.

Suppose now that another choice of points and boxes yields the function $v_{k,n}^n$. By Lemma 1 there is a finite open cover of $Y^n$ such that on each of its members, $v_{k,n}^n$ is a positive linear transformation of $u_{k,n}^n$. Since these open sets have nonempty open intersections, it must be the same transformation on each neighbouring set, hence on all of these sets. The function $u_{k,n}^n$ is thus unique up to positive linear transformations. That is, there are $a > 0$ and $b$ such that on $Y^n$, $v_{k,n}^n = a u_{k,n}^n + b$.

Consider now the functions $u_{k,n}^n$ and $u_{k,n}^{n+1}$. By the above arguments, we may assume, without loss of generality, that on $Y^n$, $u_{k,n}^{n+1} = u_{k,n}^n$. For $x_1 \in [x_1^n, \bar{x}_1^n]$, let $u_1(x_1) = u_{k,n}^n(x_1)$ and, for $x_1 \in [x_1^{n+1}, \bar{x}_1^{n+1}] \cup (\bar{x}_1^n, \bar{x}_1^{n+1}]$, let $u_1(x_1) = u_{k,n}^{n+1}(x_1)$. The function $u_1$ is thus defined on $(x_1, \bar{x}_1)$, and since for every $n$, $u_{k,n}^n$ is strictly increasing and continuous, so is $u_1$.

Let $x^0 \in S$. By (1), the order $\succeq$ on $R(x^0)$ can be represented by $\sum_{i=1}^{N} u_i(x_i; x^0)$. I show next that on its domain, $u_1(\cdot; x^0)$ is an increasing linear transformation of $u_1$. Let $Z$ be the projection of $R(x^0)$ on $\pi_1(S)$. Let $\alpha, \beta \in Z$, and let $n$ such that $[\alpha, \beta] \subset ([x_1^n, \bar{x}_1^n])$. For each $z \in [\alpha, \beta]$ there is, by Lemma 1, an open interval around $z$ in $Z \cap (x_1^n, \bar{x}_1^n)$ on which $u_1(\cdot; x^0)$ is an increasing linear transformation of $u_1$, and hence also of $u_1$ by the same arguments used in the construction of $u_{k,n}^n$ it is easy to verify that $u_1(\cdot; x^0)$ is an increasing linear transformation of $u_1$ on $[\alpha, \beta]$. Let $[x_{\alpha}, x_{\beta}] \rightarrow Z$ to get the desired result.

Repeat the above procedure for each coordinate. By the above arguments we have now obtained the condition for $V$ on $R(x^0)$,

$$V(x_1, \ldots, x_N) = \zeta \left( \sum_{i=1}^{N} \left[ a_i(x^0) u_i(x_i) \right] + b(x^0); x^0 \right),$$
where \( b(x^0) = \sum b_i(x^0) \). The functions \( u_i \) are unique up to positive linear transformations. Assume therefore, without loss of generality, that at a certain point \( x^* \in S \), \( a_1(x^*) = \cdots = a_N(x^*) = a(x^*) \). I now want to show that for every \( x^0 \in S \), \( a_i(x^0) = \cdots = a_N(x^0) \). Let \( x^0 \in S \), and let the curve \( L \subset S \) connect \( x^* \) and \( x^0 \) (recall that as \( S \) is open and connected, it is also path connected). Each point \( x \) on the curve defines an open box \( R(x) \), and, since \( L \) is compact, it can be covered by the finite set of open boxes \( R(x^1), \ldots, R(x^m) \), where \( x^1 = x^*, \ x^n - x^0 \), and \( R(x^j) \cap R(x^{j+1}) \neq \emptyset \), \( j = 1, \ldots, m - 1 \). On the open set \( R(x^*) \cap R(x^2) \), we obtain the two representations \( \sum_{i=1}^N [a(x^*) u_i(x_i)] + b(x^*) \) and \( \sum_{i=1}^N [a_i(x^2) u_i(x_i)] + b(x^2) \). By \([4]\) they are cardinally equivalent, that is, \( \sum_{i=1}^N [a_i(x^2) u_i(x_i)] + b(x^2) \) is a positive linear transformation of \( \sum_{i=1}^N [a(x^*) u_i(x_i)] + b(x^*) \). Hence \( a_1(x^2) = \cdots = a_N(x^2) \). By finite induction it thus follows that \( a_i(x^0) = \cdots = a_N(x^0) \).

We now obtain from (2) that on each \( R(x^0) \),

\[
V(x_1, \ldots, x_N) = \xi \left( a(x^0) \sum_{i=1}^N u_i(x_i) + b(x^0); x^0 \right).
\]

Define \( \zeta(s; x^0) = \xi(a(x^0)s + b(x^0); x^0) \) to obtain the desired representation.

Q.E.D.

Next I prove that the above representation can be used as a global one independent of \( x^0 \). For this, only the connectedness of the indifference surfaces is used. I start with the following technical lemma.

**Lemma 3.** All indifference surfaces of the order \( \succeq \) are path connected.

**Proof.** Let \( I \) be an indifference surface of \( \succeq \), and define \( f : I \to \mathbb{R}^{N-1} \) by \( f(x_1, x_2, \ldots, x_N) = (x_2, \ldots, x_N) \). This function is clearly continuous, and, since the order \( \succeq \) is strictly monotonic, \( f \) is one to one. Let \( G \) be the range of \( f \), \( G = \{ (x_2, \ldots, x_N) : \exists x_1 \text{ such that } (x_1, x_2, \ldots, x_N) \in I \} \). The inverse \( f^{-1} : G \to I \) of \( f \) is therefore well defined. The function \( f^{-1} \) is also continuous. Indeed, let \( \{(x_2^n, \ldots, x_N^n)\}_{n=0}^\infty \subset G \) such that \( (x_2^n, \ldots, x_N^n) \to (x_2^0, \ldots, x_N^0) \), but suppose that \( (x_1^0, x_2^0, \ldots, x_N^0) = f^{-1}(x_2^0, \ldots, x_N^0) \) does not converge to \( (x_1, x_2, \ldots, x_N) = f^{-1}(x_2, \ldots, x_N) \). Assume, without loss of generality, that for all \( n \), \( x_i^0 > x_i^0 + \epsilon \), \( \epsilon > 0 \). It thus follows by strict monotonicity that \( (x_1^0, x_2^0, \ldots, x_N^0) \) does not converge to \( (x_1^0, x_2^0, \ldots, x_N^0) = f^{-1}(x_2^0, \ldots, x_N^0) \). Assume, without loss of generality, that \( \forall n, x_i^0 > x_i^0 + \epsilon, \epsilon > 0 \). It thus follows by strict monotonicity that \( (x_1^0, x_2^0, \ldots, x_N^0) \) does not converge to \( (x_1^0, x_2^0, \ldots, x_N^0) = f^{-1}(x_2^0, \ldots, x_N^0) \), a contradiction.

Next I show that the set \( G \), the range of \( f \), is open in \( \mathbb{R}^{N-1} \). That is, \( \forall (x_2, \ldots, x_N) \in G \) there is \( \delta > 0 \) such that \( d((y_2, \ldots, y_N), (x_2, \ldots, x_N)) < \delta \Rightarrow (y_2, \ldots, y_N) \in G \), where \( d(\cdot, \cdot) \) is the Euclidean metric on \( \mathbb{R}^{N-1} \). Let \( (x_2, \ldots, x_N) \in G \), and let \( x = (x_1, x_2, \ldots, x_N) = f^{-1}(x_2, \ldots, x_N) \). Since \( S \) is
open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset S$, where $B(x, \varepsilon)$ is the ball of radius $\varepsilon$ around $x$. By the strict monotonicity of $\succsim$ it follows that

$$x^1 = \left(x_1 + \frac{\varepsilon}{2}, x_2, \ldots, x_N\right) \succ x \succ x^2 = \left(x_1 - \frac{\varepsilon}{2}, x_2, \ldots, x_N\right).$$

Since $\succsim$ is continuous there is $\delta > 0$ such that $B(x^1, \delta) \cup B(x^2, \delta) \subset B(x, \varepsilon)$, and for every $y^1 \in B(x^1, \delta)$ and $y^2 \in B(x^2, \delta)$, $y^1 \succ y^2$. Let $(y_2, \ldots, y_N) \in \mathbb{R}^{N-1}$ such that $d((y_2, \ldots, y_N), (x_2, \ldots, x_N)) < \delta$. It follows from the definition of $\delta$ that $(x_1 + \varepsilon/2, y_2, \ldots, y_N), (x_1 - \varepsilon/2, y_2, \ldots, y_N) \in S$ and that $(x_1 + \varepsilon/2, y_2, \ldots, y_N) \succ x \succ (x_1 - \varepsilon/2, y_2, \ldots, y_N)$. Hence by the continuity of $\succsim$ and the convexity of $B(x, \varepsilon)$, there exists $z_1$ such that $(z_1 + e/2, y_2, \ldots, y_N), (z_1 - e/2, y_2, \ldots, y_N) \in S$ and $(z_1, y_2, \ldots, y_N) \sim x$. Therefore, $(y_2, \ldots, y_N) \in G$.

The range of a continuous function on a connected set is connected [5, p. 108], and since $G$ is open, it is also path connected. The function $f^{-1}$ is continuous, $I = f^{-1}(G)$; hence the indifference surface $I$ is path connected [5, p. 115].

**LEMMA 4.** In the notations of Lemma 2, $V(x_1, \ldots, x_N) \geq V(y_1, \ldots, y_N) \iff \sum_{i=1}^N u_i(x_i) \geq \sum_{i=1}^N u_i(y_i)$.

**Proof.** Let $V(x_1, \ldots, x_N) = V(y_1, \ldots, y_N)$; that is, the two points $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are on the same indifference surface $I$ of $\succsim$. Let the curve $L \subset I$ connect these two points, and let the open boxes $R(x) = R^1, \ldots, R^m = R(y)$ be a finite open cover of $L$ in $S$ such that for every $j$, $L \cap R^j \cap R^{j+1} \neq \emptyset$ (recall that $L$ is a connected set). Let $z^j \in L \cap R^j \cap R^{j+1}, j = 1, \ldots, m - 1$. On each of the open boxes $R^1, \ldots, R^m$ the order $\succsim$ can be represented by $W(w) = \sum_{i=1}^N u_i(w_i)$; hence

$$\sum_{i=1}^N u_i(x_i) = \sum_{i=1}^N u_i(z_i^1) = \cdots = \sum_{i=1}^N u_i(z_i^{m-1}) = \sum_{i=1}^N u_i(y_i).$$

Suppose now that there are $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ such that $V(x) > V(y)$ (and $x > y$), but $\sum_{i=1}^N u_i(x_i) \leq \sum_{i=1}^N u_i(y_i)$. Let the curve $L \subset S$ connect these two points; that is, there is a continuous and onto function $f: [0, 1] \rightarrow L$ such that $f(0) = x$ and $f(1) = y$. By continuity we may assume that for every $t \in (0, 1)$, $x \succ f(t) \succ y$. Otherwise, let $s = \max\{t \in [0, 1] : x \sim f(t)\}$ and $r = \min\{t \in (s, 1] : y \sim f(t)\}$, and replace $x$ by $f(s)$ and $y$ by $f(r)$. By the first part of this proof, the value of the function $\sum_{i=1}^N u_i$ is the same at $x$ and $f(s)$, and at $y$ and $f(r)$. Define $g, h: [0, 1] \rightarrow \mathbb{R}$ by $g = (\sum_{i=1}^N u_i) \circ f$ and $h = V \circ f$. Since the functions $f, V$, and $u_1, \ldots, u_N$ are continuous, so are $g$ and $h$. Moreover, $g(0) \leq g(1)$ and $h(0) > h(1)$.

Let the open boxes $R(x) = R^1, \ldots, R^m = R(y)$ be a finite open cover of $L$.
in $S$ as in the first part of this proof. On $R(y)$ both $V$ and $\sum_{i=1}^{N} u_i$ represent the order $\succeq$. By construction, for every $z \in L \cap (R(y) \setminus \{y\})$, $z \succ y$; hence, for such $z$, $\sum_{i=1}^{N} u_i(z_i) > \sum_{i=1}^{N} u_i(y_i)$. There is, therefore, $t \in (0, 1)$ such that $g(t) > g(1) \geq g(0)$; hence the continuous function $g$ reaches a maximum $A$ on $(0, 1)$. Let $t^* = \min\{t \in [0, 1] : g(t) = A\}$.

On $R(f(t^*))$ both $V$ and $\sum_{i=1}^{N} u_i$ represent the order $\succeq$. There is $\varepsilon > 0$ such that for $t \in (t^* - \varepsilon, t^*)$, $f(t) \in R(f(t^*))$, and $g(t) < g(t^*)$, hence for $t \in (t^* - \varepsilon, t^*)$, $f(t^*) > f(t)$, and $h(t) < h(t^*)$. Since $h(0) > h(t^*)$, there exists $s \in (0, t^*)$ such that $h(s) = h(t^*)$. By the first part of this proof it follows that $g(s) = g(t^*)$, in contradiction with the definition of $t^*$. Q.E.D.

Theorem 1 now follows by Lemmas 1–4.

3. Closed Domain

The results of the last section do not hold for closed sets (see [7] for counterexamples). Following Wakker’s examples it is possible to show that Theorem 1 does not hold even when the set $S$ is bounded and equals the closure of its interior. For example, let $S = \text{Conv}\{(0, 0, 0), (10, 1, 1), (1, 10, 1), (1, 1, 10)\}$, and let $V(x_1, x_2, x_3) = x_1 x_2 x_3$. On the interior of $S$, $V(x_1, x_2, x_3) = x_1 x_2 x_3$, but this representation cannot be extended to $S$. In other words, $V$ is completely separable, but not additively separable.

Let $S \subset \mathbb{R}^N$ be compact, and define, for $i = 1, \ldots, N$,

$$S^i = \{(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) \in S : \quad (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N) \in S \Rightarrow x_i \succeq y_i\}$$

$$S_i = \{(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) \in S : \quad (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_N) \in S \Rightarrow x_i \preceq y_i\}.$$ 

**Theorem 2.** Let $N \geq 3$ and let $(S, \succeq, V)$ satisfy the following conditions:

- The set $S \subset \mathbb{R}^N$ is compact and equals the closure of its interior
- The interior of $S$, denoted by $T$, is a connected subset of $\mathbb{R}^N$
- The order $\succeq$ on $S$ is continuous and strictly monotonic
- The continuous function $V: S \to \mathbb{R}$ represents the order $\succeq$. This function is completely separable on $S$ and additively separable on $T$
- For every $i$, each of the sets $S^i$ and $S_i$ includes $y = (y_1, \ldots, y_N)$, $z = (z_1, \ldots, z_N)$, and $w = (w_1, \ldots, w_N)$ such that for every $j \neq i$, $y_j > z_j > w_j$.

Then the function $V$ is additively separable on $S$. 
Proof. The function $V$ is continuous on the compact domain $S$; hence it is bounded. By Theorem 1 there exist strictly increasing and continuous functions $u_1, ..., u_N$ such that on $T$, the interior of $S$,

$$V(x_1, ..., x_N) = \phi \left( \sum_{i=1}^{N} u_i(x_i) \right).$$

We want to show that the functions $u_i$ can be extended to $\bar{x}_i = \min \{ x_i : \exists (x_1, ..., x_i, ..., x_N) \in S \}$ and $\hat{x}_i = \max \{ x_i : \exists (x_1, ..., x_i, ..., x_N) \in S \}$, such that condition (3) is still satisfied.

Suppose that as $x_i \to \bar{x}_i$, $u(x_i) \to -\infty$. Let $y, z, w \in S_i$ be as in the fifth condition of the theorem. Since $T$ is a connected subset of $\mathbb{R}^N$, it follows that for every $j \neq i$, $z_j$ is in the domain of $u_j$, and $u_j(z_j) < \infty$. Since $S_i \subset S \setminus T$, there are sequences $\{z^n\}_{n=1}^{\infty}$ and $\{w^n\}_{n=1}^{\infty}$ in $T$ converging to $z$ and $w$, respectively. Moreover, by the fifth condition of the theorem, we may assume that for every $n$ and $j \in \{1, ..., N\}$, $z^n_j > w^n_j$. The continuous functions $u_j$ are strictly increasing; hence

$$\lim_{n \to \infty} \sum_{j=1}^{N} u_j(w^n_j) \leq \lim_{n \to \infty} \sum_{j=1}^{N} u_j(z^n_j) = -\infty.$$ 

By (3) and the continuity of $V$, it follows that $V(w) \leq V(z)$.

Since the functions $u_j$ are continuous on $T$ and $\lim_{x_i \to \bar{x}_i} u(x_i) = -\infty$, there are two sequences in $T$, $z^n \to z$ and $w^n \to w$, such that for every $n$, $\sum_{j=1}^{N} u_j(z^n_j) \leq \sum_{j=1}^{N} u_j(w^n_j)$. Again by (3) and the continuity of $V$, it follows that $V(z) \leq V(w)$; hence $V(z) = V(w)$, in contradiction with the assumption that the order $\succeq$ is strictly monotonic. The function $u_i$ can thus be extended to $\bar{x}_i$ and in a similar way to $\hat{x}_i$. Condition (3) follows by the continuity and boundedness of $V$.

Q.E.D.

The fifth condition of Theorem 2 is satisfied if, for example, for every $i$, the relative interiors of $S'$ and $S_i$ in $\mathbb{R}^{N-1}$ are not empty.

4. SOME FURTHER REMARKS

The importance of the assumption concerning the connectedness of the indifference curves is demonstrated in [7]. Wakker also shows why it is not sufficient to assume that the set $S$ is a closed set. As demonstrated by the following example, the condition that for every $i$ and $c$, the set $S(i, c)$ is a connected subset of $\mathbb{R}^N$, cannot be easily withdrawn.

**Example 1.** Let

- $A = \{ (x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 > 9, 2x_1 + x_2 < 20, 1 \leq x_3 < 4 \}$
- $B = \{ (x_1, x_2, x_3) \in \mathbb{R}^3_+ : 18 < 2x_1 + x_2 < 20, x_3 < 1 \}$
Let \( S = A \cup B \cup C \cup D \) and define a function \( V: S \rightarrow \mathbb{R} \) by

\[
V(x_1, x_2, x_3) = \begin{cases} 
  x_1 + x_2 + x_3 & \text{if } (x_1, x_2, x_3) \in A \cup B \cup D \\
  x_1 + x_2 + 2x_3 - 1 & \text{if } (x_1, x_2, x_3) \in C.
\end{cases}
\]

Define an order \( \succeq \) on \( S \) by \((x_1, x_2, x_3) \succeq (y_1, y_2, y_3)\) if and only if \( V(x_1, x_2, x_3) \geq V(y_1, y_2, y_3) \). Obviously, all the indifference surfaces of \( \succeq \) are connected subsets of \( \mathbb{R}^3 \). It is also easy to verify that \( V \) is completely separable, but it is not additively separable. Note that the set \( S(3, 2) = \{(x_1, x_2, x_3) \in S: x_3 = 2\} \) is not a connected subset of \( \mathbb{R}^3 \).

Next I show the importance of the assumption that the set \( S \) is a connected subset of \( \mathbb{R}^N \).

**Example 2.** Let \( S = (0, 1)^3 \cup (1, 2)^3 \), and let \( V: S \rightarrow \mathbb{R} \) be given by

\[
V(x_1, x_2, x_3) = \begin{cases} 
  x_1 \cdot x_2 \cdot x_3 & \text{if } (x_1, x_2, x_3) \in (0, 1)^3 \\
  (x_1 - 1)(x_2 - 1)(x_3 - 1) + 1 & \text{if } (x_1, x_2, x_3) \in (1, 2)^3.
\end{cases}
\]

\( V \) is completely separable, but not additively separable.

The aim of the next example is to demonstrate the importance of the existence of three points \( y, z, \) and \( w \) as in the fifth condition of Theorem 2.

**Example 3.** Let

- \( A = \text{Conv}\{(-\pi/2, \pi/2, \pi/2), (\pi/2, -\pi/2, \pi/2), (\pi/2, \pi/2, -\pi/2), (\pi/4, \pi/4, \pi/4)\} \)
- \( B = \text{Conv}\{(-\pi/2, -\pi/2, -\pi/2), (0, \pi/4, \pi/4), (\pi/4, 0, \pi/4), (\pi/4, \pi/4, 0)\} \)

Define \( S = A \cup B \). Let \( V(-\pi/2, -\pi/2, -\pi/2) = -\pi/2, \quad V(-\pi/2, \pi/2, \pi/2) = V(\pi/2, -\pi/2, \pi/2) = V(\pi/2, \pi/2, -\pi/2) = \pi/2, \) and at all the other points of \( S \), let

\[
V(x_1, x_2, x_3) = \arctan(\tan x_1 + \tan x_2 + \tan x_3).
\]

On the interior of \( S \), the function \( V \) is completely separable, additively separable, and continuous. On \( S \), the function \( V \) is completely separable and continuous, but not additively separable. The only non-trivial property of \( V \) is its continuity.
Let \((x_1^n, x_2^n, x_3^n) \rightarrow (-\pi/2, \pi/2, \pi/2)\) such that for every \(n\), \((x_1^n, x_2^n, x_3^n) \neq (-\pi/2, \pi/2, \pi/2)\). For a sufficiently large \(n\), \(x_1^n + x_2^n + x_3^n \geq \pi/2\). Let \(\varepsilon^n = x_1^n + \pi/2\), and it follows that either \(\pi/2 - x_2^n \leq \varepsilon^n/2\) or \(\pi/2 - x_3^n \leq \varepsilon^n/2\). By using l'Hôpital's rule together with the equivalence \(\tan x = \sin x/\cos x\) it is easy to verify that

\[
\lim_{\varepsilon \to 0} \left[ \tan \left( -\frac{\pi}{2} + \varepsilon \right) + \tan \left( \frac{\pi}{2} - \frac{\varepsilon}{2} \right) \right] = \infty;
\]

hence \(V\) is continuous on \(S\).

REFERENCES

7. P. P. Wakker, Counterexamples to additive representation on non-full cartesian products, Duke University, Fuqua School of Business, mimeo.