ANTICIPATED UTILITY: A MEASURE REPRESENTATION APPROACH*

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Abstract

This paper presents axioms which imply that a preference relation over lotteries can be represented by a measure of the area above the distribution function of each lottery. A special case of this family is the anticipated utility functional. One additional axiom implies this theory. This functional is then extended for the case of vectorial prizes.

1. Introduction

The optimal choice among a set of uncertain alternatives has been discussed since the early 18th century. Bernoulli [3] suggested that people prefer the lottery yielding the greatest expected utility rather than the lottery with the highest expected value. According to this theory, the value of the lottery \((x_1, p_1; \ldots; x_n, p_n)\), which yields \(x_i\) dollars with probability \(p_i\), \(i = 1, \ldots, n\), is \(\sum p_i u(x_i)\). Von Neumann and Morgenstern [19] first presented a formal set of axioms implying maximization of expected utility. Some experimental evidence shows, however, that people do not always behave in accordance with this theory (for a survey of these, see Machina [10]), thus challenging the descriptive value of this theory.

Recent years saw the emergence of some alternatives to expected utility theory. One of the most efficient of these new theories is anticipated utility (also known as expected utility with rank-dependent probabilities), first presented by Quiggin [11]. According to this theory, there is an increasing utility function \(u\), unique up to positive linear transformations, and a probabilities transformation function \(f\), such that the value of the lottery \((x_1, p_1; \ldots; x_n, p_n)\), where \(x_1 \leq \ldots \leq x_n\) is

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\[ u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})] f\left(\sum_{j=i}^{n} p_j\right) \]

\[ = u(x_n) f(p_n) + \sum_{i=1}^{n-1} u(x_i) \left[f\left(\sum_{j=i}^{n} p_j\right) - f\left(\sum_{j=i+1}^{n} p_j\right)\right], \]

where \( f(0) = 0 \) and \( f(1) = 1 \). Without loss of generality, one may assume that \( u(0) = 0 \). Note that when \( f(p) = p \), this function is reduced to the expected utility representation function. This theory claims that for every prize \( x_i \), the decision maker is interested not only in the winning probability of this prize, but also in the probability that he will win more than \( x_i \). This functional is not Fréchet differentiable and therefore does not satisfy Machina’s [9] conditions. However, Chew et al. [5] proved that Machina’s results hold true even when the functional is only Gateaux differentiable. Moreover, they showed that the anticipated utility functional is Gateaux differentiable.

This functional is very useful in the unravelling of some of the most famous paradoxes in uncertainty theory. The Allais paradox is solved in Quiggin [11] and Segal [16], the common ratio effect in Segal [16], and the preference reversal phenomenon in Karni and Safra [8]. It is also useful in analyzing the Ellsberg paradox (Segal [17]) and the probabilistic insurance problem (Segal [18]). So far, this theory has received several axiomatizations. Quiggin suggested a set of axioms, necessarily implying that \( f(1/2) = f(1)/2 \). However, we now know that in this theory the convexity of \( f \) is essential for risk aversion behavior (see Chew et al. [5]). Another set of axioms was suggested by Yaari [20], but his assumptions restrict the utility function \( u \) to be linear. For a further discussion of Quiggin and Yaari’s axioms, see Röell [13]. Recently, Chew and Epstein [4] discussed axiomatizations of nonexpected utility theories, including anticipated utility.

One possible way to represent preference relations over lotteries is by a measure of the epigraphs of their cumulative distribution functions. It turns out that expected value, expected utility, and rank-dependent probabilities are all special cases of this general family. The first part of this paper, therefore, presents a set of axioms implying such a general representation, which we suggest to call the measure representation (theorem 1). It follows that product measures are of a special interest, where the most general form is the anticipated utility functional. One additional axiom implies indeed that the general measure is a product measure (theorem 2). This functional is then extended to the case of non-money prizes (section 3).

2. Representation of \( \xi \)

Let \( L \) be the family of all the real random variables with outcomes in \([0, M]\). For every \( X \in L \), define the cumulative distribution function \( F_X \) by \( F_X(x) = \Pr(X < x) \).
Let $X^0 = \text{Cl}((x, p) \in R_+ \times [0, 1] : p > F_X(x))$, where $\text{Cl} A$ is the closure of $A$ (see fig. 1).

Let $L^0$ be the family of all the non-empty closed sets $X^0$ in $R_+ \times [0, 1]$ that satisfy the following conditions:

1. If $(x, p) \in X^0$, $0 \leq y \leq x$, and $p \leq q \leq 1$, then $(y, q) \in X^0$.
2. For $x > M. (x, 1) \notin X^0$.
3. $X^0 = \text{Cl} (\text{Int} X^0)$.

Obviously, there is a one-to-one correspondence between $L$ and $L^0$.

Let $L^*$ be the set of all the elements of $L$ for which the range of $F_X$ is finite. Elements of $L^*$, called prospects, will be denoted by vectors of the form $(x_1, p_1; \ldots ; x_n, p_n)$, where $x_1 \leq \ldots \leq x_n$ and $\Sigma p_i = 1$. Such a vector represents a lottery yielding $x_i$ dollars with probability $p_i$, $i = 1, \ldots , n$. Obviously, if $X = (x_1, p_1; \ldots ; x_n, p_n)$, then

$$F_X(x) = \left\{ \begin{array}{ll}
0 & x < x_1 \\
\sum_{j=1}^{i} p_j & x_i \leq x < x_{i+1} \\
1 & x \geq x_n.
\end{array} \right.$$
On $L$ (and $L^0$), we assume that there exists a binary relation $\preceq$. Let $X \sim Y$ mean $X \preceq Y$ and $Y \preceq X$, and let $X \equiv Y$ mean $X \preceq Y$ but not $Y \preceq X$. We assume that $\preceq$ is complete and transitive, and that it also satisfies the following assumptions:

(a) **Continuity**: $\preceq$ is continuous in the topology of weak convergence. That is, if $X, Y, Y_1, Y_2, \ldots \in L$, such that at each continuity point $x$ of $F_Y$, $F_{Y_i}(x) \rightarrow F_Y(x)$, and if for all $i$, $X \preceq Y_i$, then $X \preceq Y$. Similarly, if for all $i$, $Y_i \preceq X$, then $Y \preceq X$.

A lottery $X$ is said to dominate lottery $Y$ by first-order stochastic dominance if for every $x$, $F_X(x) \leq F_Y(x)$, and there exists $x$ for which $F_X(x) < F_Y(x)$.

(b) **First-order stochastic dominance axiom**: If $X$ dominates $Y$ by first-order stochastic dominance, then $X \equiv Y$.

According to this axiom, the decision maker is interested in the probability of receiving more (or less) than every possible outcome $x$. It is, therefore, a natural extension of this axiom to assume that whenever he compares $X$ and $Y$, the decision maker examines those prizes $x$ for which $\text{Pr}(X \leq x) \neq \text{Pr}(Y \leq x)$. Formally, we suggest the following "irrelevance" axiom.

(c) **Irrelevance**: Let $X, Y, X', Y' \in L$, and let $S$ be a finite union of segments. If on $S$, $F_X = F_Y$, $F_{X'} = F_{Y'}$, and on $[0, M] \setminus S$, $F_X = F_{X'}$, $F_Y = F_{Y'}$, then $X \preceq Y$ iff $X' \preceq Y'$ (see fig. 2).

This axiom suggests that decision makers first eliminate all the points $x$ for which the probability of receiving more than $x$ is the same under $X$ and $Y$. Then, they compare $X$ and $Y$ by their corresponding cumulative distribution functions at those points where these probabilities are not equal. Furthermore, they do it independently of the probabilities that are equal. This axiom resembles Savage's sure-thing principle (Savage [21]), but it is much weaker. Savage suggested the following axiom as a basic rule to be used in uncertain situations:

**Sure-thing principle**: Let $S$ be an event and let $X, Y, X'$, and $Y'$ be lotteries. If on $S$, $X = Y$, $X' = Y'$, and on "not $S$", $X = X'$, $Y = Y'$, then $X \preceq Y$ iff $X' \preceq Y'$.

Among other things, this axiom implies that the evaluation of the prizes available if $S$ happens does not depend on the prizes available if "not $S$" happens. In particular, it does not depend on whether the prizes at "not $S$" are larger or smaller than those of $S$. It is well known that this axiom is unacceptable from a descriptive point of view, apparently because of this last objection. The irrelevance axiom, on the other hand, restricts the sure-thing principle to those cases where the order of the

*This axiom is equivalent to the cancellation axiom in Segal [14].
prizes is not reversed. For example, the irrelevance axiom agrees with the sure-thing principle that $(5, 0.1; 0, 0.01; 0, 0.89) \geq (1, 0.1; 1, 0.01; 0, 0.89) \Rightarrow (5, 0.1; 0, 0.01; -1, 0.89) \geq (1, 0.1; 1, 0.01; -1, 0.89)$, but it does not agree that $(5, 0.1; 0, 0.01; 0, 0.89) \geq (1, 0.1; 1, 0.01; 0, 0.89) \Rightarrow (5, 0.1; 0, 0.01; 1, 0.89) \geq (1, 0.1; 1, 0.01; 1, 0.89)$. Therefore, the irrelevance axiom does not rule out Allais-type behavioral* patterns, as does the sure-thing principle. Indeed, the reason decision makers violate the sure-thing principle through the Allais paradox is that replacing $(0, 0.89)$ by $(1, 0.89)$ makes $(0, 0.01)$ strictly worse than all other prizes in one lottery only. For a further discussion of these issues, see Quiggin [12] and Gilboa [7].

One possible interpretation of the anticipated utility functional is to look at it as a measure. Let $X$ be a random variable. The expected value of $X$ is

$$
\int_0^M x \, dF_X(x) = xF_X(x) \bigg|_0^M - \int_0^M F_X(x) \, dx.
$$

The first term in this last expression is the area of the rectangle $[0, M] \times [0, 1]$, while the second term is the area below $F_X$. The expected value of $X$, therefore, equals the area of $X^0$.

* Allais [2] found that most people prefer $(0, 0.9; 5000000, 0.1)$ to $(0, 0.89; 1000000, 0.11)$, but $(1000000, 1)$ to $(0, 0.01; 1000000, 0.89, 5000000, 0.1)$. 
The expected utility functional can be represented as

\[ \int_0^M u(x) dF_X(x) = u(x)F_X(x) \bigg|_0^M - \int_0^M F_X(x) du(x) \]

\[ = \int_0^M \int_0^1 dp \, du(x), \]

which is a measure of the area of \( X^0 \). Moreover, it is a product measure, where the measure of \([x, y]\) on the prizes’ axis is \( u(y) - u(x) \), and the measure on the probabilities’ axis is the Lebesgue measure.

Consider now the anticipated utilities functional. Its general form is

\[ - \int_0^M u(x) dF(1 - F_X(x)) = u(x)f(1 - F_X(x)) \bigg|_0^M - \int_0^M f(1 - F_X(x)) du(x) \]

\[ = \int_0^M \int_0^1 - df(1 - p) du(x). \]

This too is a product measure of \( X^0 \). As before, the measure of \([x, y]\) is \( u(y) - u(x) \), but the measure of \([p, q]\) is \( f(1 - p) - f(1 - q) \). Yaari [20] assumed linear utility function; in his case, therefore, the measure on the prizes’ axis is the Lebesgue measure. This justifies calling his theory the dual theory. This discussion indicates the importance of representing a preference relation by a measure of \( X^0 \).

**THEOREM 1**

\( \preceq \) on \( L \) satisfies axioms (a)–(c) iff there is a measure \( \nu \) on \( \Gamma = [0, M] \times [0, 1] \) mutually absolutely continuous with respect to the Lebesgue measure on \( \Gamma \) such that \( X \preceq Y \) iff \( \nu(X^0) \geq \nu(Y^0) \).

**Proof**

It is easy to verify that if \( \preceq \) can be represented by a measure \( \nu \) then it satisfies axioms (a)–(c). Assume now that \( \preceq \) satisfies axioms (a)–(c). Let

\[ \Delta = \{ [x, y] \times [p, q] \in [0, M] \times [0, 1] : x < y, \, p < q \} \]

be the set of all the compact rectangles in \( \Gamma \), and let

\[ \Psi = \{ (X, \delta) \in L \times \Delta : \text{Int } X^0 \cap \text{Int } \delta = \emptyset, \, X^0 \cup \delta \in L^0 \}. \]

The irrelevance axiom implies that if \( (X, \delta), (Y, \delta) \in \Psi \), then \( X^0 \preceq Y^0 \) iff \( X^0 \cup \delta \preceq Y^0 \cup \delta \). Define on \( \Delta \)
partial orders $R_X$ by $\delta_1 R_X \delta_2$ iff $(X, \delta_1), (X, \delta_2) \in \Psi$ and $X^0 \cup \delta_1 \preceq X^0 \cup \delta_2$. For every $X$ and $Y$, $R_X$ and $R_Y$ do not contradict each other. That is, if $\delta_1$ and $\delta_2$ can be compared by both $R_X$ and $R_Y$, then $\delta_1 R_X \delta_2$ iff $\delta_1 R_Y \delta_2$. Indeed, let $X, Y \in L$ and $\delta_1, \delta_2 \in \Delta$ such that $(X, \delta_i), (Y, \delta_i) \in \Psi, i = 1, 2$ and let $Z^0 = X^0 \cap Y^0$. Obviously, $Z \in L$ and $(Z, \delta_1), (Z, \delta_2) \in \Psi$. There exist $\delta_1^1, \ldots, \delta_x^1$, and $\delta_2^1, \ldots, \delta_t^1$ such that

$$\forall i \sum_{k=1}^{j-1} Z^0 \cup \bigcup_{k=1}^s \delta_k^1, \delta_j^i) \in \Psi, \quad i = 1, 2,$$

and

$$Y^0 = Z^0 \cup \bigcup_{k=1}^r \delta_k^2.$$

It follows that

$$\delta_1 R_X \delta_2 \iff Z^0 \cup \delta_1^1 \cup \ldots \cup \delta_x^1 \cup \delta_1,$$

$$\lnot Z^0 \cup \delta_1^1 \cup \ldots \cup \delta_x^1 \cup \delta_2 \iff \ldots \iff Z^0 \cup \delta_1,$$

$$\lnot Z^0 \cup \delta_2 \iff \ldots \iff Z^0 \cup \delta_1 \cup \ldots \cup \delta_x^2 \cup \delta_1,$$

$$\lnot Z^0 \cup \delta_2 \cup \ldots \cup \delta_x^2 \cup \delta_2 \iff \delta_1 R_Y \delta_2.$$

Let $R = \cup R_X$. That is, $\delta_1 R \delta_2$ iff there exists $X$ such that $\delta_1 R_X \delta_2$. It is proved in the appendix that $R$ is acyclic. That is, $\delta_1 R \delta_2 R \ldots R \delta_2 R \delta_1$ imply $\delta_1 R \delta_3 R \ldots R \delta_2 R \delta_1$. Let $\lnot$ be the transitive closure of $R : \delta_1 \lnot R \delta_2$ iff there exist $\delta_3, \ldots, \delta_x$, such that $\delta_1 R \delta_3 R \ldots R \delta_2 R \delta_1$ and $\lnot$ is obviously complete and transitive.

Let $[0, x] \times [0, p] \sim [x, y_1] \times [p, 1]$ (see fig. 3), and let $\nu([0, x] \times [0, p]) = \nu([x, y_1] \times [p, 1]) = 1$. By the continuity assumption, there exist $z$ such that $\lnot x, w] \times [p, 1] \sim [0, z] \times [0, p] \sim [w, y_1] \times [p, 1]$. Define $\nu([x, w] \times [p, 1]) = \nu([w, y_1] \times [p, 1]) = 1/2$. This can be repeated again and again for the $x$ as well as for the $p$ axes. By the first-order stochastic dominance axiom, the areas of all these rectangles will become smaller and smaller. $\nu$ can thus be defined as an atomless, continuous, finitely additive measure on $[0, x] \times [0, p]$ and on $[x, y_1] \times [p, 1]$. Similarly, it can be defined for the rectangles $[y_i, y_{i+1}] \times [p, 1] \sim [0, x] \times [0, p]$, $i = 1, \ldots$. By the continuity assumption, $y_i$ is finite. Indeed, let $\lim y_i = y < M$. For all $i, ([0, y_i] \times [p, 1] \sim ([0, y_{i-1}] \times [p, 1]) \sim (([0, y_i] \times [p, 1])] \cup ([0, x] \times [0, p]))$, in contradiction to the continuity and the stochastic dominance axioms. This process
defines a finitely additive measure $\nu$ on $[x, M] \times [p, 1]$, which can be extended to $[0, M] \times [0, p]$ and to $[0, x] \times [p, 1]$, and thus to $[0, M] \times [0, 1]$. Define $\omega : L \to \mathbb{R}$ as follows. If

$$X^0 = \bigcup_{k=1}^t \delta_k$$

where

$$\forall j \left( \bigcup_{k=1}^{j-1} \delta_k, \delta_j \right) \in \Psi,$$

then

$$\omega(X) = \sum_{k=1}^t \nu(\delta_k).$$

Because $\nu$ is finitely additive, $\omega$ does not depend on the choice of $\delta_1, \ldots, \delta_t$.

Let

$$X^0 = \bigcup_{k=1}^t \delta_k \quad \text{and} \quad Y^0 = \bigcup_{\ell=1}^s \xi_{\xi}$$

where

$$\forall j \left( \bigcup_{k=1}^{j-1} \delta_k, \delta_j \right), \left( \bigcup_{\ell=1}^{j-1} \xi_{\xi}, \xi_{\xi} \right) \in \Psi,$$

such that $X \preceq Y$. Our aim is to construct two sequences $\{\delta'_{k}\}_{k=1}^t$ and $\{\xi_{\xi}\}_{\xi=1}^s$ such that

$$X^0 = \bigcup_{k=1}^t \delta'_k, \quad Y^0 = \bigcup_{\ell=1}^s \xi'_{\xi}, \quad \forall j \left( \bigcup_{k=1}^{j-1} \delta'_k, \delta'_j \right), \left( \bigcup_{\ell=1}^{j-1} \xi'_{\xi}, \xi'_{\xi} \right) \in \Psi,$$
and for every \( j \leq s' \), \( \delta_j' \sim \xi_j' \). We will do this by finite induction. If \( \delta_1 \sim \xi_1 \), then let \( \delta_1' = \delta_1 \) and \( \xi_1' = \xi_1 \). If \( \delta_1 \not\sim \xi_1 \), construct \( \delta_1' \in \Delta \) such that \( \delta_1' \sim \xi_1 \), \( \delta_2' = C(\delta_1' \setminus \xi_1') \in \Delta \), \( (\delta_1', \delta_2') \in \Psi \), and let \( \xi_1' = \xi_1 \). If \( \delta_1 \not\sim \xi_1 \), construct \( \delta_1' \), \( \xi_1' \), and \( \xi_2' \) similarly. It thus follows that in each step we can reduce the number of non-equivalent elements either in \( X \) or in \( Y \) (or in both) by one. The desired representation will thus be constructed in a finite number of steps. As \( X \not\subset Y \), it follows that \( t' \geq s' \).

Obviously,

\[
\omega \left( \bigcup_{k=1}^{s'} \delta_k' \right) = \omega \left( \bigcup_{k=1}^{s'} \xi_k' \right),
\]

and as \( t' \geq s' \), it follows that \( \omega(X) \geq \omega(Y) \).

Q.E.D.

As indicated above, the anticipated utility function is the most general form of product measure. To make sure that the epigraph measure \( \nu \) of theorem 1 is actually a product measure, one additional assumption is required. For the reasoning behind this assumption, consider fig. 4.

Assume that \( \delta_5 \sim \xi_1 \) and \( \delta_5 \cup \delta_6 \sim \xi_2 \). That is, \( X^0 \cup \delta_5 \sim X^0 \cup \delta_1 \) and \( X^0 \cup \delta_1 \cup \delta_5 \cup \delta_6 \sim X^0 \cup \delta_2 \). \( X^0 \cup \delta_5 \cup \delta_6 \sim X^0 \cup \delta_4 \) and \( \delta_5 \cup \delta_6 \sim \xi_5 \). Hence \( \delta_5 \cup \delta_6 \not\sim \xi_1 \). Since the projections of the rectangles \( \delta_1 \) and \( \delta_2 \) on the prizes' axis are the same, the \( \xi_5 \) order between them is determined by their projections on the probabilities axis. The projection of \( \delta_3 \) and \( \delta_4 \) on the prizes' axis is also the same, hence the \( \xi_5 \) order between them is defined by their projections on the probabilities axis. Since the probabilities axis projections of \( \delta_3 \) and \( \delta_4 \) equal those of \( \delta_1 \) and \( \delta_2 \),
respectively, $\delta_4 \leq \delta_3 \iff \delta_2 \leq \delta_1$. This is summarized by the following assumption:

(d) \textit{Projection independence:} For every $x < y < z$ and $0 < p < q < r < 1$, 
$[y, z] \times [p, q] \leq^* [y, z] \times [q, r]$ \iff $[x, y] \times [p, q] \leq^* [x, y] \times [q, r]$. 

\textbf{THEOREM 2}

The following two conditions are equivalent:

(1) $\leq^*$ satisfies axioms (a)-(d).

(2) There exists a continuous and strictly increasing function $u : [0, M] \to \mathbb{R}$.
unique up to positive linear transformations, and a unique continuous and
strictly increasing function $f : [0, 1] \to [0, 1]$, satisfying $f(0) = 0$ and $f(1) = 1$,
such that $X \leq Y$ iff

$$
-M \int_0^M u(x) \, dF_X(x) \geq -M \int_0^M u(x) \, dF_Y(x). 
$$

\textit{Proof}

Obviously, (2) $\Rightarrow$ (1). Assume now that $\leq^*$ satisfies axioms (a)-(d). From
theorem 1, it follows that there exists a measure $\nu$ representing $\leq^*$ on $L$. Define
$u : M \to \mathbb{R}$ by $u(x) = \nu([0, x] \times [0, 1])$. Let $f(p) = \nu([0, 1] \times [1 - p, 1])$. Axiom
(d) implies that for $0 < x < y$ and $p < q$, $\nu([x, y] \times [p, q]) = [u(y) - u(x)]$
$\times [f(1 - p) - f(1 - q)]$. Theorem 2 thus follows. 

\textbf{Q.E.D.}

\textbf{Conclusion:} On $L^*$, the relation $\leq^*$ can be represented by the function

$$
V(x_1, p_1; \ldots; x_n, p_n)
$$

$$
= u(x_n) f(p_n) + \sum_{i=1}^{n-1} u(x_i) \left[ f \left( \sum_{j=i}^{n} p_j \right) - f \left( \sum_{j=i+1}^{n} p_j \right) \right] \quad (2.2)
$$

$$
= u(x_1) + \sum_{i=2}^{n} \left[ u(x_i) - u(x_{i-1}) \right] f \left( \sum_{j=i}^{n} p_j \right) . \quad (2.3)
$$

\textbf{Remark:} If $f(p) = p$, then (2.1)-(2.3) reduce to the expected utility representation
function.
3. The case of vectorial prizes

The construction of the representation function in section 2 depends on the assumption that the set of prizes is an ordered set. When the prizes are bundles of commodities, one must explicitly outline this assumption. For the sake of simplicity, here we deal only with prospects.

Let \( L^* = \{ (x_1, p_1; \ldots ; x_n, p_n) : x_1, \ldots , x_n \in K = [0, M]^k, p_1, \ldots , p_n \geq 0, \Sigma p_i = 1 \} \) and let \( \preceq \) be a complete and transitive continuous preference relation on \( L^* \). Define on \( K \) a preference relation \( \preceq' \) by \( x \preceq' y \) iff \( (x, 1) \preceq (y, 1) \), and assume from now on that if \( (x_1, p_1; \ldots ; x_n, p_n) \in L^* \), then \( x_n \preceq' \ldots \preceq' x_1 \). To axioms (a)–(d) add now the following assumption:

(e) If \( x_i \preceq x_i' \), then \( (x_1, p_1; \ldots ; x_i, p_i; \ldots ; x_n, p_n) \)

\[ \preceq (x_1, p_1; \ldots ; x_i', p_i; \ldots ; x_n, p_n) \]

Since \( \preceq \) is continuous, so is \( \preceq' \). Hence, \( v \) can be represented by a real, order-preserving function (Debreu [6]). For the time being, we arbitrarily choose one such function, \( v \). Later, we show that the representation of \( \preceq \) does not depend on \( v \). By using the function \( v \), elements of \( L^* \) may be represented as lotteries over utilities of the form \( (v(x_1), p_1; \ldots ; v(x_n), p_n) \). On such lotteries, assumptions (a)–(d) of section 2 imply that \( \preceq \) on \( L^* \) can be represented by

\[
\bar{u}(v(x_1)) + \sum_{i=2}^{n} [\bar{u}(v(x_i)) - \bar{u}(v(x_{i-1}))] f \left( \sum_{j=i}^{n} p_j \right).
\]

Let \( u = \bar{u} \circ v \) and obtain that \( \preceq \) can be represented by

\[
V = u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})] f \left( \sum_{j=i}^{n} p_j \right),
\]

where \( f(0) = 0, f(1) = 1, u \) and \( f \) are strictly increasing and continuous. \( f \) is unique, and \( u \) is unique up to positive linear transformations.

**Theorem 3**

\( u \) and \( f \) do not depend on the choice of \( v \).

**Proof**

Assume that the relation \( \preceq \) on \( L^* \) can be represented by (3.1) and by

\[
V^* = u^*(x_1) + \sum_{i=2}^{n} [u^*(x_i) - u^*(x_{i-1})] f^* \left( \sum_{j=i}^{n} p_j \right),
\]

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where \( f(0) = f^*(0) = 0, \ f(1) = f^*(1) = 1, \ u(0) = u^*(0) = 0. \) Therefore, there exists a continuous function \( h \) such that \( V^* = h(V) \) (see (3.1)–(3.2)). In particular, for every \( x \) and \( p \)

\[
u^*(x)f^*(p) = h(u(x)f(p)).
\]

(3.3)

Substitute \( p = 1 \) and obtain

\[
u^*(x) = h(u(x)).
\]

Substitute \( x \) such that \( u(x) = 1 \) and obtain

\[
h(1)f^*(p) = h(f(p)).
\]

Hence, by (3.3),

\[
\frac{h(u(x))h(f(p))}{h(1)} = h(u(x)f(p)).
\]

The solution of this functional equation is \( h(a) = a a^b \) (see Aczél [1]), hence \( u^*(x) = a [u(x)]^b \) and \( f^*(p) = [f(p)]^b \).

We now prove that \( b = 1 \). Let \( x \preceq y \) and let \( X = u(x), \ Y = u(y) \) and \( P = f(p) \). There exists \( z \) such that \( (z, 1) \sim (y, 1 - p; x, p) \), hence

\[
u(z) = u(y) + [u(x) - u(y)] f(p) = Y + [X - Y] P.
\]

By (3.2), we obtain

\[
u^*(z) = u^*(y) + [u^*(x) - u^*(y)] f^*(p).
\]

Since \( u^*(z) = a [u(z)]^b \), it follows that

\[
\]

Differentiating with respect to \( X \), we obtain, for \( P \neq 0 \),

\[
[Y + (X - Y) P]^{b^{-1}} = (XP)^{b^{-1}}.
\]

For \( b \neq 1 \), this last equality implies that \( Y(1 - P) = 0 \), in contradiction to the assumption that (3.4) holds true for every \( X, Y, \) and \( P \). It thus follows that \( b = 1 \).

Q.E.D.
4. Discussion

Section 2 developed a set of axioms implying that the preference relation $\mathcal{Z}$ can be represented by the anticipated utility functional. This set of axioms emphasized the fact that this functional is actually a measure. Indeed, as is claimed in theorem 1, assumptions (a)–(c) imply that the value of a lottery $X$ is a measure of the area in $\Gamma$ above the graph of $F_X$. Given this, one may try to add other axioms which will imply more specific measures. One such attempt is presented in this paper. As is proved in theorem 2, axiom (d) guarantees that this measure is a product measure. Other axioms may imply other functional forms. Alternatively, one may analyze the general representation function obtained by theorem 1. In that case, it seems essential to assume strict first-order stochastic dominance, to avoid unreasonable relations like $V(X) = \sup x : F_X(x) \leq 1 - x/M$.

This paper assumed that all lotteries are simple, one-stage lotteries. A possible source for new axioms may be obtained by extending the space of lotteries to include two-stage lotteries. For this, see Segal [15].

Appendix

Claim

$R = \bigcup_X R_X$ is acyclic.

Proof

Let $\delta_1 R \delta_2 R \ldots R \delta_i R \delta_1$ and let $\delta_{t+1} = \delta_1$. There are $X_1, \ldots, X_t$ such that $X_i^0 \cup \delta_i \preceq X_i^0 \cup \delta_{i+1}$, $i = 1, \ldots, t$. If $X_1 \sim \ldots \sim X_t$, then by the transitivity of $\preceq$ and by the irrelevance axiom it follows that $X_i^0 \cup \delta_i \sim X_i^0 \cup \delta_{i+1}$, $i = 1, \ldots, t$, hence $\delta_1 R \delta_t R \ldots R \delta_i R \delta_1$ and $R$ is acyclic.

Let $\delta_1 P \delta_2$ mean $\delta_1 R \delta_2$ but not $\delta_2 R \delta_1$, and let $\delta_1 \not\sim \delta_2$ mean $\delta_1 R \delta_2$ and $\delta_2 R \delta_1$. Suppose now that $\delta_1 R \delta_2 R \ldots R \delta_i P \delta_1$. Of course, there do not exist $X_1, \ldots, X_t$ as above. However, in that case it is possible to find $\xi_{t+1} = \xi_1, \ldots, \xi_t \in \Delta$ such that $\xi_1 R \xi_2 R \ldots R \xi_i P \xi_1$, and there is $Y \in I$ such that $Y^0 \cup \xi_i \preceq Y^0 \cup \xi_{i+1}$, $i = 1, \ldots, t$, which contradicts, of course, the transitivity assumption.

Let $\delta \in \Delta$. $\{\delta^1, \delta^2\}$ is a horizontal (vertical) partition of $\delta$ if $\delta^1, \delta^2 \in \Delta$, $\delta^1 \cup \delta^2 = \delta$, $\text{Int} \delta^1 \cap \text{Int} \delta^2 = \emptyset$, $\delta^1$ is above (to the left of) $\delta^2$ and $\delta, \delta^1$, and $\delta^2$ have the same projection on the horizontal (vertical) axis.

Let $\delta_1 P \delta_2$. There are horizontal partitions of $\delta_1$ and $\delta_2$ into $\delta^1_1, \delta^2_1$ and $\delta^1_2, \delta^2_2$, respectively, such that $\delta^1_1 P \delta^1_2$ and $\delta^2_1 P \delta^2_2$. Indeed, there is $X$ such that $X^0 \cup \delta^1_1 \cup \delta^2_1 \preceq X^0 \cup \delta^1_2 \cup \delta^2_2$. Suppose that $X^0 \cup \delta^1_1 \cup \delta^2_1 \sim X^0 \cup \delta^1_2 \cup \delta^2_2$. It thus follows that $\delta^1_2 P \delta^2_2$. By the continuity and the stochastic dominance axiom, it follows that these partitions could be constructed such that $\delta^1_1 P \delta^2_2$. The same
argument holds for vertical partitions. It thus follows that $\delta_1, \ldots, \delta_r$ may be assumed to be sufficiently small, with no side panels on the boundary of $[0, M] \times [0, 1]$; and such that for $i \neq j$, $\delta_i \cap \delta_j = \emptyset$. Obviously, $\delta_i$ and $\delta_{i+1}$ can be compared by the relation $R$ iff either the lower left-hand corner of $\delta_{i+1}$ is above and to the right of the upper right-hand corner of $\delta_i$, or if the lower left-hand corner of $\delta_i$ is above and to the right of the upper right-hand corner of $\delta_{i+1}$. It therefore follows that each $\delta_i$ can be replaced by $\xi_i$ such that $\delta_i \xi_i$, $i = 1, \ldots, r$, and such that the lower left-hand corner of $\xi_{i+1}$ is above and to the right of the upper right-hand corner of $\xi_i$. Moreover, these $\xi_1, \ldots, \xi_r$ keep the $R$-relation between $\delta_1, \ldots, \delta_r$. The required $Y$ exists, and the contradiction proves the claim.

Q.E.D.

References
